Inverse Italian domination in graphs

Wilma L. D'Souza[1,✉], V. Chaitra[2]

[1] *Department of Mathematics, St Joseph's University, Bengaluru-560027, India*

[2] *Department of Mathematics, B.M.S. College of Enigineering, Bengaluru-560019, India*

ABSTRACT

For a graph G, an Italian dominating function (IDF) is a function $f : V(G) \to \{0, 1, 2\}$ such that all vertices labeled with 0 must have at least two neighbors assigned the label 1 or at least one neighbor assigned the label 2. The weight of f, denoted by $w(f)$, is calculated by summing all the labels assigned by the function. Let f be an IDF on G with a minimum weight, denoted as $\gamma_I(G)$. If S is the set of vertices where $f(v) > 0$, then an Inverse Italian Dominating Function (IIDF) f' is defined as an IDF on G such that $f'(v) = 0$ for all $v \in S$. The notation $\gamma_{iI}(G)$ represents the Inverse Italian Domination Number of the graph G, which is the minimum weight among all IIDFs on G. In this paper, we find $\gamma_{iI}(G)$ of graphs and characterize the graphs for which $\gamma_I(G) = 2$ and 3, as well as those with $\gamma_{iI}(G) = 2$ and 3. Additionally, we provide a characterization of trees and graphs that achieve the largest possible $\gamma_{iI}(G)$.

Keywords: domination, Italian domination, Roman {2}-domination, inverse domination, inverse Italian domination

2020 Mathematics Subject Classification: 05C69, 05C75.

1.   Introduction

The domination concept is commonly used to solve security and facility location concerns. The introduction of Roman domination resulted from studies into potential strategies for saving the Roman empire. The Italian domination was initiated as the improvement of Roman domination. In this paper, we introduce the inverse Italian dominating function

---

in graphs.

The graphs used here are simple and cannot have isolated vertices. Two non-adjacent vertices v_1 and v_2 are said to be 'identified' if vertex v_1 is deleted and all vertices adjacent to v_1 are made adjacent to v_2 [8]. An edge yz is said to be subdivided, if we remove the edge yz and add a new vertex x along with edges yx and xz.

A vertex x in a tree is considered a descendant of a vertex y if y lies in the unique path that connects x to the root of the tree. The induced subgraph $\langle S \rangle$ where S contains a vertex u and its descendants is denoted by T_u.

A set $S \subseteq V$ in a graph G is called a dominating set if every vertex of G is either in S or adjacent to a vertex of S. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set in G [3]. Cockayne and Hedeteimi introduced this notation $\gamma(G)$ in [3] for the domination number which is widely accepted now. The books [10] and [9] can be referred for more details.

Kulli and Sigarkanti introduced the concept of inverse dominating set in [12]. "Let S be a minimum dominating set of G. If $V - S$ contains a dominating set S' of G, then S' is called an inverse dominating set of G with respect to S. The inverse domination number of G is the minimum cardinality of an inverse dominating set of G, denoted by $\gamma_i(G)$ [12]."

A dominating set D of a graph G, is said to be a two dominating set of G, if every vertex in $V - D$ is adjacent to at least two vertices of D. A minimum two dominating set is denoted by γ_2 - set.

In 2016, Chellali et al. published [2] in which he introduced the Roman {2}-dominating function, which is now known as the Italian dominating function (IDF). "An IDF is a function $f : V(G) \rightarrow \{0, 1, 2\}$ where every vertex v labeled 0 must satisfy $\sum_{u \in N(v)} f(u) \geq 2$. The weight of f denoted by $w(f)$ is equal to the sum of all of its labelings. The Italian domination number (IDN) of G is the minimum weight among them. It is denoted by $\gamma_I(G)$ [2]." An IDF with weight $\gamma_I(G)$ is called a γ_I -function . Let V_0, V_1, V_2 be the set of vertices with label $0, 1$ and 2 respectively under an IDF.

If f is a γ_I - function of G, then $S = \{v \in V : f(v) > 0\}$ is the corresponding γ_I - set of f. We define another Italian dominating function $f' : V(G) \rightarrow \{0, 1, 2\}$, such that $f'(v) = 0$ for all $v \in S$. This is an inverse Italian dominating function (IIDF) of f. We find IDFs f_i and its corresponding IIDF f_{ij} of the graph. The inverse Italian domination number $\gamma_{iI}(G)$ of the graph G is given by $\gamma_{iI}(G) = min\{w(f_{ij}) : i > 0, j > 0\}$. An IIDF that has a weight of $\gamma_{iI}(G)$ is referred to as a γ_{iI} -function.

Some graphs, like K_1, do not have IIDF for any γ_I -function. The graph K_2 does not have an IIDF for the γ_I -function which assigns 1 to every vertex, where as it has an IIDF of weight 2 if the γ_I -function assigns 2 to one vertex and 0 to another. Figure 1 shows few IDF and IIDF of $K_{4,5}$. In this Figure 1, f_1 is an IDF and $f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}$ are few minimum IIDF of f_1. Furthermore, f_2 is an IDF and f_{21} is the only possible IIDF of f_2. We observe that, $w(f_{11}) = w(f_{12}) = w(f_{13}) = w(f_{14}) = w(f_{15}) = w(f_{16}) = w(f_{17}) = 4$ and $w(f_{21}) = 5$. We find all γ_I -functions f_i and its corresponding IIDFs f_{ij}. The lowest weight among all IIDF is $\gamma_{iI}(G)$. In this example $\gamma_{iI}(K_{4,5}) = 4$.
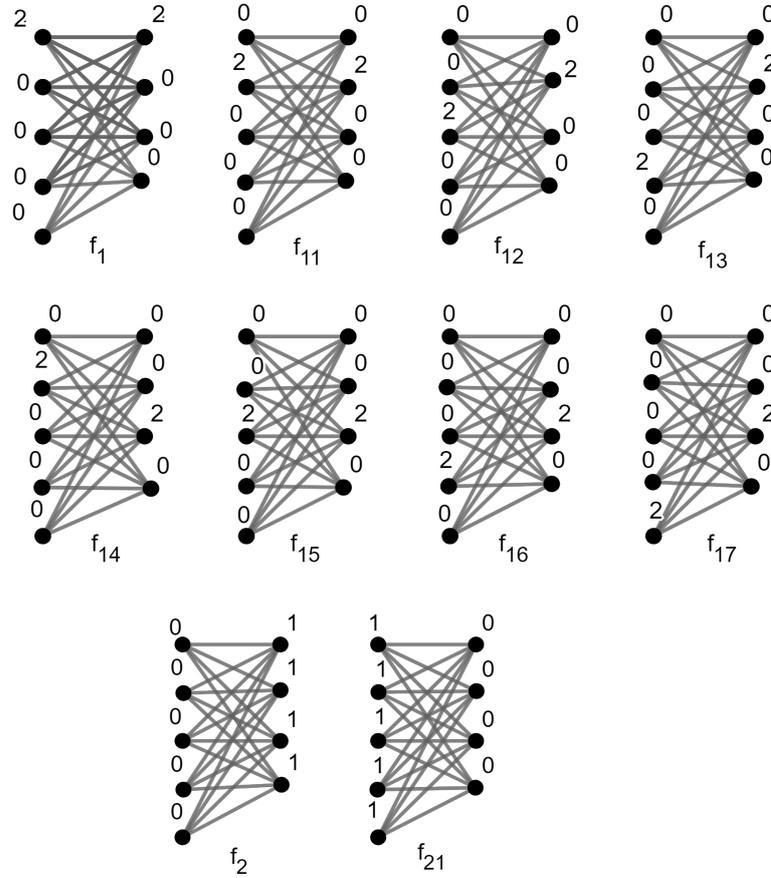
Fig. 1. Illustrations of graphs with IIDF

