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$k-$division Fibonacci-Pell sequences and their applications

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ABSTRACT

In this paper, we present a method for constructing a new sequence, which we call $k-$division sequence and denoted by $h_n(k)$. Using Fibonacci and Pell sequences, we define the $k-$division Fibonacci-Pell sequence obtain its properties, and prove that this sequence is periodic. Then, as an application of the sequence, we define $1-$division Fibonacci-Pell sequence on finite groups and study it in the groups G_m and $H_{(u,l,m)}$ groups.

Keywords: period, Fibonacci sequence, Pell sequence, $p-$group

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1. Introduction

Sequences are important used in mathematics and other sciences (see [2, 7, 18, 16]). Among the very important sequences that can be used are Fibonacci, Pell, and Balancing numbers and their generalizations ([14, 15, 13]).

The Fibonacci and Pell sequences are defined as follows.

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 0, \tag{1}$$

with initial conditions $F_0 = 0,\ F_1 = 1$.

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The Pell sequence $\{P_n\}_{n=0}^{\infty}$ defined by

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 0, \tag{2}$$

with initial conditions $P_0 = 0,\ P_1 = 1$.

The characteristic polynomials of Fibonacci and Pell sequences are $x^2 - x - 1$ and $x^2 - 2x - 1$, respectively.

Number theoretic properties were investigated of the $k-$step generalized Pell sequence by Kilic and Tasci [11]. The authors of [10] defined the generalized Fibonacci and Pell sequence on Hessenberg. [4], Deveci and Karaduman redefined this sequence by group elements. In their paper Deveci and Karaduman studied in detail in finite groups on the $k-$step generalized Pell sequence. In [5], introduced the quaternion-Pell sequence and studied some properties of it. In [19], the connections between Pell numbers and Fibonacci $p-$numbers were studied. A generalized order $k-$Pell sequence was obtained for some special types of nilpotency in [8]. In [12], studied the $k-$Pell sequences of the 2-generator $p-$groups of nilpotency class 2.

For $m, u, l \in \mathbb{N}$, we consider the finitely presented group G_m and $H_{(u,l,m)}$ as follows

$$G_m =< a, b | a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b] >,\ m \geq 2.$$
$$H_{(u,l,m)} =< a, b, c \mid a^u = b^l = c^m = 1, [a, b] = c, [a, c] = [b, c] = 1 > .$$

Lemma 1.1. *Every element of G_m can be written uniquely in the form $a^i b^j [a, b]^k$, where $0 \leq i, j, k \leq m - 1$. (see [6]).*

Lemma 1.2. [17] *Every element of the Heisen- berg group $H_{(u,l,m)}$ can be written uniquely as $a^i b^j c^k$ with $1 \leq i \leq u$, $1 \leq j \leq l$ and $1 \leq k \leq m$.*

Motivated by the above results, we construct new sequences. Using characteristic polynomials, we introduce a new sequence. Here, we use the Fibonacci and Pell sequences and give new sequences. Then, we study some results. Also, we use these sequences in group theory.

The remainder of this paper is organized as follows. In Section 2, we introduce the $k-$division Fibonacci-Pell sequences and prove that these are periodic. The $k-$division Fibonacci-Pell sequences in a finite group are presented in Section 3 and studied on G_m and $H_{(u,l,m)}$ groups.

2. The $k-$division Fibonacci-Pell sequences

In this section, first, we present a new method for constructing sequences and named the $k-$division sequence. Then, we define the $k-$division Fibonacci-Pell sequences and get some results.

Definition 2.1. Let $f(x)$ and $g(x)$ be the characteristic polynomials of two arbitrary sequences and degree u and m where $m \geq u$, respectively. For $k \in \mathbb{N}$, the $k-$division sequence, $\{h_n(k)\}_{n=0}^{\infty}$ is definded by

$$h_n(k) = x^k(g(x)) + t(x), \ n \geq k+m, \tag{3}$$

where $t(x)$ is the remainder of division $\dfrac{x^k(g(x))}{f(x)}$ and with initial conditions $h_0(k) = h_1(k) = \cdots = h_{m+k-2}(k) = 0, \ h_{m+k-1}(k) = 1.$

Here, we define the 1−division Fibonacci-Pell sequence as follows. We consider Fibonacci and Pell sequences. Suppose that $k = 1$. We have $f(x) = x^2 - x - 1$ and $g(x) = x^2 - 2x - 1$. By Definition 2.1, we define a new sequence as follows.

