UTILITAS MATHEMATICA 122 (2025) 41–52

Utilitas Mathematica
www.combinatorialpress.com/um

The covering cover pebbling number for some acyclic graphs

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ABSTRACT

The covering cover pebbling number, $\sigma(G)$, of a graph G, is the smallest number such that some distribution $D \in \mathscr{K}$ is reachable from every distribution starting with $\sigma(G)$ (or more) pebbles on G, where $\mathscr{K}$ is a set of covering distributions. In this paper, we determine the covering cover pebbling number for two families of graphs those do not contain any cycles.

Keywords: pebbling number, covering set, acyclic graph

2020 Mathematics Subject Classification: 05B40, 05C05, 05C70, 05C99.

1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [2] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Given a connected graph G, distribute certain number of pebbles on its vertices in some configuration. Precisely, a configuration on a graph G, is a function from $V(G)$ to $N \cup \{0\}$ representing a placement of pebbles on G. The size of the configuration is the total number of pebbles placed on the vertices. A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In (regular) pebbling, the target is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that regardless of their initial placement and regardless of the target vertex, we can pebble that target vertex is called the pebbling number of G, denoted by $f(G)$. In cover pebbling, the aim is to cover all the vertices with at least one pebble, when the pebbling process ends. The minimum number of pebbles required such that regardless of their initial placement on G, there is a sequence of pebbling moves, at the end of which, every vertex

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Received 25 August 2024; accepted 25 February 2025; published 22 March 2025.

DOI: 10.61091/um122-03

has at least one pebble on it, is called the cover pebbling number of G. The following definitions are stated from [3]:

Definition 1.1. A *distribution* or *configuration* D is a function $D : V(G) \longrightarrow N$ where $D(v)$ represents the number of pebbles on the vertex v. Also, for every distribution D and every positive integer, we define tD as the distribution given by $(tD)(v) = tD(v)$ for every vertex v in G.

Definition 1.2. Given two distributions D' and D'' on a graph G, we say that D'' *contains* D' if $D'(v) \leq D''(v)$ for every vertex $v \in V(G)$.

Definition 1.3. Given two distributions D and D' on a graph G, we say that D' is *reachable* from D if it is possible to use a sequence of pebbling moves to go from D to a distribution D'' that contains D.

Definition 1.4. Let $\mathcal{S}$ be a set of distributions on a graph G. The *pebbling number of* $\mathcal{S}$ in G, denoted $\pi(G, \mathcal{S})$, is the smallest number such that every distribution $D \in \mathcal{S}$ is reachable from every distribution that starts with $\pi(G, \mathcal{S})$ (or more) pebbles on G.

We find similar definitions for the following concepts in [3]:
(i) pebbling number of a distribution D, i.e., $\pi(G, D)$,
(ii) t-pebbling number of a vertex in G, i.e., $\pi_t(G, v)$,
(iii) t-pebbling number of a graph G, i.e., $\pi_t(G)$.

Definition 1.5. In a distribution on a graph G, a vertex with $D(v) \geq 1$ pebbles is called an *occupied vertex*.

Now we are going to define *covering cover pebbling number* of a graph G, using Definition 1.2 and Definition 1.3. A set $K \subseteq V(G)$ is a covering [1], if every edge of G has at least one end in K. The concept of covering cover pebbling number was first introduced by Lourdusamy et al. [11], and they determined the covering cover pebbling number for complete graphs, paths, wheel graphs, complete r-partite graphs and binary trees in [11]. For more results on covering cover pebbling number, please refer to [7, 8, 9, 11, 10, 4, 5, 6]. Let us now define some specific distribution and set of distributions that would be helpful in formulating Definition 1.8.

Definition 1.6. For a set $K \subseteq V(G)$ and a vertex $x \in V(G)$, we define the distribution χ_K on G as the function:

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases}$$

where the set K forms a covering for G.

Definition 1.7. We also let $\mathscr{K} = \{\chi_K : K \subseteq V(G) \text{ is a covering set}\}$. That is, $\mathscr{K}$ is the set of covering distributions.

Definition 1.8. The covering cover pebbling number, $\sigma(G)$, of a graph G, is the smallest number such that some distribution $D \in \mathscr{K}$ is reachable from every distribution starting with $\sigma(G)$ (or more) pebbles on G.

Theorem 1.9. [11] *For a Star graph $K_{m,1}$ $(m \geq 2)$, $\sigma(K_{m,1}) = m$.*

Theorem 1.10. [11] $\sigma(B_0) = 1$, $\sigma(B_1) = 2$, $\sigma(B_2) = 12$, $\sigma(B_3) = 86$, $\sigma(B_4) = 634$ *and for* $n \geq 2$,

$$\sigma(B_n) = 2^{n-1} + \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \left(2^{2i+1} + \sum_{j=1}^{n-2i-2} 2^{j-1} 2^{2i+2j+1} \right) + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} 2^{2n-2k+2} + 2^{2\lfloor \frac{n-1}{2} \rfloor + 1}.$$

2. The covering cover pebbling number for an m-ary tree

In this section, we are going to determine the covering cover pebbling number of an m-ary tree $(m \geq 2)$, using Definition 1.8.

Definition 2.1. A complete m-ary tree, denoted by M_n, is a tree of height n with m^i vertices at distances i from the root. Each vertex of M_n has m children except for the set of m^n vertices that are at distance n away from the root, none of which have children. The root is denoted by R_n.