2.   Some standard results

Proposition 2.1. i) *For any K_n with $n \geq 2$, $\gamma_I(K_n) = \gamma_{iI}(K_n) = 2$;*
  ii) *For any P_n, $n > 3$, $\gamma_{iI}(P_n) = \left\lceil \frac{n+2}{2} \right\rceil$;*
  iii) *For any C_n, $n > 3$, $\gamma_{iI}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+3}{2} & \text{if } n \text{ is odd} \end{cases}$;*
  iv) *For any $K_{r,s}$ where $s \geq r > 3$, $\gamma_I(K_{r,s}) = \gamma_{iI}(K_{r,s}) = 4$;*
  v) *For any $K_{1,n-1}$ with $n > 2$, $\gamma_{iI}(K_{1,n-1}) = n - 1$.*

Proposition 2.2. *A graph G does not have an IIDF if G has an isolated vertex or there exists no γ_I -set of G which is a proper subset of $V(G)$.*

Proof. Let G be a graph that has an isolated vertex v. Then $f(v) > 0$ for any γ_I -function f. If there exists any IIDF f', then $f'(v) = 0$ and as v is isolated, it is not dominated by any other vertex. Hence, we can not define an IIDF.

Let G be a graph having no γ_I -set of G which is a proper subset of $V(G)$. If there exists any IIDF f', then $f'(x) = 0$ for all $x \in V(G)$. Hence, we can not define an IIDF. $\square$

Proposition 2.3. *If* $G = K_1 + M$, *then* $\gamma_{iI}(G) = \gamma_I(M)$ *for any graph* M.

Proof. Let f be a γ_I-function such that K_1 is assigned 2 and 0 is assigned to the remaining vertices. In any minimum IIDF f', K_1 is assigned 0 which gives $\gamma_{iI}(G) = \gamma_I(M)$. $\qquad \square$

3. Characterization of graphs having small $\gamma_{iI}(G)$

The characterization of $\gamma_I(G) = 2$ is given by Volkmann [15] as a corollary of a theorem. In this section we have given a proof for the same. We also characterize the graphs with $\gamma_I(G) = 3$. Using these results, we characterize the graphs having $\gamma_{iI}(G) = 2, 3$.

3.1. Graphs with $\gamma_I(G) = 2$

Theorem 3.1. [15] *For any connected graph* G *and for graph* H, $\gamma_I(G) = 2$ *if and only if* $G \cong K_1 + H$ *or* $G \cong \overline{K_2} + H$.

Proof. If $\gamma_I(G) = 2$, then either $|V_2| = 1$ or $|V_2| = 0$ and $|V_1| = 2$. If $|V_2| = 1$, then all vertices should be adjacent to the vertex labelled 2. Hence $G \cong K_1 + H$. If $|V_2| = 0$ and $|V_1| = 2$, then all vertices should be adjacent to two vertices x and y of V_1. If the vertices x and y are adjacent, then $G \cong K_1 + H$. If x and y are not adjacent, then $G \cong \overline{K_2} + H$.
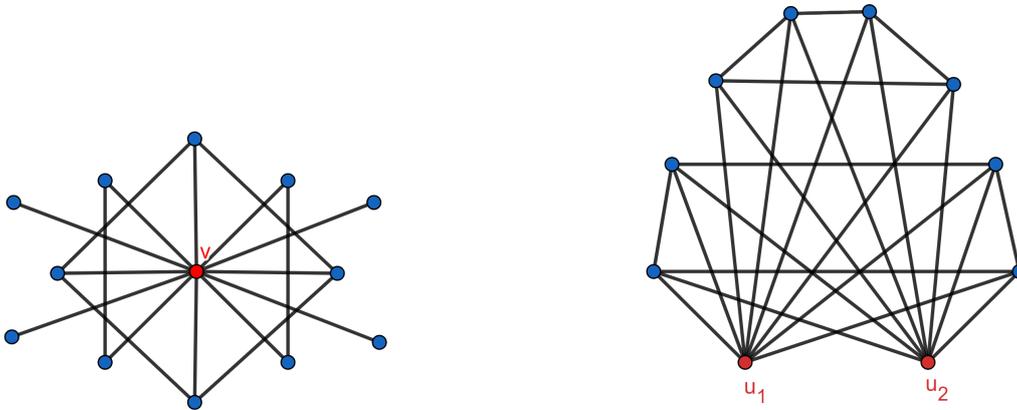
Fig. 2. Illustration of graphs having $\gamma_I(G) = 2$

Conversely, if $G \cong K_1 + H$, then the only minimum $\gamma_I(G)$ assigns 2 to vertex K_1 and the remaining vertices can be assigned 0 which gives $\gamma_I(G) = 2$. If $G \cong \overline{K_2} + H$, then vertices of $\overline{K_2}$ are assigned 1 and the vertices of H can be assigned 0.

The Figure 2 shows graphs having $\gamma_I(G) = 2$. The vertex v of the first graph is of degree $n - 1$. The second graph is $\overline{K_2} + H$ where u_1, u_2 forms $\overline{K_2}$. $\qquad \square$

3.2. Graphs with $\gamma_I(G) = 3$

Theorem 3.2. *For any connected graph* G, $\gamma_I(G) = 3$ *if and only if it satisfies any of the following properties:*

i) $\Delta(G) = n - 2$ *where* $G \ncong \overline{K_2} + H$ *for any graph* H
ii) G *has a* γ_2 *- set* S *with* $|S| = 3$ *where* $\Delta(G) < n - 2$.

Proof. If $\gamma_I(G) = 3$, then either $|V_2| = 1$ and $|V_1| = 1$ or $|V_2| = 0$ and $|V_1| = 3$. If $|V_2| = 1$ and $|V_1| = 1$, then all vertices are adjacent to the vertex labelled 2 except one, which implies $\Delta(G) = n - 2$. Furthermore, $G \neq \overline{K_2} + H$ otherwise $\gamma_I(G) = 2$. If $|V_2| = 0$ and $|V_1| = 3$, then all the vertices not in V_1 should be adjacent to any two vertices of V_1 which makes V_1 a two dominating set of G. This two dominating set is γ_2 - set because if there is a γ_2 - set of cardinality 2, then $\gamma_I(G) = 2$.
Conversely, if $\Delta(G) = n - 2$ with $G \neq \overline{K_2} + H$, then the vertex with degree $n - 2$ can be assigned 2 and the vertex not adjacent to it can be assigned 1, which gives $\gamma_I(G) = 3$. If there is a γ_2 - set S of G with $|S| = 3$, then the vertices of S can be assigned 1 which gives $\gamma_I(G) = 3$.
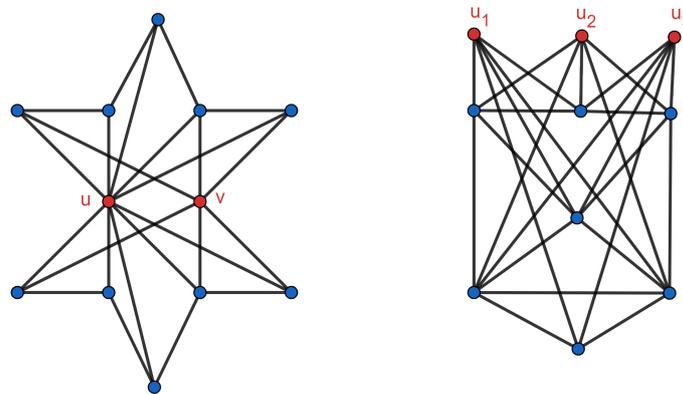