Definition 2.2. If $k = 1$, the 1−division Fibonacci-Pell sequence defined as follows

$$h_n(1) = 2h_{n-1}(1) + 2h_{n-2}(1) + h_{n-3}(1), \ n \geq 3, \tag{4}$$

with initial conditions $h_0(1) = h_1(1) = 0, \ h_2(1) = 1$.

The 1−division Fibonacci-Pell sequence modulo α,

$$\{h_n^\alpha(1)\} = \{h_0^\alpha(1), h_1^\alpha(1), \cdots, h_i^\alpha(1), \cdots\},$$

where $h_i^\alpha(1) = h_i(1) \ (mod \ \alpha)$.

Theorem 2.3. *The* 1−*division Fibonacci-Pell Sequence* $\{h_n^\alpha(1)\}$ *is simply periodic.*

Proof. Suppose that $X_3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{N} \ and \ 1 \leq x_i \leq \alpha\}$. So that we have $\mid X_3 \mid = \alpha^3$. Since there are α^3 distinct 3−tuples of elements of Z_α, at least one of the 3−tuples appears twice in the sequence $\{h_n^\alpha(1)\}$. Then the subsequence follows this 3−tuple. Thus, it is obvious that the sequence $\{h_n^\alpha(1)\}$ is periodic.

Hence, it is cearly for $w \geq 0$, there exist $w \geq v$ such that

$$h_w^\alpha(1) \equiv h_v^\alpha(1), \ h_{w+1}^\alpha(1) \equiv h_{v+1}^\alpha(1), \ h_{w+2}^\alpha(1) \equiv h_{v+2}^\alpha(1).$$

By definition of 1−division Fibonacci-Pell sequence, we have

$$h_n(1) = 2h_{n-1}(1) + 2h_{n-2}(1) + h_{n-3}(1).$$

Thus, we can easily derive that

$$h_{w-v}^\alpha(1) \equiv h_0^\alpha(1), \ h_{w-v+1}^\alpha(1) \equiv h_1^\alpha(1), \ h_{w-v+2}^\alpha(1) \equiv h_2^\alpha(1),$$

which indicate that 1−division Fibonacci-Pell sequence is simply periodic. □

We use $Kh_m(1)$ to denoted the minimal period of 1−division Fibonacci-Pell sequence modulo m. In Table 1, we calculate $Kh_m(1)$, for $2 \leq m \leq 31$.

Lemma 2.4. *For integers* s, n *and* $m \geq 2$, *we have*

(i) $h_{Kh_m(1)+n}(1) \equiv h_n(1) \ \ (mod \ m)$,

(ii) $h_{s \times (Kh_m(1))+n}(1) \equiv h_n(1) \ \ (mod \ m)$.

Table 1. $Kh_m(1)$, for $2 \le m \le 31$

m	$Kh_m(1)$	m	$Kh_m(1)$	m	$Kh_m(1)$
2	3	12	78	22	399
3	13	13	168	23	22
4	6	14	171	24	156
5	8	15	104	25	40
6	39	16	24	26	168
7	57	17	307	27	741
8	12	18	39	27	342
9	39	19	360	29	67
10	24	20	24	30	312
11	133	21	741	31	993

Proof. The result follows by using the definition of the period of the $1-$division Fibonacci-Pell sequence modulo m. (ii) We have

$$\begin{aligned} h_{s\times(Kh_m(1))+n}(1) \equiv& h_{(Kh_m(1))+(s-1)\times(Kh_m(1))+n}(1) \\ \equiv& h_{(s-1)(Kh_m(1))+n}(1) \\ \equiv& \cdots \equiv h_n(1) \pmod{m}. \end{aligned}$$

The result is obtained. □

Lemma 2.5. *Let s be an integer and $m \ge 2$ is a positive number. If*

$$\begin{cases} h_s(1) \equiv 0 & (mod\ m), \\ h_{s+1}(1) \equiv 0 & (mod\ m), \\ h_{s+2}(1) \equiv 1 & (mod\ m), \end{cases}$$

then $Kh_m(1) \mid s$.

