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Facial entire coloring of K_4-minor-free graphs

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ABSTRACT

Let G be a plane graph. If two edges are adjacent and consecutive on the boundary walk of a face of G, then they are said to be facially adjacent. We call G facially entire k-colorable if there is a mapping from $V(G) \cup E(G) \cup F(G)$ to a k color set so that any two facially adjacent edges, adjacent vertices, adjacent faces, and incident elements receive different colors. The facial entire chromatic number of G is defined to be the smallest integer k such that G is facially entire k-colorable. In 2016, Fabrici, Jendrol' and Vrbjarová conjectured that every connected, loopless, bridgeless plane graph is facially entire 7-colorable. In this paper, we give a positive answer to this conjecture for K_4-minor-free graphs. More specifically, we shall prove that every K_4-minor-free graph is facially entire 7-colorable.

Keywords: K_4-minor-free graph, entire coloring, facial entire coloring, facial entire chromatic number

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1. Introduction

Throughout our paper, we only consider finite, simple and undirected graphs. A graph G is called *planar* if it can be embedded in the plane such that any two edges intersect only at their ends. For a plane graph G, let $V(G)$, $E(G)$, $F(G)$, $|E(G)|$, $\delta(G)$ and $\Delta(G)$ denote the vertex set, edge set, face set, size, minimum degree and maximum degree of G, respectively. For $v \in V(G)$, let $d_G(v)$ denote the *degree* of v in G and call v a k-vertex if $d(v) = k$ and a k^+-vertex if $d(v) \geq k$. For a face $f \in F(G)$, we use $b(f)$ to denote its boundary walk and write $f = [v_1 v_2 \ldots v_n]$ if $v_1, v_2, \ldots, v_n$ are lying on $b(f)$.

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An *entire* k-coloring of a plane graph G is a mapping π: $V(G) \cup E(G) \cup F(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G) \cup F(G)$ receive distinct colors. The *entire chromatic number*, denoted $\chi_{vef}(G)$, of G is the least integer k such that G has an entire k-coloring. This concept was first introduced in [6] by Kronk and Mitchem who conjectured that every simple plane graph is entirely $(\Delta(G)+4)$-colorable. Moreover, in the same paper, the authors positively confirmed this conjecture for $\Delta(G) \leq 3$. We notice that for a complete graph K_4, $\chi_{vef}(K_4) = 7 = \Delta(G) + 4$, and thus the bound $\Delta(G) + 4$ is tight. In 1989, Borodin [1] affirmed the conjecture for $\Delta(G) \geq 12$, and later strengthened this to $\Delta(G) \geq 7$ in [2]. Sanders and Zhao [8] further provided a proof of $\Delta(G) \geq 6$ for the conjecture. However, there is a mistake in [8] which has been corrected by Wang and Zhu [11]. Finally, Wang and Zhu [11] positively settled the whole conjecture.

As a relaxation of the entire coloring of plane graphs, the concept of the facial entire coloring of plane graphs was first investigated by Fabrici, Jendrol' and Vrbjarová [5] in 2016. Let G be a plane graph. Two edges $e_1, e_2 \in E(G)$ are called *facially adjacent* if they are adjacent and consecutive on the boundary walk of a face of G. A *facial entire k-coloring* of G is a mapping π from $V(G) \cup E(G) \cup F(G)$ to a k color set $\{1, 2, \ldots, k\}$ such that any two facially adjacent edges, adjacent vertices, adjacent faces, and incident elements in $V(G) \cup E(G) \cup F(G)$ receive distinct colors. The *facial entire chromatic number*, denoted $\bar{\chi}_{vef}(G)$, of G is the smallest integer k such that G has a facial entire k-coloring. It is obvious that $\bar{\chi}_{vef}(G) \leq \chi_{vef}(G)$. Moreover, the parameter $\bar{\chi}_{vef}(G)$ is independent on $\Delta(G)$. The reader is referred to [3, 4, 9, 10, 12] for some results on edge-face coloring and facially edge-face coloring.

Fabrici, Jendrol' and Vrbjarová [5] showed us that $\bar{\chi}_{vef}(G) \leq 8$ for any connected, loopless and bridgeless plane graph G. Moreover, in [5], they characterized the facial entire chromatic number of wheel graphs, and put forward the following conjecture.