Fig. 3. Illustration of graphs having $\gamma_I(G) = 3$

The Figure 3 shows the graphs having $\gamma_I(G)$ equal to 3. In the first graph, u is the vertex of degree $n-2$ and v is the vertex not adjacent to u. In the second graph $\{u_1, u_2, u_3\}$ form the γ_2 - set of the graph. $\qquad\qquad\square$

3.3.   Graphs with $\gamma_{iI}(G) = 2$

Theorem 3.3. *For any connected graph* G, $\gamma_{iI}(G) = 2$ *if and only if it satisfies any of the following properties:*
i) $G \cong K_2 + H$,
ii) $G \cong P_3 + H$,
iii) $G \cong C_4 + H$,
for any graph H.

Proof. For any connected graph G, $\gamma_{iI}(G) = 2$ if and only if $\gamma_I(G) = 2$. From Theorem 3.1, we have, $\gamma_I(G) = 2$ if and only if $G \cong K_1 + M$ or $G \cong \overline{K_2} + M$ for any graph M.
   *Case 1.* When $G \cong K_1 + M$.
   If $G \cong K_1 + M$, then by Proposition 2.3, $\gamma_{iI}(G) = \gamma_I(M)$ and $\gamma_I(M) = 2$ if and only if $M = K_1 + M_1$ or $M \cong \overline{K_2} + M_2$ for any graphs M_1 and M_2. If $G \cong K_1 + M$ and

$M \cong K_1 + M_1$, then $G \cong K_2 + H$ for some graph H. If $G \cong K_1 + M$ and $M \cong \overline{K_2} + M_2$, then $G \cong P_3 + H$ for some graph H.

Case 2. When $G \cong \overline{K_2} + M$.

In this case, $\gamma_{iI}(G) = \gamma_I(M)$ and $\gamma_I(M) = 2$ if and only if $M \cong K_1 + H_1$ or $M \cong \overline{K_2} + H_2$ for any graphs H_1 and H_2. If $G \cong \overline{K_2} + M$ and $M \cong K_1 + H_1$, then $G \cong P_3 + H$ for some graph H. If $G \cong \overline{K_2} + M$ and $M \cong \overline{K_2} + H_2$, then $G \cong C_4 + H$ for some graph H.
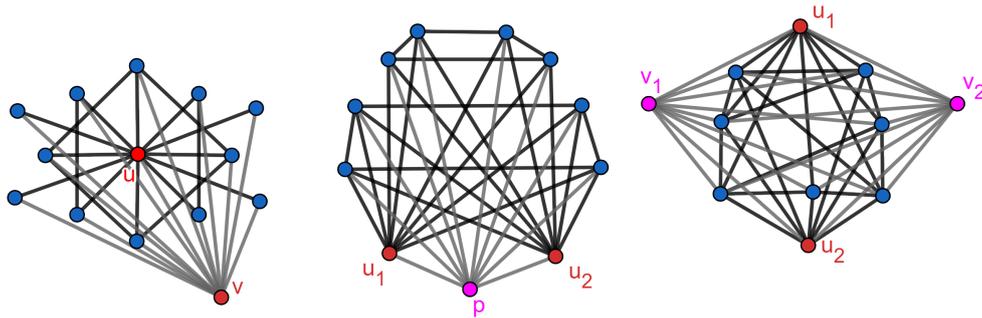
Fig. 4. Illustration of graphs having $\gamma_{iI}(G) = 2$

Conversely, Suppose $G \cong K_2 + H$, then one vertex of K_2 will be labelled 2 under any γ_I - function and the rest will be labelled 0. The IIDF will assign 2 to another vertex of K_2 and the rest will be labelled 0. If $G \cong P_3 + H$, then the vertex with degree $n - 1$ is labelled 2 under any γ_I - function and the rest will be labelled 0. The IIDF will assign 1 to the vertices of P_3 having degree $n - 2$ and the rest will be labelled 0. If $G \cong C_4 + H$, then any γ_I - function will assign 1 to two non-adjacent vertices of C_4 and the rest will be labelled 0. Any IIDF will assign 1 to the other two non-adjacent vertices of C_4 and the rest will be labelled 0. In all the above cases we get $\gamma_{iI}(G) = 2$.

The Figure 4 shows the graphs *(i)* $G \cong K_2 + H$, *(ii)* $G \cong P_3 + H$, and *(iii)* $G \cong C_4 + H$, respectively. In the first graph u and v forms K_2. In the second graph u_1, p, u_2 forms P_3. In the third graph v_1, v_2, u_1, u_2 forms C_4. $\qquad\qquad\square$

3.4.   Graphs with $\gamma_{iI}(G) = 3$

Theorem 3.4. *For any connected graph G with $n \geq 3$, $\gamma_{iI}(G) = 3$ if and only if it satisfies any of the following properties:*

i) *$G \cong K_1 + H$ or $G \cong \overline{K_2} + H$, where $\Delta(H) = |V(H)| - 2$ and $H \ncong \overline{K_2} + M$ for any graph M and H,*

ii) *$G \cong K_1 + H$ or $G \cong \overline{K_2} + H$, where H has a γ_2 - set S with $|S| = 3$ and $\Delta(H) < |V(H)| - 2$,*

iii) *G has a γ_2 - set S with $|S| = 3$. Also G has a γ_2 - set R in $V - S$ with $|R| = 3$ where $\Delta(G) < n - 2$,*

iv) *G has at least 2 vertices x, y of maximum degree $n - 2$ with $N(x) \neq N(y)$ or $G - \{x, u\}$ has a γ_2 - set S of G with $|S| = 3$, where u is the vertex non-adjacent to x.*

**Proof.** Let G be a connected graph with $\gamma_{iI}(G) = 3$. This implies $\gamma_I(G) = 2$ or 3.

Case 1. Consider the case $\gamma_I(G) = 2$.

From Theorem 3.1 we have, for any graph G, which is connected, $\gamma_I(G) = 2$ if and only if $G \cong K_1 + H$ or $G \cong \overline{K_2} + H$. In any case, $\gamma_{iI}(G) = \gamma_I(H)$. The rest of the proof follows by the Theorem 3.2.

Case 2. Consider the case $\gamma_I(G) = 3$.