Proof. There exists $0 \le i \le Kh_m(1)$ such that $s = t \times (Kh_m(1)) + i$. Also, $\forall\ 0 \le j \le 2$, $h_{s+j}(1) \equiv h_{i+j}(1)(mod\ m)$, then we have the following equations

$$\begin{cases} h_i(1) \equiv 0 & (mod\ m), \\ h_{i+1}(1) \equiv 0 & (mod\ m), \\ h_{i+2}(1) \equiv 1 & (mod\ m), \end{cases}$$

So that i is a period of $1-$division Fibonacci-Pell sequence modulo m. $i.e.$ $Kh_m(1) \mid i$. Since $0 \le i < Kh_m(1)$, we have $i = 0$. Therefore, we get the results. □

Now, For $k \ge 2$, we generlized $k-$division Fibonacci-Pell Sequence. By using Definition

2.1, we have

$$\begin{aligned}
&k=2,\ h_n(2)=2h_{n-1}(2)+h_{n-2}(2)+2h_{n-3}(2)+h_{n-4}(2),\ n\geq 4,\\
&k=3,\ h_n(3)=2h_{n-1}(3)+h_{n-2}(3)+3h_{n-4}(3)+2h_{n-5}(3),\ n\geq 5,\\
&k=4,\ h_n(4)=2h_{n-1}(4)+h_{n-2}(4)+5h_{n-5}(4)+3h_{n-6}(4),\ n\geq 6,\\
&k=5,\ h_n(5)=2h_{n-1}(5)+h_{n-2}(5)+8h_{n-6}(5)+5h_{n-7}(5),\ n\geq 7,\\
&k=6,\ h_n(6)=2h_{n-1}(6)+h_{n-2}(6)+13h_{n-7}(6)+8h_{n-8}(6),\ n\geq 8,\\
&\vdots
\end{aligned}$$

In general,

$$k=m,\ h_n(m)=2h_{n-1}(m)+h_{n-2}(m)+F_{m+1}h_{n-(m+1)}(m)+F_mh_{n-(m+2)}(m),\ n\geq m+2,$$

with initial conditions $h_0(m)=h_1(m)=0=\cdots=h_m(m)=0,\ h_{m+1}(m)=1$. Since, we need to show that

$$h_m(X)=X^m(X^2-2X-1)+F_{m+1}X+F_m, \tag{5}$$

is divisible by $f(X)=X^2-X-1$. Let $\gamma\frac{1+\sqrt{5}}{2}$ be the golden section. Then $\gamma^2=\gamma+1$. Also,

$$\gamma^{m+1}=F_{m+1}\gamma+F_m\ for\ all\ m\geq 1, \tag{6}$$

a formula which can be proved by induction. But then

$$\begin{aligned}
h_m(\gamma)&=\gamma^m(\gamma^2-2\gamma-1)+F_{m+1}\gamma+F_m\\
&=\gamma^m((\gamma^2-2\gamma-1)-\gamma)+F_{m+1}\gamma+F_m\\
&=-\gamma^{m+1}+F_{m+1}\gamma+F_m\\
&=0,
\end{aligned}$$

which shows that the polynomail shown at (5) has γ as a root. Since it has integer coefficients, it is divisible by the minimal polynomail of γ which is X^2-X-1, so indeed $F_{m+1}\gamma+F_m$ is the required $t(X)$.

In Table 2, we obtain $h_n(k)$, for $2\leq k\leq 8$ and $0\leq n\leq 10$.