Conjecture 1.1. *Every connected, loopless and bridgeless plane graph is facially entire 7-colorable.*

As far as we know, Conjecture 1.1 is widely open. In this paper, we focus on studying the facial entire coloring of K_4-minor-free graphs. A graph H is said to be a *minor* of a graph G if H can be obtained from G by a series of vertex deletions, edge deletions, and edge contractions. Further, G is called *H-minor-free* if G does not contain H as a minor. It is known that the class of K_4-minor free graphs is a special type of planar graphs.

2. A structural lemma

For $u, v \in V(G)$, The *distance* between u and v, denoted $d_G(u, v)$, is the length of a shortest path connecting them. For convenience, let $N_i(u) = \{v \mid v \in V(G) \text{ and } d_G(u, v) = i\}$. Moreover, we let $T(u, v)$ denote the set of all 2-vertices in G which are adjacent to both u and v. Let G be a K_4-minor-free graph. We define:

$D_G(u) = \{v \mid d_G(v) \geq 3$ such that either $uv \in E(G)$ or there is a 2-vertex x such that

$ux, xv \in E(G)\}$.

The following key structural lemma appeared in [7].

Lemma 2.1. *Let G be a K_4-minor-free graph. Then G contains one of the following configurations (C1), (C2) and (C3).*

(C1) $\delta(G) \le 1$;

(C2) *there exist two adjacent 2-vertices;*

(C3) *there is a 3^+-vertex u with $|D_G(u)| \le 2$.*

3. Facial entire coloring

This section is devoted to establishing the main result of this paper.

Theorem 3.1. *If G is a K_4-minor-free graph, then $\bar{\chi}_{vef}(G) \le 7$.*

Proof. We prove the theorem by induction on $|E(G)|$. Let $\mathcal{C} = \{1, 2, \dots, 7\}$ denote a set of seven colors. W.l.o.g., we may assume that G is a connected graph embedded in the plane. If $|E(G)| \le 3$, then $\Delta(G) \le 3$, we deduce that $\bar{\chi}_{vef}(G) \le \chi_{vef}(G) \le \Delta(G) + 4 \le 7$. So assume that G is a K_4-minor-free graph with $|E(G)| \ge 4$. By Lemma 2.1, G contains one of the configurations (C1), (C2) and (C3). In each of following cases, we will construct a graph G^* such that $|E(G^*)| < |E(G)|$. Firstly, we have the following:

Claim 1. G is 2-connected.

Proof. Suppose to the contrary that there exists a cut vertex $v \in V(G)$ and let G_1 and G_2 be two connected subgraphs such that $V(G_1) \cap V(G_2) = \{v\}$ and $G_1 \cup G_2 = G$. Clearly, G_1 and G_2 are both connected K_4-minor-free graphs and have at least two vertices, respectively. For $i = 1, 2$, by the induction hypothesis, each G_i admits a facial entire 7-coloring, say π_i. Now assume w.l.o.g., that $d_{G_1}(v) = m$ and $d_{G_2}(v) = k$ such that $k \ge m$. Note that $k \ge m \ge 1$.

Denote by f_1 and f_2 the outer faces of G_1 and G_2, respectively. For convenience, we use $v_{11}, \dots, v_{1m}$ to denote all neighbors of v in G_1 such that vv_{11} and vv_{1m} are lying on the boundary of f_1. Similarly, let $v_{21}, \dots, v_{2k}$ denote all neighbors of v in G_2 such that vv_{21} and vv_{2k} are lying on the boundary of f_2. First permute the colors in G_2 such that $\pi_2(v) = \pi_1(v)$ and $\pi_2(f_2) = \pi_1(f_1)$. We have to discuss two cases in light of the value of m and k.

• $m = 1$.

Notice that it might be the case that $k = 1$. If $\pi_2(vv_{21}) \ne \pi_1(vv_{11})$ and $\pi_2(vv_{2k}) \ne \pi_1(vv_{11})$, then $\pi_1 \cup \pi_2$ is just a facial entire 7-coloring of G. Otherwise, w.l.o.g., assume that $\pi_2(vv_{21}) = \pi_1(vv_{11})$. Then we can switch the colors $\pi_2(vv_{21})$ and $\alpha \in \mathcal{C} \setminus \{\pi_1(vv_{11}), \pi_2(v), \pi_2(f_2), \pi_2(vv_{2k})\}$. One may verify that the obtained coloring of G is a facial entire 7-coloring.

• $m \ge 2$.