From the Theorem 3.2 we have, $\gamma_I(G) = 3$ if and only if the maximum degree $\Delta(G) = n - 2$ with $G \neq \overline{K_2} + H$ or $\Delta(G) < n - 2$ and G has a γ_2 - set S with $|S| = 3$. If $\Delta(G) = n - 2$ with $G \ncong \overline{K_2} + H$ then $\gamma_{iI}(G) = \gamma_I(G - \{x, u\})$, where x is a vertex having degree $n - 2$ and u is the vertex not adjacent to it. If G has a γ_2 - set S of G with $|S| = 3$ with $\Delta(G) < n - 2$ then $\gamma_{iI}(G) = \gamma_I(G - S)$. The rest of the proof follows by the Theorem 3.2. We can observe that $\gamma_I(G - \{x, u\}) = 3$ implies there is another vertex y with degree $n - 2$ and a vertex v not adjacent to it, which implies there are two vertices of degree $n - 2$ where $N(x) \neq N(y)$.
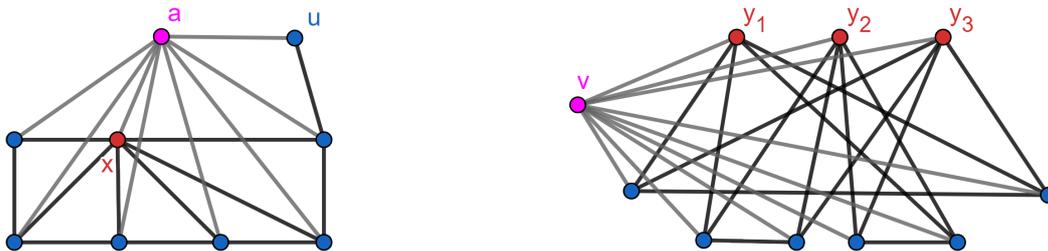
Fig. 5. Illustration of graphs having $\gamma_{iI}(G) = 3$

Conversely, if $G \cong K_1 + H$ then any IDF assigns 2 to the vertex of K_1 and 0 to the remaining vertices. Similarly, if $G \cong \overline{K_2} + H$, then any IDF assigns 1 to the vertices of $\overline{K_2}$ and 0 to the rest of the vertices. Hence, $\gamma_{iI}(G) = \gamma_I(H)$ and $\gamma_I(H) = 3$ by Theorem 3.2.

Consider G that has a γ_2 - set S with $|S| = 3$. Also G has a γ_2 - set R in $V - S$ with $|R| = 3$ where $\Delta(G) < n - 2$. In this case, $\gamma_{iI}(G) = \gamma_I(G - S)$ and $\gamma_I(G - S) = 3$ by Theorem 3.2.

Consider G that has a vertex x of maximum degree $n - 2$ and $G - \{x, u\}$ has a γ_2 - set S of G with $|S| = 3$, where u is the vertex non-adjacent to x. Then $\gamma_{iI}(G) = \gamma_I(G - \{x, u\})$ and $\gamma_I(G - \{x, u\}) = 3$ by Theorem 3.2.

Consider G that has at least 2 vertices x, y of maximum degree $n - 2$ with $N(x) \neq N(y)$. In this case any γ_I - function assigns 2 to x, 1 to its non-adjacent vertex and 0 to the rest of the vertices. A γ_{iI} - function assigns 2 to y, 1 to its non-adjacent vertex and 0 to the rest of the vertices which implies $\gamma_{iI}(G) = 3$.
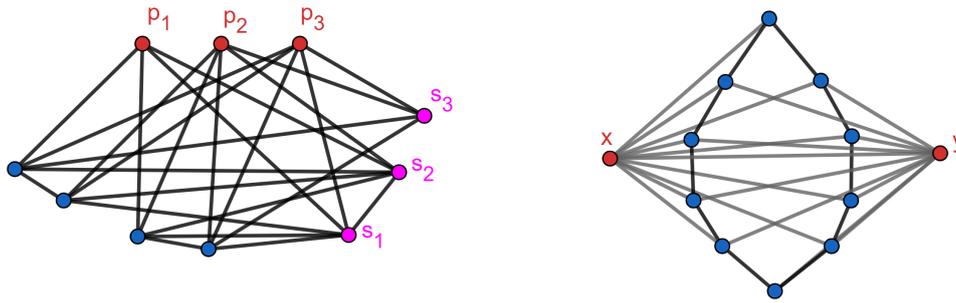
Fig. 6. Illustration of graphs having $\gamma_{iI}(G) = 3$

The Figure 5 shows the graphs of $\gamma_{iI}(G) = 3$. In the first graph, the vertex a is K_1, the vertex x is of degree $n-2$, which is not adjacent only to u. In the second graph, the vertex v is K_1 and $\{y_1, y_2, y_3\}$ forms the γ_2 - set S.

The Figure 6 shows the graphs of graphs of $\gamma_{iI}(G) = 3$. In the first graph, $\{p_1, p_2, p_3\}$ forms the γ_2 - set S and $\{s_1, s_2, s_3\}$ forms the γ_2 - set R in $V - S$. In the second graph, x, y are the vertices of degree $n-2$ with $N(x) \neq N(y)$. $\square$

4.   Characterization of graphs having largest $\gamma_{iI}(G)$

A wounded spider is a graph obtained by subdividing at most $n-2$ edges of a star $K_{1,n-1}$. A healthy spider is obtained by subdividing all the edges of a star. Let $d_T(u)$ be the degree of u in the tree T. Let T_v be the maximal subtree containing v and its descendants.
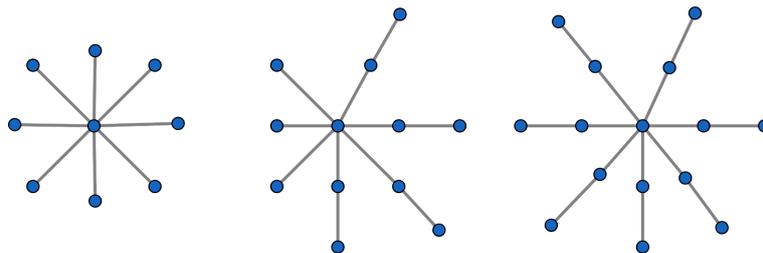

Fig. 7. A star, a wounded spider and a healthy spider

Theorem 4.1. *For any tree T of order $n \geq 3$,*

$$\gamma_{iI}(T) \leq n - 1.$$

Proof. We prove this by induction hypothesis on n, the number of vertices of T. We can observe that $diam(T) \geq 2$ since $n \geq 3$. When $diam(T) = 2$, then T has to be a

star, which implies $\gamma_{iI}(T) = n - 1$. When $diam(T) = 3$, then T has to be a double star which implies $\gamma_{iI}(T) \le n - 1$. Hence, we consider $diam(T) \ge 4$ implying $n \ge 5$. By induction hypothesis, there exists a γ_{iI} - function h' for any subtree T' of order m where $3 \le m < n$ having the property $\gamma_{iI}(T') = w(h') \le m - 1$. Consider a diametrical path $(u_1, u_2, \ldots, u_k), (k \ge 5)$ of T, so that the degree of u_2 is maximum possible in T. Let T_v be the maximal subtree containing v and its descendants. Let u_k be the root of T and proceed to the cases that follow.

Case 1. When $d_T(u_2) \ge 3$.

Consider the case $d_T(u_2) \ge 3$. Let T' be the subtree with m vertices obtained by deleting u_2 and its leaves from T. By mathematical induction, the result holds for T' and hence there is a γ_{iI} - function h' on $V(T')$ such that $w(h') = \gamma_{iI}(T') \le m - 1$. Since u_2 is a support vertex with $l \ge 3$ leaf neighbors, there exists a γ_I - function on T' which assigns 2 to u_2 and 0 to its leaf neighbours.