Table 2. $h_n(k)$, for $2\leq k\leq 8$and $0\leq n\leq 10$

n	0	1	2	3	4	5	6	7	8	9	10
$h_n(2)$	0	0	0	1	2	5	14	38	102	272	742
$h_n(3)$	0	0	0	0	1	2	5	12	32	84	219
$h_n(4)$	0	0	0	0	0	1	2	5	12	29	75
$h_n(5)$	0	0	0	0	0	0	1	2	5	12	29
$h_n(6)$	0	0	0	0	0	0	0	1	2	5	12
$h_n(7)$	0	0	0	0	0	0	0	0	1	2	5
$h_n(8)$	0	0	0	0	0	0	0	0	0	1	2

Lemma 2.6. *Let $g_{h_n(m)}$ be the generating function of $m-$division Fibonacci-Pell sequence. Then,*

$$g_{h_n(m)} = \frac{x^{m+1}}{1 - 2x - x^2 - F_{m+1}x^{m+1} - F_m x^{m+2}}. \tag{7}$$

Proof. Let $g_{h_n(m)}$ be the generating function of the generating function of $m-$division Fibonacci-Pell sequence. We have

$$\begin{aligned}
g_{h_n(m)} &= \sum_{n=1}^{\infty} h_n(m)x^n \\
&= h_1(m)x + h_2(m)x^2 + \cdots + h_{m+1}(m)x^{m+1} + \sum_{n=m+2}^{\infty} h_n(m)x^n \\
&= x^{m+1} + \sum_{n=m+2}^{\infty} (2h_{n-1}(m) + h_{n-2}(m) + F_{m+1}h_{n-(m+1)}(m) + F_m h_{n-(m+2)}(m))x^n \\
&= x^{m+1} + 2\sum_{n=m+2}^{\infty} h_{n-1}(m)x^n + \sum_{n=m+2}^{\infty} h_{n-2}(m)x^n + F_{m+1}\sum_{n=m+2}^{\infty} h_{n-(m+1)}(m)x^n \\
&\quad + F_m \sum_{n=m+2}^{\infty} h_{n-(m+2)}(m)x^n \\
&= x^{m+1} + 2x\sum_{n=1}^{\infty} h_n(m)x^n + x^2\sum_{n=1}^{\infty} h_n(m)x^n + F_{m+1}x^{m+1}\sum_{n=1}^{\infty} h_n(m)x^n \\
&\quad + F_m x^{m+2}\sum_{n=1}^{\infty} h_n(m)x^n \\
&= x^{m+1} + 2xg_{h_n(m)} + x^2 g_{h_n(m)} + F_{m+1}x^{m+1}g_{h_n(m)} + F_m x^{m+2}g_{h_n(m)}.
\end{aligned}$$

Thus,

$$g_{h_n(m)} = \frac{x^{m+1}}{1 - 2x - x^2 - F_{m+1}x^{m+1} - F_m x^{m+2}}.$$

□

Lemma 2.7. *The generating function of $m-$division Fibonacci-Pell sequence has the following exponential representation*

$$g_{h_n(m)} = x^{m+1} exp \sum_{i=1}^{\infty} \frac{x^i}{i}(2 + x + F_{m+1}x^m + F_m x^{m+1})^i,$$

where $m \geq 2$.

Proof. By using (7), we have

$$ln \frac{g_{h_n(m)}}{x^{m+1}} = -ln(1 - 2x - x^2 - F_{m+1}x^{m+1} - F_m x^{m+2}).$$

Thus,

$$-\,ln(1-2x-x^2-F_{m+1}x^{m+1}-F_mx^{m+2}) = -[-x(2+x+F_{m+1}x^m+F_mx^{m+1})$$
$$-\frac{1}{2}x^2(2+x+F_{m+1}x^m+F_mx^{m+1})^2-\cdots-\frac{1}{i}x^i(2+x+F_{m+1}x^m+F_mx^{m+1})^i-\cdots],$$

result has been achieved. □

3. The $1-$division Fibonacci-Pell sequence in a finite group

In this section, we define the $1-$division Fibonacci-Pell sequence in a finite group and prove that the $1-$division Fibonacci-Pell sequence in a finite group is simply periodic. Also, we study this sequence on G_m and $H_{(u,l,m)}$. First, we define the $1-$division Fibonacci-Pell sequence in a finite group as follows.