Obviously, $\sigma(M_0) = 1$, and $\sigma(M_1) = m$ [11], since $M_0 \cong K_1$, the complete graph on one vertex, and $M_1 \cong K_{1,m}$, the star graph on $m + 1$ vertices.

Remark 2.2. Note that M_2 has $m - M_1$'s as subtrees on it. We label them, as M_{11}, M_{12}, $\cdots$, M_{1m} and their corresponding roots are R_{11}, R_{12}, $\cdots$, R_{1m}. So, in general, the complete m-ary tree M_n has $m - M_{n-1}$'s as subtrees on it and hence we label them as $M_{(n-1)1}$, $M_{(n-1)2}$, $\cdots$, $M_{(n-1)m}$ and we denote their corresponding roots by $R_{(n-1)1}$, $R_{(n-1)2}$, $\cdots$, $R_{(n-1)m}$. Let v be the rightmost bottom vertex of M_n.

Lemma 2.3. *We can send a pebble to R_n, the root of M_n, at a cost of at most 2^n pebbles,*

1) *when $n = 2$ and there exists a M_{1i} $(1 \leq i \leq m)$, a subtree of M_2, such that $p(M_{1i}) \geq m + 3$,*

2) *when $n = 3$ and there exists a M_{2i} $(1 \leq i \leq m)$, a subtree of M_3, such that $p(M_{2i}) \geq m^2 + 7m$,*

3) *when $n = 4$ and there exists a M_{3i} $(1 \leq i \leq m)$, a subtree of M_4, such that $p(M_{3i}) \geq m^3 + 31m^2 - 24m + 14$.*

Proof. 1). Let $n = 2$ and $p(M_{1i}) \geq m + 3$ for a subtree M_{1i} of M_2. If $p(R_{1i}) \geq 2$ or a vertex of $M_{1i} - R_{1i}$ has more than three pebbles or two verices of $M_{1i} - R_{1i}$ contain at least two pebbles each on them, then we can send one pebble to the root R_2 of M_2 easily at a cost of at most 4 pebbles. If not, then $p(M_{1i}) \leq 3 + (m - 1) = m + 2$ - a contradiction to our assumption.

2). Let $n = 3$ and $p(M_{2i}) \geq m^2 + 7m$ for a subtree M_{2i} of M_3. If $p(R_{2i}) \geq 2$ then clearly we can move a pebble to R_3. So assume that $p(R_{2i}) \leq 1$. Assume $p(R_{2i}) = 0$ (otherwise, $\left\lceil \frac{m^2+7m-1}{m} \right\rceil \geq m+3$ and hence we can move one more pebble to $p(R_{2i})$ by (1)). Let $p(R_{2i}) = 0$. Clearly any one of the subtree of M_{2i} must contain at least $\left\lceil \frac{m^2+7m}{m} \right\rceil \geq m + 7$. By (1), we can move a pebble to R_{2i} and then the remaining number of pebbles on the subtree of M_{2i} is at least $m + 3$. Again by (1), we can move another one pebble to R_{2i} and hence we move a pebble to R_3 using at most eight pebbles.

3). Let $n = 4$ and $p(M_{3i}) \geq m^3 + 31m^2 - 24m + 14$ for a subtree M_{3i} of M_4. If $p(R_{3i}) \geq 2$ then we can move a pebble to R_4. Assume $p(R_{3i}) = 0$ (otherwise we are done). Then any one of the subtree of M_{3i} must contain at least $\left\lceil \frac{m^3+31m^2-24m+14}{m} \right\rceil \geq m^2+31m-24$. By (2), we can move a pebble (at a cost of at most 16 pebbles) to R_4 through R_{3i}, since the subtree of M_{3i} contains at least m^2+7m+8 pebbles. □

Theorem 2.4. *For the m-ary tree M_2, $\sigma(M_2) = m^2 + 7m - 6$.*

Proof. First, we place $m^2 - m$ pebbles on the bottom vertices such that no m pebbles of which share a parent and we did not put any pebbles on the vertex v. And then we place $8m - 7$ pebbles on the vertex v, then no distribution of $\mathscr{K}$ is reachable. Thus $\sigma(M_2) \geq m^2 + 7m - 6$.

Now, consider the distribution of $m^2 + 7m - 6$ pebbles on the vertices of M_2. According to the distributions of $p(M_2)$ pebbles, we can partite them into three cases.

Case 1. Let $p(M_{1i}) \geq m$ for all $1 \leq i \leq m$.

If $p(R_2) \geq 1$ then there exists a distribution of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_1) = m$. Let $p(R_2) = 0$. Any one of the subtree, say M_{11}, must contain at least $\left\lceil \frac{m^2+7m-6}{m} \right\rceil \geq m + 4$ pebbles and hence we can move a pebble to R_2 by Lemma 2.3 (1). The remaining number of pebbles on M_{11} is at least m and thus there exists a distribution of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_1) = m$.

Case 2. Let $p(M_{1i}) \leq m - 1$ for all $1 \leq i \leq m$.

Clearly $p(R_2) \geq p(M_2) - m(m - 1) = 8m - 7 \geq 2m$ and hence we put one pebble each on the vertices $R_{11}, R_{12}, \cdots, R_{1m}$ from the pebbles at R_2. Thus, the distribution $\chi_{\{R_{11},R_{12},\cdots,R_{1m}\}}$ of $\mathscr{K}$ is reachable.

