

Dynamical of Kirchhoff type BBM equations on unbounded domains

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ABSTRACT

This paper is concerned with the pullback attractors for the Kirchhoff type BBM equations defined on unbounded domains. Sobolev embeddings are invalid on unbounded domains. We obtain the pullback asymptotic compactness of such non-autonomous BBM equations by using the method of uniform tail-estimates.

Keywords: Pullback attractors, BBM equations, Kirchhoff type, unbounded domains

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1. Introduction

Consider the following Kirchhoff type BBM equation in unbounded domains:

$$u_t - \Delta u_t - \nu \Delta u + \nabla \cdot \vec{F}(u) - M(\|\nabla u\|^2) \Delta u = g(x, t), \quad (1)$$

with the initial value conditions

$$u(x, \tau) = u_\tau(x), u(x, t) = 0. \quad (2)$$

Assume $M(\cdot) \in C^1(\mathbb{R})$, $M'(s) \geq 0$

$$0 \leq M(s) \leq C, \quad (3)$$

where $x \in \mathcal{O}$, and ν is positive constant, \mathcal{O}_0 is a bounded open subset in \mathbb{R}^2 , $\mathcal{O} = \mathcal{O}_0 \times \mathbb{R}$ is a unbounded channel, $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathcal{O}))$ is a deterministic time-dependent external

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forcing term, \vec{F} is a nonlinear vector function give by

$$\vec{F}(s) = (F_1(s), F_2(s), F_3(s)), s \in \mathbb{R}. \quad (4)$$

Benjamin, Bona and Mahony proposed the BBM equation in [3], which describes the mathematical model of long waves propagation through nonlinear dispersion and dissipation effects. Then many scholars have studied the asymptotic behavior of BBM equations in [2, 14, 15, 16] on bounded domains and in [6, 13, 17] on unbounded domains. The existence and uniqueness of solutions to the BBM equation have been studied in [1, 5, 9, 10]. In recent years, Chen studied the existence and uniqueness of pullback random attractors of BBM equations driven by additive noise, white noise or colored noise, see [7, 6, 17, 12]. Other equations with Kirchhoff terms have been studied by many scholars. In [18], the authors showed the asymptotic behavior for kirchhoff type stochastic plate equations on unbounded domains. In [19], the authors showed the random attractors of Kirchhoff-type reaction-diffusion equations without uniqueness driven by nonlinear colored noise. With the help of reference [14, 4, 11]. We studied the pullback attractors for the Kirchhoff type BBM equations defined on unbounded channels. Due to the Sobolev embedding theorem is noncompact on unbounded domain, to conquer this difficulty we use the uniform tail estimation of the solution.

The structure of our article is as follows. In Section 2, we review some results of the pullback attractor. In Section 3, we discuss the uniform tail-estimates of solutions. In Section 4, we prove the existence of the pullback attractor of the Kirchhoff BBM equation.

In this paper, we use $\|\cdot\|$ and (\cdot, \cdot) to denote the norm and the inner product of $L^2(\mathcal{O})$, respectively. If $u \in H_0^1(\mathcal{O})$, we define $\|u\|_{H_0^1(\mathcal{O})} = \|\nabla u\|$ and $\|u\|_{H^1(\mathcal{O})} = (\|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}$. For easy of writing, each C used in this article represents a different positive constant.

2. Preliminaries and some results

In this paper, we will use the following Poincaré inequality :

$$\|\nabla u\|^2 \geq \lambda \|u\|^2, \quad \forall u \in H_0^1(\mathcal{O}), \quad (5)$$

where,

$$\gamma = \min \left\{ \nu, \frac{1}{4} \nu \lambda \right\}. \quad (6)$$

We define that $G_i (i = 1, 2, 3), F_i (i = 1, 2, 3)$ meet the following conditions:

$$G_i(u) = \int_0^u F_i(t) dt, \quad \vec{G}(u) = (G_1(u), G_2(u), G_3(u)), \quad u \in \mathbb{R}, \quad (7)$$

$$|F_i(u)| \leq |u| + C|u|^2, |F_i'(u)| \leq 1 + C|u|, |G_i(u)| \leq C|u|^2 + C|u|^3, \quad u \in \mathbb{R} \quad (8)$$

where $t, u \in \mathbb{R}, 0 \leq p < 1$.

By [14], we get that for $\beta \geq 0$ and $0 \leq \alpha < \frac{\gamma}{2}$ such that

$$\|g(t)\|^2 \leq \beta e^{\alpha|t|} \tag{9}$$

and

$$\int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr < \infty \quad \text{for all } \tau \in \mathbb{R}, \tag{10}$$

$$\int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr < \infty \quad \text{for all } \tau \in \mathbb{R}. \tag{11}$$

Furthermore

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\tau} \int_{|x_3| \geq k} e^{\gamma r} |g(x, r)|^2 dx dr = 0 \quad \text{for all } \tau \in \mathbb{R}. \tag{12}$$

Let $P = \mathbb{R}$ and define $\Phi : \mathbb{R}^+ \times P \times H_0^1(\mathcal{O}) \rightarrow H_0^1(\mathcal{O})$ by

$$\Phi(t, \tau, u_\tau) = u(t + \tau, \tau, u_\tau), \tag{13}$$

and

$$\Phi(t + s, \tau, u_\tau) = \Phi(t, s + \tau, \Phi(s, \tau, u_\tau)), \quad t, s \geq 0, \tau \in \mathbb{R}, u_\tau \in H_0^1(\mathcal{O}). \tag{14}$$

Set $\mathcal{R}_\gamma = \{r : \mathbb{R} \rightarrow (0, \infty) : \lim_{s \rightarrow -\infty} e^{\gamma s} r^2(s) = 0\}$ and

$$\mathcal{B}_\gamma = \{B = \{B(s)\}_{s \in \mathbb{R}} | B(s) \subset \overline{D}(0, r(s)) \text{ for some } r \in \mathcal{R}_\gamma\}, \tag{15}$$

where $\overline{D}(0, r(s))$ denotes the closed ball in $H_0^1(\mathcal{O})$.