We can define an IIDF f' on $V(T)$ such that, it assigns 1 to its l leaf neighbours, $f'(u_2) = 0$, and $f'(x) = h'(x)$ for all $x \in V(T')$. Hence $\gamma_{iI}(T) \le w(f') = w(h') + l$ which implies, $\gamma_{iI}(T) \le (m - 1) + l \le n - (l + 1) - 1 + l = n - 2 < n - 1$. In this case, we proved that $\gamma_{iI}(T) < n - 1$.

Case 2. When $d_T(u_2) = d_T(u_3) = 2$.

Let a tree T' of order m be obtained by deleting u_1, u_2 and u_3. We have assumed that $diam(T) \ge 4$ implying $n \ge 5$ hence $m \ge 2$. If $m = 2$, then $T = P_5$ which implies, $\gamma_{iI}(T) = 4 = n - 1$. So, let us assume that $m \ge 3$. By the induction hypothesis, there is a γ_{iI} - function h' on $V(T')$ such that $w(h') = \gamma_{iI}(T') \le m - 1$. There exists a γ_I - function f on $V(T)$ such that, $f(u_2) = 2$ and $f(u_1) = f(u_3) = 0$. We can define an IIDF f' on $V(T)$ such that, $f'(u_2) = 0$, $f'(u_1) = f'(u_3) = 1$ and $f'(x) = h'(x)$ for all $x \in V(T')$. Hence $\gamma_{iI}(T) \le w(f') = w(h') + 2$ which implies, $\gamma_{iI}(T) \le (m - 1) + 2 \le n - 2 < n - 1$. In this case, we proved that $\gamma_{iI}(T) < n - 1$.

Case 3. When $d_T(u_2) = 2$ and $d_T(u_3) \ge 3$.

The vertex u_2 in the diametrical path is chosen such that, every child of u_3 is a vertex similar to u_2 or it is a leaf.

Subcase 3.1. When the vertex u_4 is a support vertex with $deg(u_4) = 2$.

If u_4 is a support vertex and $deg(u_4) = 2$, then T is a wounded spider or a healthy spider where $q \ge 2$ edges of a star are subdivided.

i) Suppose u_3 has no leaf as neighbours.

In this case T is a healthy spider. There exists an IDF f on $V(T)$ which assigns 1 to u_3 and to all vertices at a distance 2 from u_3 and 0 to the rest of the vertices. The only possible IIDF f' of f is defined such that $f'(u) = 2$ if u is at a distance 1 from u_3 and 0 to the rest of the vertices. Then we have $\gamma_{iI}(T) = w(f') = 2q = \frac{2(n-1)}{2} = n - 1$.

ii) Suppose u_3 has l -leaves as neighbours.

In this case T is a wounded spider. There exists an IDF f on $V(T)$ which assigns 2 to u_3, 1 to all vertices at a distance 2 from u_3 and 0 to the remaining vertices. The only possible IIDF f' of f is defined such that it assigns 1 to the leaf neighbours of u_3 and 2 to the remaining neighbours of u_3. All the rest of the vertices of T will be assigned 0 under f'. Then $\gamma_{iI}(T) = w(f')$ and $w(f') = 2q + l = \frac{2(n-l-1)}{2} + l = n - 1$.

In *Subcase 3.1*, we proved that $\gamma_{iI}(T) = n - 1$.

Subcase 3.2. When the vertex u_4 is not a support vertex of degree 2.

Let $deg(u_4) \neq 2$ and let u_4 is not a support vertex of T. Suppose $T' = T - V(T_{u_3})$ and $m \geq 3$. By induction hypothesis, there exists an IIDF h' on $V(T')$ such that $\gamma_{iI}(T') = w(h') \leq m - 1$. We observe that T_{u_3} is obtained by subdividing $q \geq 1$ edges of a star.

i) Suppose u_3 has no leaf as neighbours.

In this case, T_{u_3} is a healthy spider. There exists an IDF f on $V(T)$ which assigns 1 to u_3 and to every grandchild of u_3 and 0 to every child of u_3. We can define an IIDF f' of f where $f'(x) = h'(x)$ for all $x \in V(T')$, it assigns 2 to every child of u_3 and 0 to the rest of the vertices. This implies,

$\gamma_{iI}(T) \leq w(h') + 2q \leq m - 1 + 2q \leq (n - 2q - 1) - 1 + 2q = n - 2 < n - 1$.

ii) Suppose u_3 has l-leaves as neighbours.

In this case T_{u_3} is a wounded spider. There exists an IDF f on $V(T)$ which assigns 2 to u_3 and 1 to every grandchild of u_3 and 0 to every child of u_3. We can define an IIDF f' of f where $f'(x) = h'(x)$ for all $x \in V(T')$, it assigns 1 to every leaf neighbour of u_3 and 2 to the remaining neighbours of u_3 and 0 to the rest of the vertices. This implies,

$\gamma_{iI}(T) \leq w(h') + 2q + l \leq m - 1 + 2q + l = (n - 2q - l - 1) - 1 + 2q + l = n - 2 < n - 1$.

In *Subcase 3.2*, we proved that $\gamma_{iI}(T) < n - 1$. This proves the upper bound. □

Theorem 4.2. *For any tree T of order $n \geq 3$, $\gamma_{iI}(T) = n - 1$ if and only if T is a star, a wounded spider or a healthy spider.*

Proof. Consider a tree with $\gamma_{iI}(T) = n - 1$. In the Theorem 4.1, we have proved that $\gamma_{iI}(T) \leq n - 1$. From the proof of the theorem 4.1, we can observe that, it attains equality only when T is a star, $T = P_4$, $T = P_5$ and T as in *Subcase 3.1*. All these trees are either stars, wounded spiders or healthy spiders.

Conversely, let T be a tree which is a star, a healthy spider or a wounded spider with its $q \geq 0$ edges subdivided. Let v be the vertex of maximum degree in T.

(i) If T is a star, then $\gamma_{iI}(T) = n - 1$.

(ii) If T is a healthy spider, then the only γ_I - function, f assigns 1 to v and all vertices at a distance 2 from it. The only γ_{iI} - function, f' assigns 2 to all vertices at a distance 1 from v and 0 to the remaining vertices. Hence $\gamma_{iI}(T) = n - 1$.