Definition 3.1. The $1-$division Fibonacci-Pell sequence in a finite group is a sequence of group elements $x_0, x_1, \cdots, x_n, \cdots$ for which, given an initial (seed) set in $X = \{a_1, a_2, \cdots, a_j\}$, each element is defined by

(i) If $j = 2$, then $x_0 = a_1,\ x_1 = a_2$ and $x_2 = a_1^2a_2^2$,

$$x_n = x_{n-3}(x_{n-2})^2(x_{n-1})^2, \quad n \geq 3. \tag{8}$$

(ii) If $j \geq 3$, then $x_0 = a_1,\ x_1 = a_2,\ \cdots, x_j = a_j$ and

$$x_n = x_{n-3}(x_{n-2})^2(x_{n-1})^2, \quad n \geq j+1. \tag{9}$$

The element of $1-$division Fibonacci-Pell sequence in group are denoted by $Q(G, X)$ and its period is denoted by $hQ(G, X)$.

Theorem 3.2. *The $1-$division Fibonacci-Pell sequence in group is simply periodic.*

Proof. Let G be a $i-$generator group and let $(a_0,\ a_1,\ \cdots,\ a_{i-1})$ be a generating $i-$tuple for G. If $\mid G \mid= m$, then there are m^i distinict $i-$tuple of elements of G. Thus, at least one of the $i-$tuple appers twice in $Q(G, X)$. Because of the repetition, the sequence $Q(G, X)$ is periodic. Now, we show simply periodic. It is cearly, there is r and s in $\mathbb{N}$, with $s > r$ such that $x_{s+1} = x_{r+1},\ x_{s+2} = x_{r+2}, x_{s+3} = x_{r+3}$. Form (9), we written $x_{s-1} = x_{r-1},\ x_{s-2} = x_{r-2}, x_{s-r} = a_0 = x_0 = x_{r-s}$. Thus, $hQ(G, X)$ is simply periodic. □

For $m \in \mathbb{N}$, we consider the finitely presented group G_m as follows

$$G_m =< a, b | a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b] >,\ m \geq 2.$$

Set $h_n := h_n(1)$. Now, we obtain sequence w_n and c_n as follows

$$w_n = h_{n+1} - (h_{n-1} + h_{n-2}),$$
$$c_0 = c_1 = c_2 = 0,$$
$$c_n = c_{n-3} + 2c_{n-2} + 2c_{n-1} - (h_{n-2}w_{n-2} + (h_{n-2} + h_{n-1})w_{n-2} + (h_{n-2} + 2h_{n-1})w_{n-1}$$
$$+ (h_{n-2} + 2h_{n-1} + h_n)w_{n-1}), \quad n \geq 3,$$

Lemma 3.3. *Every element of* $Q(G_m, X)$ *may be presented by* $x_n = a^{w_n} b^{h_{n+1}} [a, b]^{c_n}$, $n \geq 3$.

Proof. For $n = 2$ and $n = 3$, we have $x_2 = a^2 b^2$ and $x_3 = ab^2(a^2b^2)^2 = a^5 b^6 [a, b]^{-12}$. Now, by induction on n, we have

$$\begin{aligned}
x_n &= x_{n-3}(x_{n-2})^2(x_{n-1})^2 \\
&= a^{w_{n-3}} b^{h_{n-2}} [a, b]^{c_{n-3}} (a^{w_{n-2}} b^{h_{n-1}} [a, b]^{c_{n-2}})^2 (a^{w_{n-1}} b^{h_n} [a, b]^{c_{n-1}})^2 \\
&= a^{w_{n-3}} b^{h_{n-2}} [a, b]^{c_{n-3}} a^{w_{n-2}} b^{h_{n-1}} [a, b]^{c_{n-2}} a^{w_{n-2}} b^{h_{n-1}} [a, b]^{c_{n-2}} (a^{w_{n-1}} b^{h_n} [a, b]^{c_{n-1}})^2 \\
&= a^{w_{n-3}+w_{n-2}} b^{h_{n-2}+h_{n-1}} [a, b]^{c_{n-3}+c_{n-2}-h_{n-2}w_{n-2}} a^{w_{n-2}} b^{h_{n-1}} [a, b]^{c_{n-2}} (a^{w_{n-1}} b^{h_n} [a, b]^{c_{n-1}})^2 \\
&= a^{w_{n-3}+2w_{n-2}} b^{h_{n-2}+2h_{n-1}} [a, b]^{c_{n-3}+c_{n-2}-h_{n-2}w_{n-2}-(h_{n-2}+h_{n-1})w_{n-2}} (a^{w_{n-1}} b^{h_n} [a, b]^{c_{n-1}})^2 \\
&= \cdots \\
&= a^{w_{n-3}+2w_{n-2}+2w_{n-1}} b^{h_{n-2}+2h_{n-1}+2h_n} \\
&\quad [a, b]^{c_{n-3}+2c_{n-2}+2c_{n-1}-(h_{n-2}w_{n-2}+(h_{n-2}+h_{n-1})w_{n-2}+(h_{n-2}+2h_{n-1})w_{n-1}+(h_{n-2}+2h_{n-1}+h_n)w_{n-1})} \\
&= a^{w_n} b^{h_{n+1}} [a, b]^{c_n}.
\end{aligned}$$