Case 3. Let $p(M_{1i}) \leq m - 1$ for some i.

Let h subtrees contain at most $m - 1$ pebbles each on them, where $1 \leq h \leq m - 1$. We prove this case by induction on $h \geq 1$. Let $h = 1$, that is, only one subtree, say M_{1m}, has at most $m-1$ pebbles on it. So, our aim is to provide two pebbles to the root R_2 from the subtrees those have totally at least $m^2 + 6m - 5$ pebbles, so that we can move one pebble to R_{1m}. Clearly, any one of the subtree, say M_{11}, must contain at least $\left\lceil \frac{m^2+6m-5}{m-1} \right\rceil \geq m + 8$ and hence we can move two pebbles to R_2 while retaining m pebbles, by Lemma 2.3 (1). Thus, the distribution $\chi_{\{K\}} \cup \chi_{\{R_{1m}\}} = \chi_{\{K \cup R_{1m}\}}$ of $\mathscr{K}$ is reachable, where $\chi_{\{K\}}$ is a distribution of $\mathscr{K}$ which is reachable from those subtrees having at least m pebbles each on them and $\chi_{\{R_{1m}\}}$ is a reachable distribution of $\mathscr{K}$ for $V(M_{1m}) \cup R_2$. So assume the result is true for $h \leq m-2$. Let $h = m-1$. WLOG, let M_{11} be the subtree that contains at least m pebbles. Clearly, $p(M_{11}) \geq p(M_2) - (m - 1)(m - 1) \geq 9m - 7$. We have to retain $m + 1$ pebbles on M_{11} and thus M_{11} has $8m - 8$ extra pebbles on it. Now, we need at most eight pebbles from M_{11} to put one pebble on a root vertex, say R_{12} of the subtree M_{12} , by induction and by Lemma 2.3 (1). After using eight pebbles (at most) from M_{11}, the remaining number of pebbles is at least $(m + 1) + 8(m - 2)$ and therefore we can move one pebble to every root vertex R_{1j} $(j \neq 1, 2)$ of M_{1j} by induction and by Lemma 2.3 (1). Thus, M_{11} has at least $m + 1$ pebbles on it and hence we can move one pebble to R_{11} easily. So the distribution $\chi_{\{R_{11},R_{12},\cdots,R_{1m}\}}$ of $\mathscr{K}$ is reachable.

Thus $\sigma(M_2) \leq m^2 + 7m - 6$. □

Theorem 2.5. *For the m-ary tree M_3, $\sigma(M_3) = m^3 + 31m^2 - 24m + 2$.*

Proof. First, we place $m^3 - m^2$ pebbles on the bottom vertices such that no m pebbles of which share a parent and we did not put any pebbles on the vertex v. This leaves the rightmost bottom vertex v unpebbled; and then we place $32m^2-24m+1$ pebbles on the vertex v. Then no distribution of $\mathscr{K}$ is reachable. Thus $\sigma(M_3) \geq m^3+31m^2-24m+2$.

Now, consider the distribution of $m^3+31m^2-24m+2$ pebbles on the vertices of M_3. According to the distributions of $p(M_3)$ pebbles, we can partite them into three cases.

Case 1. Let $p(M_{2i}) \geq m^2+7m-6$ for all $1 \leq i \leq m$.

If $p(R_3) \geq 1$ then there exists a distribution $\chi_{\{K \cup R_3\}}$ of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_2) = m^2+7m-7$, where $K \subseteq \bigcup_{i=1}^{m} V(M_{2i})$. Let $p(R_3)=0$. Any one of the subtree, say M_{21}, must contain at least $\left\lceil \frac{m^3+31m^2-24m+2}{m} \right\rceil \geq m^2+7m+2$ pebbles and hence we can move a pebble to R_3 by Lemma 2.3 (2). The remaining number of pebbles on M_{21} is at least m^2+7m-6 and thus there exists a distribution $\chi_{\{K \cup R_3\}}$ of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_2) = m^2+7m-6$, where $K \subseteq \bigcup_{i=1}^{m} V(M_{2i})$.

Case 2. Let $p(M_{2i}) \leq m^2+7m-7$ for all $1 \leq i \leq m$.

Clearly $p(R_3) \geq p(M_3) - m(m^2+7m-7) = 24m^2-17m+2 \geq 4m^2+1$ and hence we can put $2m$ pebbles each on the vertices $R_{21}, R_{22}, \cdots, R_{2m}$ from the pebbles at R_3. Thus, the distribution $\chi_{\{R_3 \cup K\}}$ of $\mathscr{K}$ is reachable, where $K = \{v : d(v, R_3) = 2\} \subseteq \bigcup_{i=1}^{m} V(M_{2i})$.

Case 3. Let $p(M_{2i}) \leq m^2+7m-7$ for some i.