(iii) If T is a wounded spider, then $1 \leq q \leq n - 2$. Let v have $l \geq 1$ leaves as neighbors. A γ_I - function, f assigns 2 to v and 1 to all vertices at a distance 2 from it and 0 to the rest of the vertices. The IIDF, f' assigns 0 to v and all vertices at a distance 2 from it. All leaves of v are assigned 1 and the rest of the neighbors of v can be assigned 2 under the function f'. Hence $\gamma_{iI}(T) = 2q + l = 2(\frac{n-l-1}{2}) + l = n - 1$. □

5. The family $\mathcal{G}$

A graph having two non-adjacent pendant edges attached to C_3 is called a bull graph, denoted by B. We define a family G of connected graphs, which includes the graphs C_5

and $\mathcal{G}(t, r, s, m)$.
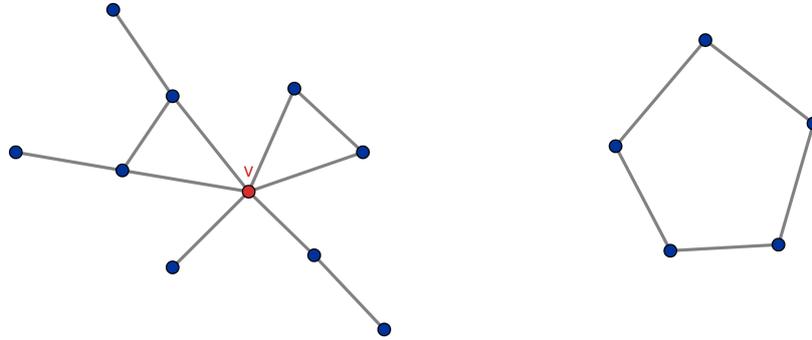
Fig. 8. Graphs $\mathcal{G}(1, 1, 1, 1)$ and C_5 of graph family $\mathcal{G}$

Graph $\mathcal{G}(t, r, s, m)$: In this graph we can have $t \geq 0$ number of P_2, $r \geq 0$ number of P_3, $s \geq 0$ number of C_3, $m \geq 0$ number of Bull graphs B, where one vertex from each graph is identified/merged into a single vertex v. From Bull graph B we select the vertex of degree 2 and from P_3 we select the vertex of degree 1 to identify with vertex v. Bull graph is a graph C_3 having two non-adjacent pendant edges. Figure 8 depicts the graph $\mathcal{G}(1, 1, 1, 1)$ and C_5 of graph family $\mathcal{G}$. Few graphs are excluded from the graph family $\mathcal{G}(t, r, s, m)$. They are $\mathcal{G}(0, 0, 0, 1)$, $\mathcal{G}(0, 0, 0, 2)$, $\mathcal{G}(0, 1, 0, 1)$ which are shown in Figure 9.
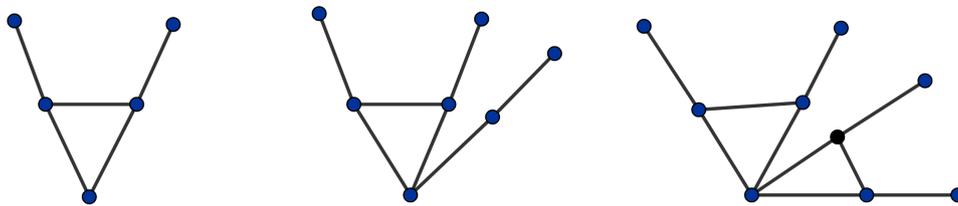
Fig. 9. Graphs excluded from the graph family $\mathcal{G}(t, r, s, m)$

Theorem 5.1. *For a connected graph G of order $n \geq 3$,*

$$\gamma_{iI}(G) \leq n - 1.$$

Proof. Consider a graph G, which is connected and has $n \geq 3$ vertices. If we remove any edge of G, then its $\gamma_{iI}(G)$ will either increase or remain the same but never decrease. Therefore, $\gamma_{iI}(G) \leq \gamma_{iI}(T)$ for any spanning tree T of G. Hence, by Theorem 4.1, we have $\gamma_{iI}(G) \leq \gamma_{iI}(T) \leq n - 1$. $\square$

Theorem 5.2. *For a connected graph G of any order $n \geq 3$, $\gamma_{iI}(G) = n - 1$ if and only if $G \in \mathcal{G}$.*

Proof. Consider G which is connected and $\gamma_{iI}(G) = n - 1$. If T is any spanning tree of G, then $\gamma_{iI}(T) = n - 1$, because deletion of edges will not decrease the $\gamma_{iI}(G)$ and it cannot increase since it is the maximum. By Theorem 4.2, T is a star, a wounded spider or a healthy spider. So G is a super graph of T having T as a spanning tree. By adding all possible types of edges to T, the $\gamma_{iI}(G)$ will not decrease, and we can consequently derive the family $\mathcal{G}$.

i) Let T be a star and $k_1, k_2, \ldots, k_{n-1}$ be the leaves of T. We add all possible types of edges to T so that its $\gamma_{iI}(G)$ remains the same. If we add the edge $k_i k_{i+1}$ to T, then it results in $\mathcal{G}(n - 3, 0, 1, 0)$ which has $\gamma_{iI}(G)$ equals $n - 1$. If we add the edges $k_i k_{i+1}$ and $k_{i+1} k_{i+2}$, the resulted graph will have $\gamma_{iI}(G)$ less than $n - 1$ as it has C_4 as its subgraph. Figure 10 shows these graphs. Hence the only possible graphs of $\gamma_{iI}(G)$ equal to $n - 1$ are $\mathcal{G}(t, 0, s, 0)$ and $\mathcal{G}(0, 0, s, 0)$.
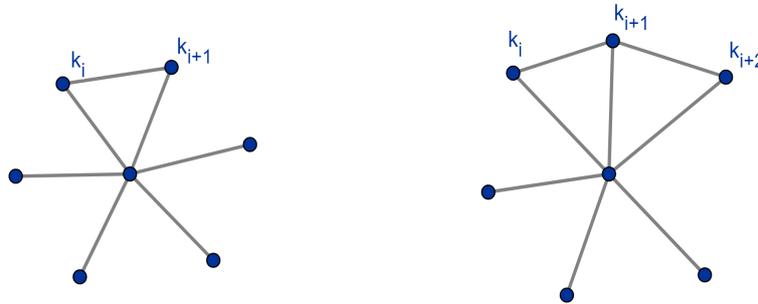

Fig. 10. Graphs obtained from a star

ii) Let T be a healthy spider. Let v be the vertex of maximum degree and $k_1, k_2, \ldots, k_p$ be the adjacent vertices of v. Let $l_1, l_2, \ldots, l_p$ be the vertices of degree 1. If we add the edge $k_i k_{i+1}$ to T, then it results in $\mathcal{G}(0, p - 2, 0, 1)$ which has $\gamma_{iI}(G)$ equals $n - 1$. If we add the edge $l_i v$ to T, then it results in $\mathcal{G}(0, p - 1, 1, 0)$ which has $\gamma_{iI}(G)$ equals $n - 1$. Figure 11 shows these graphs.
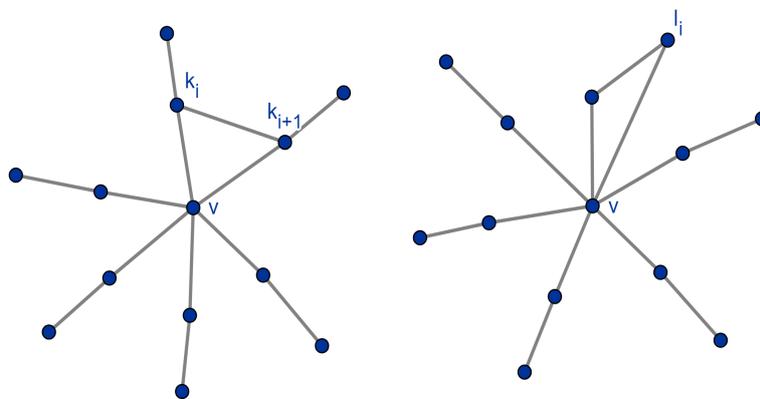