Set P be a non-empty set and X a metric space with distance $d(\cdot, \cdot)$.

Definition 2.1. A family of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ from P to itself is called a family of shift operators on P , if $\{\theta_t\}_{t \in \mathbb{R}}$ satisfies the following group properties:

- (i) $\theta_0(p) = p$ for all $p \in P$,
- (ii) $\theta_{t+\tau}(p) = \theta_t(\theta_\tau(p))$ for all $p \in P$ and $t, \tau \in \mathbb{R}$.

Definition 2.2. Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a family of shift operators on P . Then a continuous θ -cocycle Φ on X is a mapping $\Phi : \mathbb{R}^+ \times P \times X \rightarrow X$ satisfying, for all $p \in P$ and $t, \tau \in \mathbb{R}$,

- (i) $\Phi(0, q, \cdot)$ is the identity on X ;
- (ii) $\Phi(t + \tau, p, \cdot) = \Phi(t, \theta_\tau(p), \Phi(\tau, p, \cdot))$;
- (iii) $\Phi(t, p, \cdot) : X \rightarrow X$ is continuous.

Suppose \mathcal{B} is a nonempty class of parameterized sets $B = \{B(p)\}_{p \in P}, B(P) \subset X$ for every $p \in P$.

Definition 2.3. It is said $D = \{D(p)\}_{p \in B} \in \mathcal{B}$ is pullback \mathcal{B} -absorbing for Φ in \mathcal{B} if for every $p \in P$ and $B \in \mathcal{B}$, there exists $T(p, B) > 0$ such that

$$\Phi(t, \theta_{-t}(p), B(\theta_{-t}(p))) \subset D(p), \quad \forall t \geq T(p, B). \tag{16}$$

Definition 2.4. The cocycle Φ is said to be pullback \mathcal{B} -asymptotically compact in X , if for every $p \in P$, $\Phi(t_n, \theta_{-t_n}(p), x_n)$ has a convergent subsequence in X whenever $t_n \rightarrow +\infty$ and $x_n \in B(\theta_{-t_n}(p))$ with $\{B(p)\}_{p \in P} \in \mathcal{B}$.

Definition 2.5. A family $C = \{C(p)\}_{p \in P} \in \mathcal{B}$ is said to be pullback \mathcal{B} -attracting if

$$\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, \theta_{-t}(p), B(\theta_{-t}(p))), C(p)) = 0 \quad \text{for all } p \in P, B \in \mathcal{B}, \tag{17}$$

where $\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$ is the Hausdorff semi-distance between X and Y .

Definition 2.6. It is said that $A = \{A(p)\}_{p \in P} \in \mathcal{B}$ is a pullback \mathcal{B} -attractor if it satisfies

- (1) $A(p)$ is compact in X for any $p \in P$;
- (2) A is pullback \mathcal{B} -attracting;
- (3) A is invariant, that is, $\Phi(t, p, A(p)) = A(\theta_t(p))$ for all $(t, p) \in \mathbb{R}^+ \times P$.

Theorem 2.7. Let Φ be a continuous θ -cocycle on X and there exists a family of pullback \mathcal{B} -absorbing sets $\{D(p)\}_{p \in P} \in \mathcal{B}$. If Φ is a pullback \mathcal{B} -asymptotically compact in X , then Φ has a unique pullback \mathcal{B} -attractor $\{A(p)\}_{p \in P} \in \mathcal{B}$ defined by

$$A(p) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}(p), D(\theta_{-t}(p)))}. \tag{18}$$

3. Uniform estimates of solutions

Lemma 3.1. For every $\tau \in \mathbb{R}$, $B = \{B(s)\}_{s \in \mathbb{R}} \in \mathcal{B}_\gamma$ and $u_0(\tau - t) \in B(\tau - t)$, there exists $T = T(\tau, B) > 0$ such that, for all $t \geq T$

$$\begin{aligned} \|u(\tau, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 &+ \int_{\tau-t}^s e^{\gamma r} (M(\|\nabla u\|^2) \|\nabla u\|_{H_0^1(\mathcal{O})}^2) dr \\ &\leq \frac{2}{\nu \lambda} e^{-\gamma \tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr, \end{aligned} \tag{19}$$

$$\int_{\tau-t}^{\tau} e^{\gamma r} \|u(r, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 dr \leq \frac{2}{\nu \lambda \gamma} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr, \tag{20}$$

and

$$\begin{aligned} \|u(s, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 &+ \int_{\tau-t}^s e^{\gamma r/2} (M(\|\nabla u\|^2) \|\nabla u\|_{H_0^1(\mathcal{O})}^2) dr \\ &\leq \frac{2}{\nu \lambda} e^{-\gamma s/2} \int_{-\infty}^s e^{\gamma r/2} \|g(r)\|^2 dr, \quad s \in [\tau - t, \tau]. \end{aligned} \tag{21}$$