Then $k \ge 2$. Similarly, if $\pi_2(vv_{21}) \ne \pi_1(vv_{11})$ and $\pi_2(vv_{2k}) \ne \pi_1(vv_{1m})$, then $\pi_1 \cup \pi_2$ itself is a facial entire 7-coloring of G. Otherwise, assume w.l.o.g., that $\pi_2(vv_{21}) = \pi_1(vv_{11})$. Noting that $\pi_2(vv_{2k}) \neq \pi_2(vv_{21})$, in order to establish a facial entire 7-coloring of G, it suffices to switch colors $\pi_2(vv_{21})$ and $\pi_2(vv_{2k})$ in G_2. □

Given a partial facial entire coloring π of G and an uncolored element $x \in V(G) \cup E(G) \cup F(G)$, let $F(x)$ denote the set of colors forbidden to use on x, and let $f(x) = |F(x)|$. By Claim 1, we are sure that G does not contain (C1).

Case 1. G contains (C2): two adjacent 2-vertices u and v.

Let u_1 denote the neighbor of u other than v, and v_1 the neighbor of v other than u. We have two subcases below.

Subcase 1.1. $u_1 = v_1$.

Write $u_1 = v_1 = u^*$. Then uvu^*u is a 3-cycle. By Claim 1, we ensure that u^* is a 2-vertex, implying that $|E(G)| = 3$, which contradicts the assumption that $|E(G)| \geq 4$.

Subcase 1.2. $u_1 \neq v_1$.

Let G^* be a graph obtained from G by contracting the edge uv into a new vertex w. Obviously, G^* is a K_4-minor-free graph with $|E(G^*)| < |E(G)|$. By the induction hypothesis, G^* has a facial entire 7-coloring π^* by using the color set $\mathcal{C}$.

To extend π^* to the graph G, we first assign $\pi^*(u_1w)$ to u_1u, $\pi^*(v_1w)$ to v_1v, and $\pi^*(w)$ to v. Then we color uv and u in order with a color in $\mathcal{C} \setminus F(uv)$ and $\mathcal{C} \setminus F(u)$, respectively. This is available because $f(uv) \leq 5$ and $f(u) \leq 5$.

Case 2. G contains (C3): There is a 3^+-vertex u with $|D_G(u)| \leq 2$.

By the proof of Case 1, we may assume that $|D_G(u)| \geq 1$. Let $d_G(u) = k$ and denote $u_1, u_2, \ldots, u_k$ be adjacent to u in cyclic order. It remains us to deal with two subcases in terms of $|D_G(u)|$.

Subcase 2.1. $|D_G(u)| = 1$, say $D_G(u) = \{v\}$.

It means that all neighbors of u are either v or some neighbors of v. Namely, $v \in N_1(u) \cup N_2(u)$. The following claims will be helpful in the rest of our paper.

Claim 2. Let u be a 3^+-vertex in G and let $u_1, u_2, \ldots, u_{d_G(u)}$ consecutively be adjacent to u. If there exist a 3^+-vertex u_i and a 2-vertex u_{i+1} such that $[uu_iu_{i+1}]$ is a 3-face, then G is facially entire 7-colorable.

Proof. Let $G^* = G - u_{i+1}$. Then G^* is a K_4-minor-free graph and $|E(G^*)| < |E(G)|$. By the induction hypothesis, G^* has a facial entire 7-coloring π^* by using $\mathcal{C}$. Since $f([uu_iu_{i+1}]) \leq 5$, $f(uu_{i+1}) \leq 4$, $f(u_iu_{i+1}) \leq 4$ and $f(u_{i+1}) \leq 3$, one may color $[uu_iu_{i+1}]$, uu_{i+1}, u_iu_{i+1}, and u_{i+1} in succession. It is easy to check that the obtained coloring of G is a facial entire 7-coloring and thus we are done. □

Claim 3. Let u be a 3^+-vertex in G and let $u_1, u_2, \ldots, u_{d_G(u)}$ consecutively be adjacent to u. If there exists a vertex $v \notin N_1(u)$ such that $d_G(u_i) = d_G(u_{i+1}) = 2$ and $[uu_ivu_{i+1}]$ is a 4-face, then G is facially entire 7-colorable.

Proof. Let $G^* = G - \{u_i, u_{i+1}\} + uv$. Obviously, G^* is a K_4-minor-free graph with $|E(G^*)| < |E(G)|$. Thus, G^* has a facial entire 7-coloring π^* by the induction hypothesis. We will show how to extend π^* to the whole graph G.