Let h subtrees contain at most m^2+7m-7 pebbles each on them, where $1 \leq h \leq m-1$. We prove this case by induction on $h \geq 1$. Let $h=1$, that is, only one subtree, say M_{2m}, has at most m^2+7m-7 pebbles on it. So, our aim is to provide $4m+1$ pebbles to the root R_3 from the subtrees M_{2i} $(i \neq m)$, those have totally at least $m^3+30m^2-31m-5$ pebbles, so that we can move $2m$ pebbles to R_{2m} from R_3. Also, note that, any preexisting pebbles on R_3 or any pebbles on M_{2m} other than the $m(m-1)$ pebbles on the bottom vertices of M_{2m}, $m-1$ pebbles each with a different parent, only make our strategy easier to implement, so assume that $p(M_{2m}) \leq m(m-1)$. Clearly, any one of the subtree, say M_{21}, must contain at least $\left\lceil \frac{m^3+30m^2-23m+2}{m-1} \right\rceil \geq m^2+30m-23$. Let $x \geq 0$ pebble(s) is/are sent by the other subtrees M_{2j} $j \neq 1, m$ to the root R_3 at a cost of at most $8x$ pebbles by Lemma 2.3 (2). So, the remaining number of pebbles on M_{21} is at least $m^3+30m^2-23m+2-(m-2)(m^2+7m+1)-8x \geq 25m^2-10m+4-8x$, and hence we can move $4m-x+1$ pebbles to the root R_3 from the subtree M_{21}, by Lemma 2.3 (1) & (2). Assume the result is true for $h \leq m-2$. Let $h = m-1$. WLOG, let M_{21} be the subtree that has at least m^2+7m-6 pebbles. As we said earlier in this case, we also assume that the other subtrees only contain at most $m(m-1)$ pebbles each on them. So, the subtree M_{21} contains at least $p(M_3)-(m-1)m(m-1) \geq 33m^2-25m+2$ pebbles and hence we can move $4m^2-4m+1$ pebbles to the root R_3 from the subtree M_{21}, by applying induction and by Lemma 2.3 (1) & (2). Thus, the distribution $\chi_{\{R_3 \cup K \cup L\}}$ of $\mathscr{K}$ is reachable, where $K = \{v : d(v, R_3) = 2\} \subseteq \bigcup_{i=2}^{m} V(M_{2i})$ and $L \subseteq V(M_{21})$.

Thus $\sigma(M_3) \leq m^3+31m^2-24m+2$. □

Theorem 2.6. *For the m-ary tree M_4, $\sigma(M_4) = m^4+127m^3-96m^2+8m-30$.*

Proof. First, we place m^4-m^3 pebbles on the bottom vertices such that no m pebbles of which share a parent. This should leave the rightmost bottom vertex unpebbled; and then we place $128m^3-96m^2+8m-31$ pebbles on the vertex v. Then no distribution of $\mathscr{K}$ is reachable. Thus $\sigma(M_4) \geq m^4+127m^3-96m^2+8m-30$.

Now, consider the distribution of $m^4+127m^3-96m^2+8m-30$ pebbles on the vertices of M_4.

According to the distributions of $p(M_4)$ pebbles, we can partite them into three cases.

Case 1. Let $p(M_{3i}) \geq m^3 + 31m^2 - 24m + 2$ for all $1 \leq i \leq m$.

If $p(R_4) \geq 1$ then there exists a distribution $\chi_{\{K \cup R_4\}}$ of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_3) = m^3 + 31m^2 - 24m + 2$, where $K \subseteq \bigcup_{i=1}^{m} V(M_{3i})$. Let $p(R_4) = 0$. Any one of the subtree, say M_{31}, must contain at least

$$\left\lceil \frac{m^4 + 127m^3 - 96m^2 + 8m - 30}{m} \right\rceil \geq m^3 + 31m^2 - 24m + 18,$$

pebbles and hence we can move a pebble to R_4 by Lemma 2.3 (3). The remaining number of pebbles on M_{31} is at least $m^3 + 31m^2 - 24m + 2$ and thus there exists a distribution $\chi_{\{K \cup R_4\}}$ of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_3) = m^3 + 31m^2 - 24m + 2$, where $K \subseteq \bigcup_{i=1}^{m} V(M_{3i})$.

Case 2. Let $p(M_{3i}) \leq m^3 + 31m^2 - 24m + 1$ for all $1 \leq i \leq m$.

Clearly $p(R_4) \geq p(M_4) - m(m^3 + 31m^2 - 24m + 1) \geq 8m^3 + 2m$ and hence we put $4m^2 + 1$ pebbles each on the vertices R_{31}, R_{32}, $\cdots$, R_{3m} from the pebbles at R_4. Thus, the distribution $\chi_{\{R_{31}, R_{32}, \cdots, R_{3m} \cup K\}}$ of $\mathscr{K}$ is reachable, where $K = \{v : d(v, R_4) = 3\} \subseteq \bigcup_{i=1}^{m} V(M_{3i})$.

Case 3. Let $p(M_{3i}) \leq m^3 + 31m^2 - 24m + 1$ for some i.