Fig. 11. Supergraphs of a healthy spider which are included in the graph family

If we add the edges $k_i k_{i+1}$ and $k_{i+1} k_{i+2}$, the resulted graph will have $\gamma_{iI}(G)$ less than $n - 1$ as it has C_4 as its subgraph. Moreover, adding the edges $l_i l_{i+1}$ or $l_i k_{i+1}$ also will

decrease the $\gamma_{iI}(G)$. Figure 12 shows these graphs. Hence, the only possible graphs that can have $\gamma_{iI}(G)$ equal to $n-1$ are $\mathcal{G}(0, r, s, m)$ where $r \geq 0, s \geq 0, m \geq 0$. Furthermore, $\mathcal{G}(0,0,0,1)$, $\mathcal{G}(0,0,0,2)$ and $\mathcal{G}(0,1,0,1)$ are excluded as they have $\gamma_{iI}(G)$ less than $n-1$.

iii) Let T be a wounded spider. Let v be the vertex of maximum degree, $b_1, b_2, \ldots, b_p$ be the vertices of degree 2 and l_i be the leaves adjacent to b_i. Let $k_1, k_2, \ldots, k_q$ be the leaves adjacent to v. We omit the cases, which have already been discussed. Adding the edges $k_i k_j$ or $b_i b_j$ will keep the $\gamma_{iI}(G)$ same which is shown in Figure 14.

Adding the edge $b_i k_j$ decreases the $\gamma_{iI}(G)$, when the degree of v is more than 2. When the degree of v is equal to 2, we get the graph $\mathcal{G}(1, 0, 1, 0)$ which is discussed in *Case 1*. Adding the edge $l_i k_j$ also decreases the $\gamma_{iI}(G)$ as it has C_4 as its subgraph. This is shown in Figure 13. Hence, the only possible graphs that can have $\gamma_{iI}(G)$ equal to $n-1$ are $\mathcal{G}(t, r, s, m)$ where $t \geq 0, r \geq 0, s \geq 0, m \geq 0$.
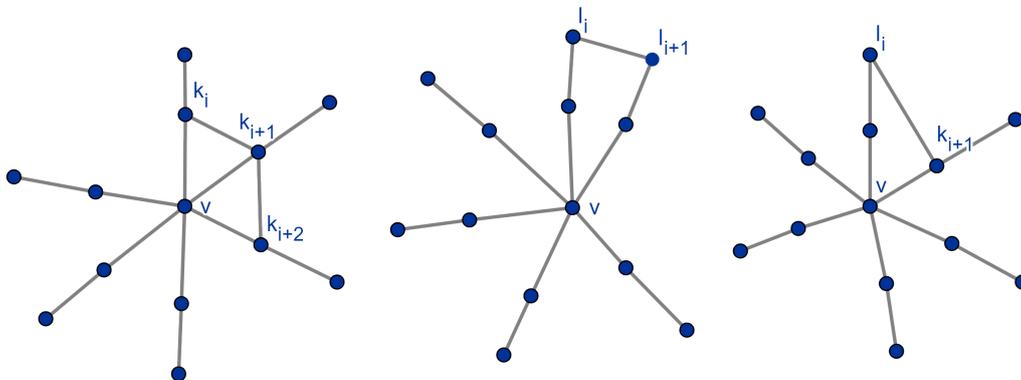

Fig. 12. Supergraphs of a healthy spider which are excluded from the graph family $\mathcal{G}$

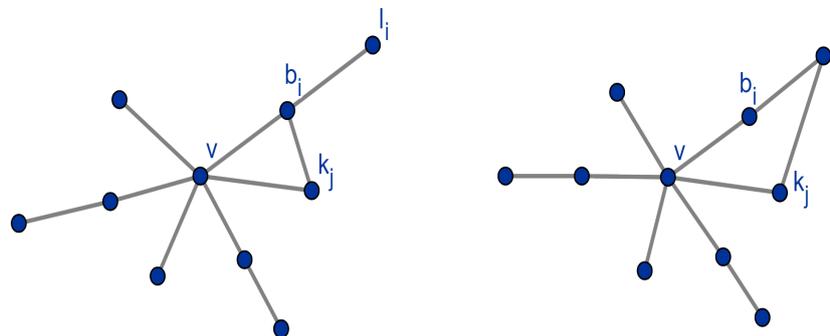**Fig. 13.** Supergraphs of a wounded spider which are excluded from the graph family $\mathcal{G}$

Conversely, let $G \in \mathcal{G}$. The only possible γ_I - function f to the graph $\mathcal{G}(t, r, s, m)$ assigns 2 to v and 0 to all vertices adjacent to v. Rest of the vertices are assigned 1 under f. A γ_{iI} - function f' is defined in such a way that, all vertices u with $f(u) > 0$ are

assigned $f'(u) = 0$. All the vertices adjacent to v and belonging to B or P_3 are assigned 2 and the remaining vertices are assigned 1 under f'. This gives $\gamma_{iI}(\mathcal{G}(t, r, s, m)) = n - 1$. From Proposition 2.1, $\gamma_{iI}(C_5) = 4 = n - 1$. $\qquad\square$
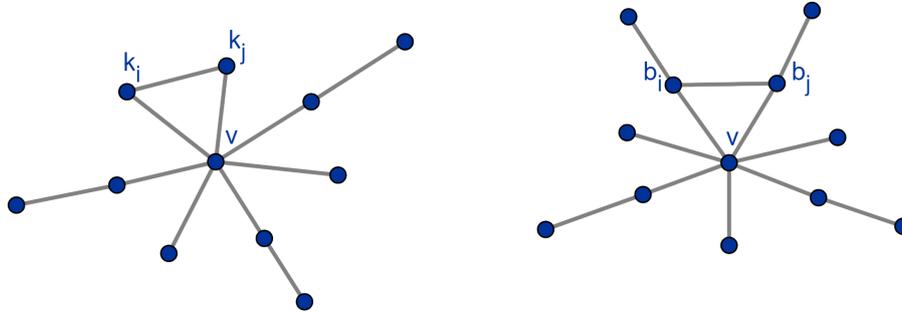

Fig. 14. Supergraphs of a wounded spider which are included in the graph family $\mathcal{G}$

6. Some bounds on $\gamma_{iI}(G)$

Proposition 6.1. *For every graph G,*

$$\gamma(G) \leq \gamma_I(G) \leq \gamma_{iI}(G),$$

and the upper bound is sharp for K_n with $n \geq 2$.

Proof. For any graph G, a minimum dominating set or γ - set is always the subset of some γ_I - set. As a result $\gamma(G) \leq \gamma_I(G)$. An IIDF is also an IDF. Hence $\gamma_I(G) \leq \gamma_{iI}(G)$. For $G \cong K_n$, we have $\gamma_I(G) = \gamma_{iI}(G)$. $\qquad\square$

Proposition 6.2. [2] *For any graph G of order n,*

$$\gamma(G) \leq \gamma_I(G) \leq 2\gamma(G).$$

Proposition 6.3. *For every graph G of order n,*

$$\gamma_i(G) \leq \gamma_{iI}(G) \leq 2\gamma_i(G).$$

Theorem 6.4. *For every graph G of order $n \geq 2$,*

$$\gamma_I(G) + \gamma_{iI}(G) \leq 2n,$$

and this bound is sharp for K_2.