Proof. Suppose $\tau \in \mathbb{R}$, $t \geq 0$, $B = \{B(s)\}_{s \in \mathbb{R}} \in \mathcal{B}_\gamma$, $u_0(\tau - t) \in B(\tau - t)$, and define $u(r) = u(r, \tau - t, u_0(\tau - t))$ for $r \geq \tau - t$. Then by (1) $u(r)$ in $L^2(\mathcal{O})$, we get

$$\frac{1}{2} \frac{d}{dr} (\|u\|^2 + \|\nabla u\|^2) + \nu \|\nabla u\|^2 + M(\|\nabla u\|^2) \|\nabla u\| + \int_{\mathcal{O}} (\nabla \cdot \vec{F}(u)) u dx = (g, u). \quad (22)$$

We have

$$\int_{\mathcal{O}} (\nabla \cdot \vec{F}(u)) u dx = - \int_{\mathcal{O}} \vec{F}(u) \cdot \nabla u dx = - \int_{\mathcal{O}} \nabla \cdot \vec{G}(u) dx = 0. \quad (23)$$

By Young's inequality, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dr} (\|u\|^2 + \|\nabla u\|^2) + \nu \|\nabla u\|^2 + M(\|\nabla u\|^2) \|\nabla u\| &\leq \frac{\nu\lambda}{2} \|u\|^2 + \frac{1}{2\nu\lambda} \|g\|^2 \\ &\leq \frac{\nu}{2} \|\nabla u\|^2 + \frac{1}{2\nu\lambda} \|g\|^2, \end{aligned} \quad (24)$$

Set $s \in [\tau - t, \tau]$. By Gronwall lemma and integrating it over $(\tau - t, s)$, we get

$$\begin{aligned} &\|u(s, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 + \int_{\tau-t}^s e^{\gamma r} (M(\|\nabla u\|^2) \|\nabla u\|_{H_0^1(\mathcal{O})}^2) dr \\ &\quad + \gamma \int_{\tau-t}^s e^{\gamma r} \|u(r, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 dr \\ &\leq e^{\gamma(\tau-t)} \|u_0(\tau - t)\|_{H_0^1(\mathcal{O})}^2 + \frac{1}{\nu\lambda} \int_{-\infty}^s e^{\gamma r} \|g(r)\|^2 dr. \end{aligned} \quad (25)$$

Note that $u_0(\tau - t) \in B(\tau - t)$ and $B = \{B(s)\}_{s \in \mathbb{R}} \in \mathcal{B}_\gamma$, $T = T(s, B) > 0$,

$$e^{\gamma(\tau-t)} \|u_0(\tau - t)\|_{H_0^1(\mathcal{O})}^2 \leq \frac{1}{\nu\lambda} \int_{-\infty}^s e^{\gamma r} \|g(r)\|^2 dr, \quad t \geq T. \quad (26)$$

By (25), we get

$$\begin{aligned} &\|u(s, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 + \int_{\tau-t}^s e^{\gamma r} (M(\|\nabla u\|^2) \|\nabla u\|_{H_0^1(\mathcal{O})}^2) dr \\ &\leq \frac{2}{\nu\lambda} e^{-\gamma s} \int_{-\infty}^s e^{\gamma r} \|g(r)\|^2 dr. \end{aligned} \quad (27)$$

By (27), we know that

$$\begin{aligned} &\|u(s, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 + \int_{\tau-t}^s e^{\gamma r} (M(\|\nabla u\|^2) \|\nabla u\|_{H_0^1(\mathcal{O})}^2) dr \\ &\leq \frac{2}{\nu\lambda} e^{-\gamma s/2} \int_{-\infty}^s e^{\gamma r/2} \|g(r)\|^2 dr, \quad s \in [\tau - t, \tau]. \end{aligned} \quad (28)$$

This completes the proof of Lemma 3.1. \square

Lemma 3.2. *For every $\tau \in \mathbb{R}$, $B = \{B(s)\}_{s \in \mathbb{R}} \in \mathcal{B}_\gamma$ and $u_0(\tau - t) \in B(\tau - t)$, there exists $T = T(\tau, B) > 0$ such that, for all $t \geq T$*

$$\int_{\tau-t}^{\tau} e^{\gamma r} \|u_r\|_{H_0^1(\mathcal{O})}^2 dr \leq C \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr + C \int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr. \quad (29)$$

Proof. Taking the inner product of (1) with $u_r(r)$ in $L^2(\mathcal{O})$ and using Young's inequality, we get

$$\begin{aligned} \|u_r\|^2 + \|\nabla u_r\|^2 &= -\nu(\nabla u, \nabla u_r) - M(\|\nabla u\|^2)(\nabla u, \nabla u_r) - \int_{\mathcal{O}} (\nabla \cdot \vec{F}(u))u_r dx + (g, u_r) \\ &\leq \frac{1}{4}\|\nabla u_r\|^2 + \nu^2\|\nabla u\|^2 + \frac{1}{4}\|\nabla u_r\|^2 + M(\|\nabla u\|^2)^2\|\nabla u\|^2 \\ &\quad - \int_{\mathcal{O}} (\nabla \cdot \vec{F}(u))u_r dx + \frac{1}{2}\|g\|^2 + \frac{1}{2}\|u_r\|^2. \end{aligned} \quad (30)$$