Therefore, the assertion holds. □

Lemma 3.4. *If* $hQ(G_m, X) = s$ *then* s *is the least integer such that all of the congruences*

$$\begin{cases}
w_s \equiv 1 & (mod\ m), \\
w_{s+1} \equiv 0 & (mod\ m), \\
w_{s+2} \equiv 2 & (mod\ m), \\
h_{s+1} \equiv 0 & (mod\ m), \\
h_{s+2} \equiv 1 & (mod\ m), \\
h_{s+3} \equiv 2 & (mod\ m), \\
c_s \equiv 0 & (mod\ m), \\
c_{s+1} \equiv 0 & (mod\ m), \\
c_{s+2} \equiv 0 & (mod\ m),
\end{cases}$$

hold. Morover, $Kh_m(1)$ *divides* $hQ(G_m, X)$.

Proof. By Lemma 3.3, we obtain $x_n = a^{w_n} b^{h_{n+1}} [a, b]^{c_n}$. Since $x_s = a$, $x_{s+1} = b$ and $x_{s+2} = a^2 b^2$, by Lemma 1.1, we have

$$\begin{cases}
w_s \equiv 1 & (mod\ m), \\
w_{s+1} \equiv 0 & (mod\ m), \\
w_{s+2} \equiv 2 & (mod\ m), \\
h_{s+1} \equiv 0 & (mod\ m), \\
h_{s+2} \equiv 1 & (mod\ m), \\
h_{s+3} \equiv 2 & (mod\ m), \\
c_s \equiv 0 & (mod\ m), \\
c_{s+1} \equiv 0 & (mod\ m), \\
c_{s+2} \equiv 0 & (mod\ m).
\end{cases}$$

Thus, Lemma 2.5 proved that $Kh_m(1) \mid hQ(G_m, X)$. □

Here, we consider Heisenberg group $H_{(u,l,m)} =< a, b, c \mid a^u = b^l = c^m = 1, [a, b] = c, [a, c] = [b, c] = 1 >$ and define sequences g_n and S_n as follows.

$$\begin{aligned}
&g_n = h_n - (h_{n-2} + h_{n-3}),\\
&S_0 = 0,\ S_1 = 0,\ S_2 = 1,\\
&S_n = S_{n-3} + 2S_{n-2} + 2S_{n-1} - (g_{n-3}h_{n-3} + h_{n-3}(g_{n-3} + g_{n-2}) + h_{n-2}(g_{n-3} + 2g_{n-2})\\
&\qquad + h_{n-2}(g_{n-3} + 2g_{n-2} + g_{n-1})),\ n \geq 3.
\end{aligned}$$

Lemma 3.5. *Every element of $Q(H_{(u,l,m)}, X)$ may be presented by $x_n = a^{h_{n-1}} b^{g_n} c^{S_n}$, $n \geq 3$.*