First, assign $\pi^*(uv)$ to both uu_i and $u_{i+1}v$. Then, one can color $[uu_ivu_{i+1}]$, uu_{i+1}, u_iv, u_i and u_{i+1} in order successfully due to the fact that $f([uu_ivu_{i+1}]) \leq 5$, $f(uu_{i+1}) \leq 4$, $f(u_iv) \leq 4$ and $f(u_i) = f(u_{i+1}) = 4$. Thus, G is facially entire 7-colorable. □

The following proof of Subcase 2.1 is divided into two cases below, according to the situation of v.

• $v \in N_1(u)$.

W.l.o.g., assume that $v = u_1$. Then $d_G(u_i) = 2$ for all $i = 2, 3, \ldots, k$, and thus $T(u, v) = \{u_2, \ldots, u_k\}$. By Claim 1, we are sure that v cannot be adjacent to any other vertices apart from $u, u_2, \ldots, u_k$. This guarantees us that G has been fixed, as depicted in the left configuration of Figure 1.

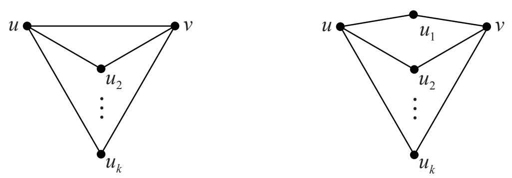

Fig. 1. The fixed graph G in two possibilities

For this case, we notice that u and v are both 3^+-vertices, and u_2 is a 2-vertex such that $[uvu_2]$ is a 3-face. So by Claim 2, one can conclude that G is facially entire 7-colorable.

• $v \in N_2(u)$.

Then $uv \notin E(G)$. It follows immediately that $T(u, v) = \{u_1, \ldots, u_k\}$. By Claim 1, neither u nor v can be a cut vertex, and hence G has been fixed, as shown in the right configuration of Figure 1.

At this moment, we observe that both u and v are 3^+-vertices, and $d_G(u_1) = d_G(u_2) = 2$ such that $[uu_1vu_2]$ is a 4-face. Moreover, $uv \notin E(G)$. Consequently, G is facially entire 7-colorable by Claim 3.

Subcase 2.2. $|D_G(u)| = 2$, say $D_G(u) = \{v, w\}$.

It follows that $d_G(v) \geq 3$ and $d_G(w) \geq 3$. Moreover, the neighbors of u is either v, or w, or a 2-vertex that is adjacent to v or w. That is, $v, w \in N_1(u) \cup N_2(u)$. To obtain a facial entire 7-coloring of the graph G, we have to consider three possibilities according to the situations of v and w.

• $N_2(u) = \{v, w\}$.

Then $uv \notin E(G)$ and $d_G(u_i) = 2$ for all $i = 1, 2, \ldots, k$. By the planarity of G, assume w.l.o.g., that $T(u, v) = \{u_1, \ldots, u_j\}$ and $T(u, w) = \{u_{j+1}, \ldots, u_k\}$, and suppose that $j \geq 2$ due to $k \geq 3$. Note that $[uu_1vu_2]$ (if $j = 2$ then $u_2 = u_j$) is a 4-face; otherwise, v has some neighbor inside of the 4-cycle uu_1vu_2u and thus v becomes a cut vertex of G, which contradicts Claim 1. Hence, by Claim 3, G is facially entire 7-colorable.

• $N_1(u) = \{v, w\}$.

As $k \geq 3$, we have that either $|T(u, v)| \geq 1$ or $|T(u, w)| \geq 1$. W.l.o.g., assume that $|T(u, v)| \geq 1$. It implies that there is a 2-vertex $u_i \in T(u, v)$ such that $[uu_iv]$ is a 3-face, and therefore G is facially entire 7-colorable by applying Claim 2.

• By symmetry, suppose that $v \in N_1(u)$ and $w \in N_2(u)$.

If $T(u,v) \neq \emptyset$, then there exists a 2-vertex $u_i \in T(u,v)$ such that $[uu_iv]$ is a 3-face. Hence, by Claim 2, G is facially entire 7-colorable. Otherwise, assume that $|T(u,w)| \geq 2$. Namely, there are consecutive two 2-vertices, say $u_j, u_{j+1} \in T(u,w)$, so that $[uu_jwu_{j+1}]$ is a 4-face. By Claim 3, we conclude that G is facially entire 7-colorable.

This completes the proof of Theorem 3.1. □

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