Using (8), we obtain

$$\begin{aligned} - \int_{\mathcal{O}} (\nabla \cdot \vec{F}(u))u_r dx &= \int_{\mathcal{O}} \vec{F}(u) \cdot \nabla u_r dx \\ &\leq C \int_{\mathcal{O}} |u| |\nabla u_r| dx + C \int_{\mathcal{O}} |u|^2 |\nabla u_r| dx \\ &\leq \frac{1}{2}\|\nabla u_r\|^2 + C(\|u\|^2 + \|u\|_4^4) \\ &\leq \frac{1}{2}\|\nabla u_r\|^2 + C(\|u\|_{H_0^1(\mathcal{O})}^2 + \|u\|_{H_0^1(\mathcal{O})}^4). \end{aligned} \quad (31)$$

Therefore, we have

$$\|u_r\|^2 + \|\nabla u_r\|^2 \leq C(\|u\|_{H_0^1(\mathcal{O})}^2 + \|g\|^2 + \|u\|_{H_0^1(\mathcal{O})}^4). \quad (32)$$

Multiplying this by $e^{\gamma r}$, integrating it over $(\tau - t, \tau)$, we have

$$\int_{\tau-t}^{\tau} e^{\gamma r} (\|u_r\|^2 + \|\nabla u_r\|^2) dr \leq C \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr + C \int_{\tau-t}^{\tau} e^{\gamma r} \|u(r)\|_{H_0^1(\mathcal{O})}^4 dr. \quad (33)$$

By (21), we get

$$\begin{aligned} \int_{\tau-t}^{\tau} e^{\gamma r} \|u(r)\|_{H_0^1(\mathcal{O})}^4 dr &\leq C \int_{\tau-t}^{\tau} e^{\gamma r} \left(e^{-\gamma r/2} \int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr \\ &\leq C \int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr. \end{aligned} \quad (34)$$

□

Lemma 3.3. For every $\epsilon > 0, \tau \in \mathbb{R}$ and $B = \{B(s)\}_{s \in \mathbb{R}} \in \mathcal{B}_\delta$, there exist $T = T(\tau, B, \epsilon) > 0$ and $K = K(\tau, \epsilon) > 1$ such that for all $t \geq T$ and $k \geq K$,

$$\int_{|x_3| \geq k} (|u(x, \tau, \tau - t, u_0(\tau - t))|^2 + |\nabla u(x, \tau, \tau - t, u_0(\tau - t))|^2) dx \leq \epsilon, \quad (35)$$

where $u_0(\tau - t) \in B(\tau - t)$.

Proof. Let $k \geq 1$ and take a smooth function φ such that $0 \leq \varphi \leq 1$ for all $s \in \mathbb{R}^+$ and

$$\varphi(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2. \end{cases} \tag{36}$$

Let $|\varphi'(s)| \leq C$ for all $s \in \mathbb{R}^+$. Taking the inner product of (1) by $\varphi^2\left(\frac{x_3^2}{k^2}\right)u$, we observe that

$$\begin{aligned} & \frac{d}{dr} \int_{\mathcal{O}} \varphi^2\left(\frac{x_3^2}{k^2}\right) (|u|^2 + |\nabla u|^2) dx + 2\nu \int_{\mathcal{O}} \varphi^2\left(\frac{x_3^2}{k^2}\right) |\nabla u|^2 dx \\ & + 2M(\|\nabla u\|^2) \int_{\mathcal{O}} \varphi^2\left(\frac{x_3^2}{k^2}\right) |\nabla u|^2 dx \\ & = -2 \int_{\mathcal{O}} \left(\nabla \varphi^2\left(\frac{x_3^2}{k^2}\right) \cdot \nabla u_r \right) u dx - 2\nu \int_{\mathcal{O}} \left(\nabla \varphi^2\left(\frac{x_3^2}{k^2}\right) \cdot \nabla u \right) u dx \\ & - 2M(\|\nabla u\|^2) \int_{\mathcal{O}} \left(\nabla \varphi^2\left(\frac{x_3^2}{k^2}\right) \cdot \nabla u \right) u dx \\ & - 2 \int_{\mathcal{O}} (\nabla \cdot \vec{F}(u)) \varphi^2\left(\frac{x_3^2}{k^2}\right) u dx + 2 \int_{\mathcal{O}} g \varphi^2\left(\frac{x_3^2}{k^2}\right) u dx \\ & := J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{37}$$

Then, we will handle each term in (37) we have

$$\begin{aligned} J_1 & \leq 2 \int_{\mathcal{O}} |2\varphi\varphi'\left(\frac{x_3^2}{k^2}\right)| \frac{2|x_3|}{k^2} |\nabla u_r| |u| dx \\ & \leq C \int_{k \leq |x_3| \leq \sqrt{2}k} \frac{2|x_3|}{k^2} |\nabla u_r| |u| dx \\ & \leq \frac{C}{k} \int_{k \leq |x_3| \leq \sqrt{2}k} |\nabla u_r| |u| dx \\ & \leq \frac{C}{k} (\|\nabla u_r\|^2 + \|u\|^2). \end{aligned} \tag{38}$$