Proof. For $n = 3$, $n = 4$ and $n = 5$, we have $x_3 = ab^2c^2$, $x_4 = bc^2(ab^2c^2)^2 = a^2b^5c^2$ and $x_5 = c(ab^2c^2)^2(a^2b^5c^2)^2 = a^6b^{14}c^{-19}$. Now, by induction on n, we have

$$\begin{aligned}
x_n &= x_{n-3}(x_{n-2})^2(x_{n-1})^2\\
&= a^{h_{n-4}} b^{g_{n-3}} c^{S_{n-3}} (a^{h_{n-3}} b^{g_{n-2}} c^{S_{n-2}})^2 (a^{h_{n-2}} b^{g_{n-1}} c^{S_{n-1}})^2\\
&= a^{h_{n-4}} b^{g_{n-3}} c^{S_{n-3}} a^{h_{n-3}} b^{g_{n-2}} c^{S_{n-2}} a^{h_{n-3}} b^{g_{n-2}} c^{S_{n-2}} (a^{h_{n-2}} b^{g_{n-1}} c^{S_{n-1}})^2\\
&= a^{h_{n-4}+h_{n-3}} b^{g_{n-3}+g_{n-2}} c^{S_{n-3}+S_{n-2}-g_{n-3}h_{n-3}} a^{h_{n-3}} b^{g_{n-2}} c^{S_{n-2}} (a^{h_{n-2}} b^{g_{n-1}} c^{S_{n-1}})^2\\
&= a^{h_{n-4}+2h_{n-3}} b^{g_{n-3}+2g_{n-2}} c^{S_{n-3}+2S_{n-2}-g_{n-3}h_{n-3}-h_{n-3}(g_{n-3}+g_{n-2})} (a^{h_{n-2}} b^{g_{n-1}} c^{S_{n-1}})^2\\
&= \cdots\\
&= a^{h_{n-4}+2h_{n-3}+2h_{n-2}} b^{g_{n-3}+2g_{n-2}+2g_{n-1}}\\
&\quad c^{S_{n-3}+2S_{n-2}+2S_{n-1}-(g_{n-3}h_{n-3}+h_{n-3}(g_{n-3}+g_{n-2})+h_{n-2}(g_{n-3}+2g_{n-2})+h_{n-2}(g_{n-3}+2g_{n-2}+g_{n-1}))}\\
&= a^{h_{n-1}} b^{g_n} c^{S_n}.
\end{aligned}$$

Lemma is proved. □

Theorem 3.6. *For $u \geq 1$, we have $Kh_u(1) \mid MQ(H_{(u,l,m)}, X)$.*

Proof. By Lemma 3.5, we obtain $x_n = a^{h_{n-1}} b^{g_n} c^{S_n}$. Suppose that $hQ(H_{(u,l,m)}, X) = i$. Since $x_i = a$, $x_{i+1} = b$ and $x_{i+2} = c$, by Lemma 1.2, we have

$$\begin{cases}
h_{i-1} \equiv 1 & (mod\ u),\\
h_i \equiv 0 & (mod\ u),\\
h_{i+1} \equiv 0 & (mod\ u),\\
g_i \equiv 0 & (mod\ l),\\
g_{i+1} \equiv 1 & (mod\ l),\\
g_{i+2} \equiv 0 & (mod\ l),\\
S_i \equiv 0 & (mod\ m),\\
S_{i+1} \equiv 0 & (mod\ m),\\
S_{i+2} \equiv 1 & (mod\ m).
\end{cases}$$

So, Lemma 2.5 proved that $Kh_u(1) \mid hQ(H_{(u,l,m)}, X)$. □

4. Conclusion

Considering the importance of sequences in all sciences, here we have presented a new method for constructing sequences and these sequences can be used in group theory, coding theory, cryptography, and other sciences. In this paper, we used the Fibonacci sequence and Pell sequence and defined the $k-$division Fibonacci-Pell sequence got its properties, and proved that this sequence is periodic. Then, as an application of the sequence, we defined $k-$division Fibonacci-Pell sequence in finite groups and studied it in G_m and $H_{(u,l,m)}$ groups. As future work, the new sequences presented in Definition 2.1 can be use use any sequence to obtain a $k-$division [1, 3, 9].

In conclusion, we have two open questions.

1. Prove or disprove
   i. $hQ(G_m, X) = Kh_m(1)$.
   ii. $hQ(H_{(u,l,m)}, X) = Kh_u(1)$.
2. Is it possible to prove that each finite group G divides the minimal period of $1-$division Fibonacci-Pell sequence?

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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