Let h subtrees contain at most $m^3 + 31m^2 - 24m + 1$ pebbles each on them, where $1 \leq h \leq m - 1$. We prove this case by induction on $h \geq 1$. Let $h = 1$, that is, only one subtree, say M_{3m}, has at most $m^3 + 31m^2 - 24m + 1$ pebbles on it. We have to send $8m^2 + 1$ pebbles to the root R_4 from the subtrees those have totally at least $m^4 + 126m^3 - 127m^2 + 32m - 31$ pebbles, so that we can move $4m^2$ pebbles to R_{3m} from R_4. Also, note that, any preexixting pebbles on R_4 or any pebbles on M_{3m} other than the $m^2(m-1)$ pebbles on the bottom vertices of M_{3m}, $m(m-1)$ pebbles each with a different parent, only make our strategy easier to implement, so assume that $p(M_{3m}) \leq m^2(m - 1)$. Clearly, any one of the subtree, say M_{31}, must contain at least $\left\lceil \frac{m^4+126m^3-95m^2+8m-30}{m-1} \right\rceil \geq m^3 + 126m^2 - 95m - 14$. Let $x \geq 0$ pebble(s) is/are sent by the other subtrees M_{3j} $j \neq 1, m$, to the root R_4 at a cost of at most $16x$ pebbles by Lemma 2.3 (3). So, the remaining number of pebbles on M_{31} is at least $m^4 + 126m^3 - 95m^2 + 8m - 30 - (m-2)(m^3 + 31m^2 - 24m + 13) - 16x \geq 98m^3 - 10m^2 - 53m - 4 - 16x$, and hence we can move $8m^2 + 1 - x$ pebbles to the root R_4 from the subtree M_{31}, by Lemma 2.3 (3). Assume the result is true for $h = l \leq m - 2$. Let $h = m - 1$. WLOG, let M_{31} be the subtree that has at least $m^3 + 31m^2 - 24m + 2$ pebbles. As we said earlier in this case, we also assume that the other subtrees only contain at most $m^2(m - 1)$ pebbles each on them. So, the subtree M_{31} contains at least $p(M_4) - (m - 1)m^2(m - 1) \geq 129m^3 - 97m^2 + 8m - 30$ pebbles and hence we can move $8m^3 - 8m^2$ pebbles to the root R_4 and one pebble to R_{31} from the subtree M_{31}, by applying induction and by Lemma 2.3 (3). Thus, the distribution $\chi_{K \cup L \cup M}$ of $\mathscr{K}$ is reachable, where $K = \{v : d(v, R_4) = 3\} \subseteq \bigcup_{i=2}^{m} V(M_{3i})$, $L = \{v : d(v, R_4) = 1\}$ and $M \subseteq V(M_{31}) - \{R_{31}\}$.

Thus $\sigma(M_4) \leq m^4 + 127m^3 - 96m^2 + 8m - 30$. □

Theorem 2.7. *For $n \geq 2$,*

$$\begin{aligned}\sigma(M_n) =& m^{n-1}(m-1) + (m-1)\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k} 2^{2n-2k+1} + 2^{2\lfloor \frac{n-1}{2} \rfloor + 1} \\ &+ \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \left(2^{2i+1} + (m-1) \sum_{j=1}^{n-2i-2} m^{j-1} 2^{2i+2j+1} \right).\end{aligned}$$

Before proving the above theorem, we state a theorem below to prove Theorem 2.7, which is a generalization of the Lemma 2.3.

Theorem 2.8. *We can send a pebble to R_n, the root of M_n, at a cost of at most 2^n pebbles, when $n = k$ $(k \geq 2)$ and there exists a $M_{(n-1)i}$ $(1 \leq i \leq m)$, a subtree of M_n, such that $p(M_{(n-1)i}) \geq \sigma(M_{n-1}) + 3(2^{n-2})$.*

Proof. It can be easily proved by induction on k and using the Lemma 2.3. □

Now, we are going to prove Theorem 2.7.

Proof of Theorem 2.7. Let

$$T(M_n) = m^{n-1}(m-1) + (m-1)\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{2n-2k+1} + 2^{2\lfloor \frac{n-1}{2} \rfloor + 1} + \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} \left(2^{2i+1} + (m-1)\sum_{j=1}^{n-2i-2} m^{j-1}2^{2i+2j+1} \right).$$

We place $m^{n-1}(m-1)$ pebbles on the bottom verices such that no m pebbles of which share a parent. This should leave the rightmost bottom vertex unpebbled; and then we place $T(M_n) - m^{n-1}(m-1) - 1$ pebbles on the vertex v. Then no distribution of $\mathscr{K}$ is reachable. Thus $\sigma(M_n) \geq T(M_n)$.

We now proceed to prove the upper bound by induction. Clearly, the result is true for $n = 2, 3, 4$ by Thorem 2.4, 2.5 and 2.6. Consider the distribution of $T(M_n)$ pebbles on the vertices of M_n $(n \geq 5)$. According to the distributions of $T(M_n)$ pebbles, we can partite them into three cases.

Case 1. Let $p(M_{(n-1)i}) \geq T(M_{n-1})$ for all $1 \leq i \leq m$.

If $p(R_n) \geq 1$ then there exists a distribution of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_{n-1}) = T(M_{n-1})$. Let $p(R_n) = 0$. Any one of the subtree, say $M_{(n-1)1}$, must contain at least $\left\lceil \frac{T(M_n)}{m} \right\rceil \geq T(M_{n-1}) + 2^n$ pebbles and hence we can move a pebble to R_n by the Theorem 2.8. The remaining number of pebbles on $M_{(n-1)1}$ is at least $T(M_{n-1})$ and thus there exists a distribution of $\mathscr{K}$ which is reachable by our assumption and $\sigma(M_{n-1}) = T(M_{n-1})$.

Case 2. Let $p(M_{(n-1)i}) \leq T(M_{n-1}) - 1$ for all $1 \leq i \leq m$.