Similarly,

$$J_2 \leq \frac{C}{k} (\|\nabla u\|^2 + \|u\|^2). \tag{39}$$

$$J_3 \leq \frac{C}{k} (\|\nabla u\|^2 + \|u\|^2). \tag{40}$$

We get

$$J_4 = 2 \int_{\mathcal{O}} (\vec{F}(u) \cdot \nabla u) \varphi^2\left(\frac{x_3^2}{k^2}\right) dx + 2 \int_{\mathcal{O}} \left(\vec{F}(u) \cdot \nabla \varphi^2\left(\frac{x_3^2}{k^2}\right) \right) u dx. \tag{41}$$

By Green formula and $\vec{G}(u) = \vec{0}$, we obtain

$$\begin{aligned} 2 \int_{\mathcal{O}} (\vec{F}(u) \cdot \nabla u) \varphi^2 \left(\frac{x_3^2}{k^2} \right) dx &= 2 \int_{\mathcal{O}} (\nabla \cdot \vec{G}(u)) \varphi^2 \left(\frac{x_3^2}{k^2} \right) dx \\ &= -2 \int_{\mathcal{O}} \vec{G}(u) \cdot \nabla \varphi^2 \left(\frac{x_3^2}{k^2} \right) dx. \end{aligned} \quad (42)$$

Using (42) and (7), (8), we get

$$\begin{aligned} J_4 &\leq 2 \int_{\mathcal{O}} |\vec{G}(u)| |2\varphi\varphi' \left(\frac{x_3^2}{k^2} \right)| \frac{2|x_3|}{k^2} dx + 2 \int_{\mathcal{O}} |\vec{F}(u)| |2\varphi\varphi' \left(\frac{x_3^2}{k^2} \right)| \frac{2|x_3|}{k^2} |u| dx \\ &\leq \frac{C}{k} \int_{k \leq |x_3| \leq \sqrt{2}k} |\vec{G}(u)| dx + \frac{c}{k} \int_{k \leq |x_3| \leq \sqrt{2}k} |\vec{F}(u)| |u| dx \\ &\leq \frac{C}{k} \int_{\mathcal{O}} (|u|^2 + |u|^3) dx \leq \frac{C}{k} (\|u\|^2 + \|u\|_3^3) \\ &\leq \frac{C}{k} (\|u\|^2 + \|u\|_{H_0^1(\mathcal{O})}^3) \leq \frac{C}{k} (1 + \|u\|^2 + \|u\|_{H_0^1(\mathcal{O})}^4). \end{aligned} \quad (43)$$

Using Young's inequality, we have

$$\begin{aligned} J_5 &= 2 \int_{|x_3| \geq k} g \varphi^2 \left(\frac{x_3^2}{k^2} \right) u dx \\ &\leq \frac{\nu\lambda}{4} \int_{|x_3| \geq k} \varphi^2 \left(\frac{x_3^2}{k^2} \right) u^2 dx + \frac{4}{\nu\lambda} \int_{|x_3| \geq k} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |g|^2 dx \\ &\leq \frac{\nu\lambda}{4} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) u^2 dx + \frac{4}{\nu\lambda} \int_{|x_3| \geq k} |g|^2 dx. \end{aligned} \quad (44)$$

By (38)-(44), we get

$$\begin{aligned} &\frac{d}{dr} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) (|u|^2 + |\nabla u|^2) dx + 2\nu \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u|^2 dx \\ &\leq \frac{C}{k} (1 + \|u\|_{H_0^1(\mathcal{O})}^2 + \|\nabla u_r\|^2 + \|u\|_{H_0^1(\mathcal{O})}^4) \\ &\quad + \frac{\nu\lambda}{4} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) u^2 dx + \frac{4}{\nu\lambda} \int_{|x_3| \geq k} |g|^2 dx. \end{aligned} \quad (45)$$

By [14],

$$2\nu \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u|^2 dx \geq \nu \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u|^2 dx + \frac{\nu\lambda}{2} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) u^2 dx - \frac{C}{k^2} \|u\|^2, \quad (46)$$

Similarly,

$$\begin{aligned} -2 \int_{\mathcal{O}} M(\|\nabla u\|^2) \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u(t)|^2 dx &\leq -M(\|\nabla u\|^2) \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u|^2 dx \\ &\quad - \frac{M(\|\nabla u\|^2)\lambda}{2} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |u|^2 dx + \frac{C}{k^2} \|u\|^2 \\ &\leq \frac{C}{k} \|u\|^2, \end{aligned} \quad (47)$$

we get

$$\begin{aligned} & \frac{d}{dr} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) (|u|^2 + |\nabla u|^2) dx + \nu \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) |\nabla u|^2 dx + \frac{\nu\lambda}{4} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) u^2 dx \\ & \leq \frac{4}{\nu\lambda} \int_{|x_3| \geq k} |g|^2 dx + \frac{C}{k} (1 + \|u\|_{H_0^1(\mathcal{O})}^2 + \|\nabla u_r\|^2 + \|u\|_{H_0^1(\mathcal{O})}^4). \end{aligned} \quad (48)$$