Proof. Let $f = (V_0, V_1, V_2)$ be any γ_I - function and let $f' = (V_0', V_1', V_2')$ be any γ_{iI} - function. Every vertex $v_i \in (V_1 \cup V_2)$ under γ_I - function, will be in V_0' under γ_{iI} - function. Hence, $f(v_i) + f'(v_i) \leq 2$ for all $1 \leq i \leq n$, which implies $\sum_{i=1}^{n} \left(f(v_i) + f'(v_i) \right) \leq 2n$. This proves that, $\gamma_I(G) + \gamma_{iI}(G) \leq 2n$. For K_2, this sum is $4 = 2n$. $\qquad\square$

Theorem 6.5. [11] *If G is any connected graph of order $n \geq 3$, then*

$$\gamma_I(G) \leq \frac{3n}{4}.$$

Theorem 6.6. *For every connected graph G of order $n \geq 3$,*

$$\gamma_I(G) + \gamma_{iI}(G) \leq \frac{7n}{4} - 1,$$

and it is a sharp bound for P_4.

Proof. Let G be a connected graph G with $n \geq 3$. By Theorem 6.5 we have, $\gamma_I(G) \leq \frac{3n}{4}$. By adding this inequality to the inequality given in the Theorem 5.1, we arrive at, $\gamma_I(G) + \gamma_{iI}(G) \leq \frac{3n}{4} + n - 1 \leq \frac{7n}{4} - 1$. The equality holds for P_4 where $\gamma_I(G) + \gamma_{iI}(G) = 3 + 3 = 6 = \frac{7n}{4} - 1$. $\qquad\square$

Theorem 6.7. [7] *If G is any graph, then*

$$\gamma_I(G) \geq \frac{\gamma_{dR}(G)}{2}.$$

Theorem 6.8. *For every graph G of order $n \geq 3$,*

$$\gamma_{iI}(G) \leq \frac{7n}{4} - 1 - \frac{\gamma_{dR}(G)}{2}.$$

Proof. Let G be a graph with $n \geq 3$. By Theorem 6.6 and Theorem 6.7, we have the required result. $\qquad\square$

Theorem 6.9. [2] *If G is any connected graph, then*

$$\gamma_I(G) \geq \frac{2n}{\Delta(G) + 2}.$$

Theorem 6.10. *For every connected graph G of order $n \geq 3$,*

$$\frac{2n}{\Delta(G) + 2} \leq \gamma_{iI}(G) \leq \frac{7n}{4} - \frac{2n}{\Delta(G) + 2} - 1.$$

The upper bound is tight for C_4 and the lower bound is tight for C_n when n is even.

Proof. Let G be a connected graph of order n. By Theorem 6.6 and Theorem 6.7, we have the required upper bound. Since $\gamma_I(G) \leq \gamma_{iI}(G)$, we get the lower bound. The lower bound is sharp for cycles C_n where n is even. When n is even, $\gamma_{iI}(C_n) = \frac{n}{2}$. The upper bound is sharp for C_4 where $\gamma_{iI}(C_4) = 2$. $\qquad\square$

7.   Conclusion and open problems.

This field is expansive, offering endless opportunities for exploration. We propose a few
open problems on the basis of our research.

   Open problems

- Characterize the graphs with $\gamma_I(G) = \gamma_{iI}(G)$.

- Characterize the graphs with $\gamma_{iI}(G) = 2\gamma_i(G)$.

- Characterize the graphs with $\gamma_I(G) + \gamma_{iI}(G) = \dfrac{7n}{4} - 1$ for $n \geq 3$.

- Characterize the graphs with $\gamma_I(G) + \gamma_{iI}(G) = k$ for any $k < \dfrac{7n}{4} - 1$ for $n \geq 3$.

- Explore the Nordhaus Gaddum inequalities for $\gamma_{iI}(G)$.

- Find bounds for $\gamma_{iI}(G)$ in terms of different graph parameters for arbitrary graphs
  or some classes of graphs.

- Characterize the graphs with $\gamma_{iI}(G) = k$ for some k where $4 \leq k \leq n - 2$.

- Explore the relation between $\gamma_{idR}(G)$ and $\gamma_{iI}(G)$ for graphs.

- Explore the relation between $\gamma_{iI}(G)$ and $\gamma_i(G)$ for graphs.

References

[1]   B. Chaluvaraju and V. Chaitra. On coefficients of edge domination polynomial of a graph.
      Journal of Discrete Mathematical Sciences and Cryptography, 19(2):413–423, 2016. https:
      //doi.org/10.1080/09720529.2015.1107974.

[2]   M. Chellali, T. W. Haynes, S. T. Hedetniemi, and A. A. McRae. Roman 2-domination.
      Discrete Applied Mathematics, 204:22–28, 2016. https://doi.org/10.1016/j.dam.2015.
      11.013.

[3]   E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks*,
      7(3):247–261, 1977. https://doi.org/10.1002/net.3230070305.

[4]   W. L. D'Souza, V. Chaitra, and M. Kumara. Inverse double roman domination in graphs.
      Discrete Mathematics, Algorithms and Applications, 15(06):2250144, 2023. https://doi.
      org/10.1142/S1793830922501440.

[5]   W. L. D'Souza and V. Chaitra. Inverse minus domination in graphs. *Advances and Ap-
      plications in Discrete Mathematics*, 29(1):31–44, 2022. https://doi.org/10.17654/
      0974165822004.

[6]   H. Gao, T. Feng, and Y. Yang. Italian domination in the Cartesian product of paths.
      Journal of Combinatorial Optimization, 41(2):526–543, 2021. https://doi.org/10.1007/
      s10878-020-00694-x.

[7]   M. Hajibaba and N. J. Rad. Some notes on the Roman domination number and Italian
      domination number in graphs. In *Journal of Physics: Conference Series*, volume 890 of
      number 1, page 012123. IOP Publishing, 2017. https://doi.org/10.1088/1742-6596/
      890/1/012123.

[8]   F. Harary. *Graph Theory*. Addison-Wesley Series in Mathematics. Addison-Wesley Pub. Co., 1969.

[9]   T. Haynes, S. Hedetniemi, and P. Slater. *Domination in Graphs: Advanced Topics*. Marcel Dekker,Inc.,New York, 1998.

[10]   T. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker,Inc., New York, 1998.

[11]   T. W. Haynes, M. A. Henning, and L. Volkmann. Graphs with large Italian domination number. *Bulletin of the Malaysian Mathematical Sciences Society*, 43(6):4273–4287, 2020. https://doi.org/10.1007/s40840-020-00921-y.

[12]   V. Kulli and S. Sigarkanti. Inverse domination in graphs. *National Academy Science Letters*, 14(12):473–475, 1991.

[13]   T. Tamizh Chelvam, T. Asir, and G. Grace Prema. *Inverse Domination in Graphs*. Lambert Academic Publishing, 2013.

[14]   J. Varghese and S. Aparna Lakshmanan. Italian domination on Mycielskian and Sierpinski graphs. *Discrete Mathematics, Algorithms and Applications*, 13(04):2150037, 2021. https://doi.org/10.1142/S1793830921500373.

[15]   L. Volkmann. Italian domination in digraphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 111:269–278, 2019.