Clearly $p(R_n) \geq T(M_n) - m\left(T(M_{n-1}) - 1\right) \geq m\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right) + 1$ and hence we put $\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k}$ pebbles each on the vertices $R_{(n-1)1}, R_{(n-1)2}, \cdots, R_{(n-1)m}$ from the pebbles at R_n. From the pebbles on $R_{(n-1)i}$, we move a pebble to every vertex in every second row, starting, in the subtree $M_{(n-1)i}$, with the row that is next to the bottom row. Thus, the distribution $\chi_{\{K \cup R_n\}}$ of $\mathscr{K}$ is reachable, where $K \subseteq \bigcup_{i=1}^{m} V(M_{(n-1)i})$.

Case 3. Let $p(M_{(n-1)i}) \leq T(M_{n-1}) - 1$ for some i.

Let h subtrees contain at most $T(M_{n-1}) - 1$ pebbles each on them, where $1 \leq h \leq m-1$. We prove this case by induction on $h \geq 1$. Let $h = 1$, that is, only one subtree, say $M_{(n-1)m}$, has at most $T(M_{n-1}) - 1$ pebbles on it. So, our aim is to provide $\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right)$ pebbles to the root R_n from the subtrees those have totally at least $T(M_n) - m^{n-2}(m-1)$ pebbles, so that we can move $\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k}\right)$ pebbles to $R_{(n-1)m}$. Clearly, any one of the subtree, say $M_{(n-1)1}$, must contain at least $\left\lceil \frac{T(M_n)-m^{n-2}(m-1)}{m-1} \right\rceil \geq T(M_{n-1})+2^n\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}+1\right)$ and hence we can move $\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right)+1$ pebbles to R_n while retaining $T(M_{n-1})$ pebbles, by the Theorem 2.8. Thus, the distribution $\chi_{\{K \cup L \cup R_n\}}$ of $\mathscr{K}$ is reachable, where $\chi_{\{K\}}$ is a distribution of $\mathscr{K}$ which is reachable from those subtrees having at least $T(M_{n-1})$ pebbles each on them, $\chi_{\{L\}}$ is a reachable distribution of $\mathscr{K}$ in $M_{(n-1)m}$ and $\chi_{\{R_n\}}$ is a reachable distribution of $\mathscr{K}$ for the incident edges of R_n. So assume the result is true for $h=l \leq m-2$. Let $h=m-1$. WLOG, let $M_{(n-1)1}$ be the subtree that contains at least $T(M_{n-1})$ pebbles. Clearly, $p(M_{(n-1)1}) \geq T(M_n)-(m-1)m^{n-2}(m-1) \geq T(M_{n-1})+2^n(m-1)\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right)+(m-1)m^{n-2}$. We have to retain $T(M_{n-1})$ pebbles on $M_{(n-1)1}$ and thus $M_{(n-1)1}$ has $2^n(m-1)\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right)$ extra pebbles on it. Now, we need at most 2^{n+1} pebbles from $M_{(n-1)1}$ to put one pebble on a root vertex, say $R_{(n-1)2}$ of the subtree $M_{(n-1)2}$, by induction and by the Theorem 2.8. After using at most $2^n\left(\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} m^{n-2k}2^{n-2k+1}\right)$ pebbles from $M_{(n-1)1}$, the remaining number of pebbles is at least $T(M_{n-1})+2^n(m-2)\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k+1}\right)+(m-1)m^{n-2}$ and therefore we can move $\left(\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m^{n-2k}2^{n-2k}\right)$ pebbles to every root vertex $R_{(n-1)j}$ $(j \neq 1, 2)$ of $M_{(n-1)j}$ by induction and by the Theorem 2.8. In this process, for even values of n, as we place one pebble each on the root vertex $R_{(n-1)j}$ (for all $j \neq 1$), the edges between R_n and $R_{(n-1)j}$ is also covered. The edge between R_n and $R_{(n-1)1}$ can also be covered as the root vertex $R_{(n-1)1}$ can be pebbled with at least $T(M_{n-1})=\sigma(M_{n-1})$ on the vertices of $M_{(n-1)1}$. For the case when n is odd, as there are 2^n extra pebbles (after covering all the edges of $M_{(n-1)1}$), we can pebble the root vertex R_n of M_n and thus the edge $R_nR_{(n-1)1}$ is also covered. So there exists a distribution of $\mathscr{K}$ is reachable.

Thus, $\sigma(M_n) \leq T(M_n)$. □

Corollary 2.9. [11] *$\sigma(B_0)=1$, $\sigma(B_1)=2$, $\sigma(B_2)=12$, $\sigma(B_3)=86$, $\sigma(B_4)=634$ and for $n \geq 2$,*

$$\sigma(M_n)=2^{n-1}+\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k}2^{2n-2k+1}+2^{2\lfloor \frac{n-1}{2} \rfloor+1}+\sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor}\left(2^{2i+1}+\sum_{j=1}^{n-2i-2} 2^{j-1}2^{2i+2j+1}\right).$$

Proof. Let $m=2$ in Theorem 2.7. □

3. The covering cover pebbling number for a caterpillar

In this section, we are going to determine the covering cover pebbling number of a caterpiller, using Definition 1.8.

Definition 3.1. A tree T is called a caterpillar if the deletion of all pendent vertices of the tree results in a path P', the spine of the caterpillar T. For convenience, we shall call a path P with maximum length which contains P' a body of the caterpillar, and all the edges which are incident to pendent vertices are the legs of the caterpillar T. Furthermore, the vertex $v \in V(P)$ is a joint of T provided that $deg_T(v) \geq 3$ or v is adjacent to the end vertices.

In otherwords, a tree is said to be caterpillar iff all nodes of degree three or more are surrounded by at most two nodes of degree two or greater.