Finally, Using Gronwall lemma, integrating it over $(\tau - t, \tau)$ and Lemma 3.1, Lemma 3.2 and (17), we have

$$\begin{aligned} & e^{\gamma\tau} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) (|u(x, \tau, \tau - t, u_0(\tau - t))|^2 + |\nabla u(x, \tau, \tau - t, u_0(\tau - t))|^2) dx \\ & \leq e^{\gamma(\tau-t)} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) (|u_0|^2 + |\nabla u_0|^2) dx + \frac{4}{\nu\lambda} \int_{\tau-t}^{\tau} e^{\gamma r} \int_{|x_3| \geq k} |g(x, r)|^2 dx dr \\ & \quad + \frac{C}{k} \int_{\tau-t}^{\tau} e^{\gamma r} (1 + \|u\|_{H_0^1(\mathcal{O})}^2 + \|\nabla u_r\|^2 + \|u(r)\|_{H_0^1(\mathcal{O})}^4) dr \\ & \leq e^{\gamma(\tau-t)} \int_{\mathcal{O}} \varphi^2 \left(\frac{x_3^2}{k^2} \right) (|u_0|^2 + |\nabla u_0|^2) dx + \frac{4}{\nu\lambda} \int_{\tau-t}^{\tau} e^{\gamma r} \int_{|x_3| \geq k} |g(x, r)|^2 dx dr \\ & \quad + \frac{C}{k} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr + \frac{C}{k} (e^{\gamma\tau} - e^{\gamma(\tau-t)}) + \frac{C}{k} \int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr, \end{aligned} \quad (49)$$

and

$$\begin{aligned} & \int_{|x_3| \geq \sqrt{2}k} (|u(x, \tau, \tau - t, u_0(\tau - t))|^2 + |\nabla u(x, \tau, \tau - t, u_0(\tau - t))|^2) dx \\ & \leq e^{-\gamma\tau} e^{\gamma(\tau-t)} \|u_0(\tau - t)\|_{H_0^1(\mathcal{O})}^2 + \frac{4}{\nu\lambda} e^{-\gamma\tau} \int_{\tau-t}^{\tau} e^{\gamma r} \int_{|x_3| \geq k} |g(x, r)|^2 dx dr \\ & \quad + \frac{C}{k} + \frac{C}{k} e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr + \frac{C}{k} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr. \end{aligned} \quad (50)$$

Since $u_0(\tau - t) \in B(\tau - t) \in \mathcal{B}_\delta$, for given $\epsilon > 0$, there exists $T(\tau, B, \epsilon) > 0$ such that

$$e^{-\gamma\tau} e^{\gamma(\tau-t)} \|u_0(\tau - t)\|_{H_0^1(\mathcal{O})}^2 \leq \frac{\epsilon}{3}, \quad t \geq T. \quad (51)$$

By (10)-(12), we get

$$\frac{4}{\nu\lambda} e^{-\gamma\tau} \int_{\tau-t}^{\tau} e^{\gamma r} \int_{|x_3| \geq k} |g(x, t)|^2 dx dr \leq \frac{\epsilon}{3}, \quad (52)$$

$$\frac{C}{k} + \frac{C}{k} e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr + \frac{C}{k} \int_{-\infty}^{\tau} \left(\int_{-\infty}^r e^{\gamma s/2} \|g(s)\|^2 ds \right)^2 dr \leq \frac{\epsilon}{3}. \quad (53)$$

□

This completes the proof of Lemma 3.3.

Lemma 3.4. *Set $B \in \mathcal{B}_\gamma$ and $u_0(\tau - t) \in B(\tau - t)$. Then, for every $\epsilon > 0$ and $k \geq 1$, there exist $T = T(\tau, B, \epsilon) > 0$ and an integer $N = N(\tau, \epsilon) > 0$ such that*

$$\|(I - P_n)v(\tau, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O}_{2k})} < \epsilon, \quad \text{for all } t \geq T, n \geq N. \quad (54)$$

where $v_{\tau-t} = \xi \left(\frac{x_3^2}{k^2} \right) u_{\tau-t}$.

Proof. Set $v = \mathcal{P}_n v + (I - \mathcal{P}_n)v := v_1 + v_2$. Applying $I - \mathcal{P}_n$ to (1) and taking the inner product of the resulting equation with v_2 in $L^2(\mathcal{O}_{2k})$, by [8] we have

$$\begin{aligned} & \frac{d}{dt} \|v_2\|_{H^1(\mathcal{O}_{2k})}^2 + \nu \|\nabla v_2\|^2 + M(\|\nabla u\|^2) \|\nabla v_2\|^2 + \frac{1}{4} \nu \lambda_{n+1} \|v_2\|^2 \\ & \leq C(\lambda_{n+1}^{-1} + \lambda_{n+1}^{-\frac{1}{2}}) \left(1 + \|g(t)\|^2 + \|u\|_{H^1(\mathcal{O})}^4 \right). \end{aligned} \quad (55)$$

If $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, we know that there exists $N \in \mathbb{N}$ such that

$$\lambda_{n+1} \geq \max\{\lambda, 1\} \quad \text{for } n \geq N \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_{n+1}^{-\frac{1}{2}} = 0. \quad (56)$$

Thus, by (8) and (55) we know that for $n \geq N$,

$$\frac{d}{dt} \|v_2\|_{H^1(\mathcal{O}_{2k})}^2 + \gamma \|v_2\|_{H^1(\mathcal{O}_{2k})}^2 \leq C \lambda_{n+1}^{-\frac{1}{2}} \left(1 + \|g(t)\|^2 + \|u\|_{H^1(\mathcal{O})}^4 \right). \quad (57)$$

Using the Gronwall lemma to (57), and integrating over $(\tau - t, \tau)$, and $t \geq 0$, we have

$$\begin{aligned} \|v_2(\tau, \tau - t, v_{2, \tau-t})\|_{H^1(\mathcal{O}_{2k})}^2 & \leq e^{-\gamma t} \|(I - \mathcal{P}_n)(\xi u_{\tau-t})\|_{H^1(\mathcal{O}_{2k})}^2 \\ & \quad + C \lambda_{n+1}^{-\frac{1}{2}} \int_{\tau-t}^{\tau} e^{\gamma r} \left(1 + \|g(r)\|^2 \right) dr \\ & \quad + C \lambda_{n+1}^{-\frac{1}{2}} \int_{\tau-t}^{\tau} e^{\gamma r} \|u(r, \tau - t, u_{\tau-t})\|_{H^1(\mathcal{O})}^4 dr. \end{aligned} \quad (58)$$

By $\|I - \mathcal{P}_n\| \leq 1$, $\|\xi\| \leq 1$ and $B \in \mathcal{B}$, we obtain that as $t \rightarrow +\infty$,

$$e^{-\gamma t} \|(I - \mathcal{P}_n)(\xi u_{\tau-t})\|_{H^1}^2 \leq C e^{-\gamma t} \|u_{\tau-t}\|_{H^1(\mathcal{O})}^2 \leq C e^{-\gamma t} \|B\|_{H_0^1(\mathcal{O})}^2 < \epsilon. \quad (59)$$

Note that $\lambda_{n+1}^{-\frac{1}{2}} \rightarrow 0$ as $n \rightarrow \infty$. By [6], we have

$$\|v_2(\tau, \tau - t, v_{2, \tau-t})\|_{H^1(\mathcal{O}_{2k})}^2 < \epsilon \quad \text{as } n, t \rightarrow +\infty. \quad (60)$$

This completes the proof of Lemma 3.4. \square

4. Existence of pullback attractors

Lemma 4.1. *Let $\tau \in \mathbb{R}$, $B \in \mathcal{B}_\gamma$, $t_m \rightarrow \infty$ and $u_{0m} \in B(\tau - t_m)$. If $u_m(r, \tau - t_m, u_{0m})$ is the solution of (1) with initial condition u_{0m} and*

$$v_m(r, \tau - t_m, u_{0m}) = \xi \left(\frac{x_3^2}{k^2} \right) u_m(r, \tau - t_m, u_{0m}), \quad (61)$$

where $k \geq 1$. Then the sequence $\{v_m(\tau, \tau - t_m, u_{0m})\}$ has a convergent subsequence in $H_0^1(\mathcal{O})$.

Proof. Use Lemma 3.1, we get that

$$\|u(\tau, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O})}^2 \leq \frac{2}{\nu\lambda} e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr, \quad t \geq \tilde{T}. \quad (62)$$

By (62), we know

$$\|u_m(\tau, \tau - t_m, u_{0m})\|_{H_0^1(\mathcal{O})}^2 \leq \frac{2}{\nu\lambda} e^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr, \quad m \geq \tilde{M}. \quad (63)$$

From (61), we have

$$\|v_m(\tau, \tau - t_m, u_{0m})\|_{H_0^1(\mathcal{O})}^2 \leq ce^{-\gamma\tau} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr, \quad m \geq \tilde{M}. \quad (64)$$

Now, set $\epsilon > 0$ and $k \geq 1$. By Lemma 3.4, there exist $T_2 = T_2(\tau, B, \epsilon)$ and $N = N(\tau, \epsilon)$ such that

$$\|(I - P_N)v(\tau, \tau - t, u_0(\tau - t))\|_{H_0^1(\mathcal{O}_{2k})} \leq \epsilon, \quad t \geq \tilde{T}_1,$$

so there is $\tilde{M}_1 = \tilde{M}_1(\tau, B, \epsilon)$ such that $t_m \geq \tilde{T}_1$ for $m \geq \tilde{M}_1$.

$$\|(I - P_N)v_m(\tau, \tau - t_m, u_{0m})\|_{H_0^1(\mathcal{O}_{2k})} \leq \epsilon, \quad m \geq \tilde{M}_1.$$

Since $v_m(\tau, \tau - t_m, u_{0m}) = 0$ on $\mathcal{O} \setminus \mathcal{O}_{2k}$, $\{v_m(\tau, \tau - t_m, u_{0m})\}$ is precompact in $H_0^1(\mathcal{O})$. □

Lemma 4.2. *Let $\tau \in \mathbb{R}$, $B \in \mathcal{B}_\gamma$, $t_m \rightarrow \infty$ and $u_{0m} \in B(\tau - t_m)$, then $u_m(\tau, \tau - t_m, u_{0m})$ has a convergent subsequence in $H_0^1(\mathcal{O})$.*

Proof. See, for example [14]. □

It follow that the existence of a pullback attractor for the θ -cocycle Φ .