Let $C(n, s_1, s_2, s_3, ..., s_n)$ be the caterpillar tree T such that the spine $P' : v_1v_2v_3 \cdots v_n$ has n vertices and let the vertex $v_i \in V(P')$ has $s_i \geq 0$ pendant vertices where $1 \leq i \leq n$. Clearly $s_1 \geq 1$ and $s_n \geq 1$, and hence we label a vertex as v_0 which is adjacent to v_1 and label another vertex as v_{n+1} which is adjacent to v_n. Let $I = \{v_i : s_i \geq 2\}$ where $1 \leq i \leq n$ and let $J = \{v_j : s_j = 1\}$ where $1 \leq j \leq n$. Let $I \cup J = \{v_1, v_k, v_l, \cdots, v_m, v_n : 1 < k < l < \cdots < m < n\}$.

Theorem 3.2. [11] *For the path P_n $(n \geq 2)$, $\sigma(P_n) = \lceil \frac{2^n-1}{3} \rceil$.*

Theorem 3.3. *For a caterpillar $C(n, s_1, s_2, ..., s_n)$, the covering cover pebbling number is equal to,*

$$\sum_{v_a \in I \cup J} 2^a - \begin{cases} 1 & \text{if } v_2 \in I \cup J \text{ or } |P_A^{1k}| = 0(\bmod 2) \\ 0 & \text{otherwise} \end{cases} + \sum_{v_a \in I \cup J} (s_a - 1) + \sum_{v_b, v_c \in I \cup J} 2^{b+1} T(P_A^{bc}),$$

where v_b and v_c $(b < c)$ are a pair of consecutive vertices in $I \cup J$, and $P_A^{bc} : v_{b+1}v_{b+2}...v_{c-2}v_{c-1}$ with $T(P_A^{bc}) = \sigma(P_d)$ if $|P_A^{bc}| = d \geq 2$ and 0 otherwise.

Proof. Let

$$p(C) = \sum_{v_a \in I \cup J} 2^a - \begin{cases} 1 & \text{if } v_2 \in I \cup J \text{ or } |P_A^{1k}| = 0(\bmod 2) \\ 0 & \text{otherwise} \end{cases} + \sum_{v_a \in I \cup J} (s_a - 1) + \sum_{v_b, v_c \in I \cup J} 2^{b+1} T(P_A^{bc}).$$

First we place one pebble each on the $s_i - 1$ pendant vertices of $v_i \in I$ for all i, and place zero pebbles on the pendant vertex of $v_j \in J$ for all j such that we do not place pebbles on v_0. After that we place $p(C) - \sum_{v_a \in I \cup J} (s_a - 1)$ pebbles on the vertex v_0 and zero pebbles on the other vertices of $C(n, s_1, s_2, ..., s_n)$. Thus no distribution of $\mathscr{K}$ is reachable and so $\sigma(C(n, s_1, s_2, ..., s_n)) \geq p(C)$.

Now consider the distribution of $p(C)$ pebbles on the vertices of $C(n, s_1, s_2, ..., s_n)$. Here our strategy is to move one pebble each to the vertices belong to $I \cup J$ and we have to cover the edges of in between vertices, for example v_k and v_l, of $I \cup J$ using the path P_A^{kl}, if needed. Note that if we have one pebble each at the $s_i - 1$ pendant vertices of $v_i \in I$ does not decrease the number of pebbles needed for v_i from the other vertex. If we add one more pebble to a pendant vertex of v_i, clearly all edges except $v_{i-1}v_i$ and v_iv_{i+1} incident with the vertex v_i is covered. We now prove that a worst case scenario is indeed the one in which all the $p(C) - \sum_{v_a \in I \cup J} (s_a - 1)$ pebbles are in either v_0 or v_{n+1}, first we let v_0 be the vertex: Let $\mathscr{C}$ be a worst case configuration that contains pebbles on $P' \cup \{v_{n+1}\} \cup$ {unpebbled pendant vertices} other than those on the above mentioned pendant vertices. Then moving all of these to the vertex v_0 would require us to use more pebbles to cover the edges of $C(n, s_1, s_2, ..., s_n)$, a contradiction to the fact that $\mathscr{C}$ is a worst case configuration. Also, as noted before, removing single pebble from the pebbled pendant vertices does not lessen the covering cover pebbling number starting at v_0.

Now, we put one pebble each to the vertices of $I \cup J$ from v_0 to cover the incident edge of the unpebbled pendant vertices which are adjacent to some vertices of $I \cup J$. Clearly, to achieve that we need $\sum_{v_a \in I \cup J} 2^a$ pebbles from v_0. Let v_k be the next vertex in $I \cup J$ after the vertex v_1. We have to cover the edges between the vertices v_1 and v_k in $C(n, s_1, s_2, ..., s_n)$. Clearly, we have covered the incident edges of v_1 and v_k. Consider the path $P_A^{1k} : v_2v_4 \cdots v_{k-2}v_{k-1}$. To cover the edges of this P_A^{1k} path from the vertex v_0, we need $2^3\sigma(P_d)$ pebbles if $|P_A^{1k}| = d \geq 2$ and we don't need any pebbles if $|P_A^{1k}| \leq 1$. We do the same procedure for other pairs of consecutive verices (v_k, v_l), $\cdots$, (v_m, v_n) of $I \cup J$. So, to cover the edges between the vertices of the pair of vertices of $I \cup J$, we need $\sum_{v_b, v_c \in I \cup J} 2^{b+1}T(P_A^{bc})$ pebbles from v_0. Clearly, we are done using $p(C)$ pebbles if $v_2 \notin I \cup J$ and $|P_A^{1k}| \neq 0(\text{ mod } 2)$. Suppose $v_2 \in I \cup J$ or $|P_A^{1k}| = 0(\text{ mod } 2)$, then we remove the single pebble at v_1 and put one pebble at v_0. So we subtract one pebble for the case $v_2 \in I \cup J$ or $|P_A^{1k}| = 0(\text{ mod } 2)$. Thus we have covered all the edges of $C(n, s_1, s_2, ..., s_n)$ using $p(C)$ pebbles.