Lemma 4.3. *The problem (1) has a unique pullback \mathcal{B}_γ -attractor in $H_0^1(\mathcal{O})$.*

Proof. For $\tau \in \mathbb{R}$, we get

$$(r_\gamma(\tau))^2 = \frac{2}{\nu\lambda} \int_{-\infty}^{\tau} e^{\gamma r} \|g(r)\|^2 dr,$$

and

$$D_\gamma(\tau) = \{u \in H_0^1(\mathcal{O}) : \|u\|_{H_0^1(\mathcal{O})} \leq r_\gamma(\tau)\}.$$

Then, $D_\gamma(\tau) \in \mathcal{B}_\gamma$ is pullback \mathcal{B}_γ -absorbing for Φ in $H_0^1(\mathcal{O})$ by Lemma 3.1. By Lemma 4.2, Φ is pullback \mathcal{B}_γ -asymptotically compact in H_0^1 . Thus, Φ has a unique pullback \mathcal{B}_δ -attractor from Theorem 2.7. □

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References

- [1] M. Astaburuaga, E. Bisognin, V. Bisognin, and C. Fernandez. Global attractor and finite dimensionality for a class of dissipative equations of bbm's type. *Electronic Journal of Differential Equations*, 25:1–14, 1998.
- [2] J. Avrin. The generalized benjamin-bona-mahony equation in \mathbb{R}^n with singular initial data. *Nonlinear Analysis: Theory, Methods & Applications*, 11(1):139–147, 1987. [https://doi.org/10.1016/0362-546X\(87\)90032-0](https://doi.org/10.1016/0362-546X(87)90032-0).
- [3] T. B. Benjamin, J. L. Bona, and J. J. Mahony. Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 272(1220):47–78, 1972. <https://doi.org/10.1098/rsta.1972.0032>.
- [4] E. M. Bonotto, A. N. Carvalho, M. J. Nascimento, and E. B. Santiago. Lower semicontinuity of pullback attractors for a non-autonomous coupled system of strongly damped wave equations. *Applied Mathematics & Optimization*, 90(1):27, 2024. <https://doi.org/10.1007/s00245-024-10170-1>.
- [5] A. Celebi, V. Kalantarov, and M. Polat. Attractors for the generalized benjamin-bona-mahony equation. *Journal of Differential Equations*, 157(2):439–451, 1999. <https://doi.org/10.1006/jdeq.1999.3634>.
- [6] P. Chen, B. Wang, R. Wang, and X. Zhang. Multivalued random dynamics of benjamin-bona-mahony equations driven by nonlinear colored noise on unbounded domains. *Mathematische Annalen*, 386(1):343–373, 2023. <https://doi.org/10.1007/s00208-022-02400-0>.
- [7] P. Chen, R. Wang, and X. Zhang. Asymptotically autonomous robustness of random attractors for 3d bbm equations driven by nonlinear colored noise. *SIAM Journal on Mathematical Analysis*, 56(1):254–274, 2024. <https://doi.org/10.1137/22M1529129>.
- [8] Y. Chen. Remark on the global existence for the generalized benjamin-bona-mahony equations in arbitrary dimension. *Appl Anal - National Genomics Data Center*, 30(2):1–15, 1988.
- [9] I. D. Chueshov, M. Polat, and S. Siegmund. Gevrey regularity of global attractor for generalized benjamin-bona-mahony equation. *Mathematical Physics, Analysis, Geometry*, 11(2):226–242, 2004.
- [10] J. A. Goldstein and B. J. Wichnoski. On the benjamin-bona-mahony equation in higher dimensions. *Nonlinear Analysis: Theory, Methods & Applications*, 4(4):665–675, 1980. [https://doi.org/10.1016/0362-546X\(80\)90067-X](https://doi.org/10.1016/0362-546X(80)90067-X).
- [11] G. Kirchhoff. *Vorlesungen Über Mechanik*, volume 1. BG Teubner, 1897.

-
- [12] Y. Li and R. Wang. Random attractors for 3d benjamin–bona–mahony equations derived by a laplace-multiplier noise. *Stochastics and Dynamics*, 18(01):1850004, 2018. <https://doi.org/10.1142/S0219493718500041>.
- [13] Y. Li, R. Wang, and J. Yin. Backward compact attractors for non-autonomous benjamin–bona–mahony equations on unbounded channels. *Discrete and Continuous Dynamical Systems - B*, 22(7):2569–2586, 2017. <http://dx.doi.org/10.3934/dcdsb.2017092>.
- [14] J. Y. Park and S. H. Park. Pullback attractors for the non-autonomous benjamin–bona–mahony equation in unbounded domains. *Science China Mathematics*, 54:741–752, 2011. <https://doi.org/10.1007/s11425-011-4190-0>.
- [15] M. Stanislavova, A. Stefanov, and B. Wang. Asymptotic smoothing and attractors for the generalized benjamin–bona–mahony equation on \mathbb{R}^3 . *Journal of Differential Equations*, 219(2):451–483, 2005. <https://doi.org/10.1016/j.jde.2005.08.004>.
- [16] B. Wang, D. W. Fussner, and C. Bi. Existence of global attractors for the benjamin–bona–mahony equation in unbounded domains. *Journal of Physics A: Mathematical and Theoretical*, 40(34):10491, 2007. <https://doi.org/10.1088/1751-8113/40/34/007>.
- [17] R. Wang and Y. Li. Asymptotic autonomy of random attractors for bbm equations with laplace-multiplier noise. *Journal of Applied Analysis and Computation*, 10(4):1199–1222, 2020. <http://dx.doi.org/10.11948/20180145>.
- [18] X. Yao and Y. Bai. Asymptotic behavior for kirchhoff type stochastic plate equations on unbounded domains. *Bulletin des Sciences Mathématiques*, 195:103470, 2024. <https://doi.org/10.1016/j.bulsci.2024.103470>.
- [19] Z. Zhang and X. Yao. Random attractors of kirchhoff-type reaction–diffusion equations without uniqueness driven by nonlinear colored noise. *Open Mathematics*, 22(1):20240083, 2024. <https://doi.org/10.1515/math-2024-0083>.