We relabel the vertices $v_{n+1}, v_n, \cdots, v_1, v_0$ by $v_0, v_1, \cdots, v_n, v_{n+1}$ respectively and then we do the same thing as we did above. Finally choose the one (before relabeling or after relabeling) having maximum amount of pebbles with it will be the covering cover pebbling number for $C(n, s_1, s_2, ..., s_n)$. □

For our convenience, we take $p(C) = T_{11} + T_{12} + T_{13}$, where

$$T_{11} = \sum_{v_a \in I \cup J} 2^a - \begin{cases} 1 & \text{if } v_2 \in I \cup J \text{ or } |P_A^{1k}| = 0(\text{ mod } 2), \\ 0 & \text{otherwise,} \end{cases}$$
$$T_{12} = \sum_{v_b, v_c \in I \cup J} 2^{b+1}T(P_A^{bc}),$$

and

$$T_{13} = \sum_{v_a \in I \cup J} (s_a - 1).$$

Theorem 3.4. *For the path P_n $(n \geq 4)$,*

$$\sigma(P_n) = 2^{n-2} + 4\sigma(P_{n-4}) + 2 - \begin{cases} 1 & \textit{if } |P_A^{1(n-2)}| = 0(\textit{ mod } 2), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $P_n : v_0v_1 \cdots v_{n-1}$. Clearly, $I = \emptyset$ and $J = \{v_1, v_{n-2}\}$, so $s_1 = 1$, $s_{n-2} = 1$ and $I \cup J = \{v_1, v_{n-2}\}$. Then $P_A^{1(n-2)} : v_2 \cdots v_{n-3}$. Now we are going to apply Thorem 3.3.

Case 1. $|P_A^{1(n-2)}| = 0(\text{mod } 2)$.

$$T_{11} = (2 + 2^{n-2}) - 1 = 2 + 2^{n-2} - 1,$$
$$T_{12} = 2^2 T\left(P_A^{1(n-2)}\right) = 4\sigma(P_{n-4}),$$

and

$$T_{13} = (s_1 - 1) + (s_{n-2} - 1) = 0.$$

Thus $\sigma(P_n) = 2^{n-2} + 4\sigma(P_{n-4}) + 1$.

Case 2. $|P_A^{1(n-2)}| \neq 0(\text{mod } 2)$.

$T_{11} = 2 + 2^{n-2}$, $T_{12} = 4\sigma(P_{n-4})$ and $T_{13} = 0$. Thus $\sigma(P_n) = 2^{n-2} + 4\sigma(P_{n-4}) + 2$. □

Definition 3.5. A graph is called Double Star-Path(DSP) if the end vertices of a path $P : v_1 v_2 \cdots v_n$ on n vertices adjoint to the center vertices of the star graphs $K_{1,l}$ $(l \geq 2)$ and $K_{1,m}$ $(m \geq 2)$ respectively. We denote it by $P_n(l, m)$.

Corollary 3.6. *For the graph Double Star-Path $P_n(l, m)$,*

$$\sigma(P_n(l,m)) = 2^n + 4\sigma(P_{n-2}) + l + m - \begin{cases} 1 & \text{if } |P_A^{1n}| = 0(\text{ mod } 2), \\ 0 & \text{otherwise,} \end{cases}$$

where $P_A^{1n} : v_2 v_3 \cdots v_{n-1}$ $(n \geq 4)$. Note that, we let $\sigma(P_{n-2}) = 0$ if $n \leq 3$.

Proof. Let $P_{n+2} : v_0 v_1 \cdots v_{n+1}$. Clearly, $I = \{v_1, v_n\}$ and $J = \emptyset$, so $s_1 = l$, $s_n = m$ and $I \cup J = \{v_1, v_n\}$. Then $P_A^{1n} : v_2 \cdots v_{n-1}$. Now we are going to apply Thorem 3.3.

Case 1. $|P_A^{1n}| = 0(\text{mod } 2)$.

$$T_{11} = 2 + 2^n - 1,$$
$$T_{12} = 2^2 T\left(P_A^{1n}\right) = 4\sigma(P_{n-2}),$$

and

$$T_{13} = (s_1 - 1) + (s_n - 1) = l + m - 2.$$

Thus $\sigma(P_n(l,m)) = 2^n + 4\sigma(P_{n-2}) + l + m - 1$.

Case 2. $|P_A^{1(n-2)}| \neq 0(\text{mod } 2)$.

$$T_{11} = 2 + 2^n, \quad T_{12} = 4\sigma(P_{n-2}) \quad \text{and} \quad T_{13} = l + m - 2.$$

Thus $\sigma(P_n(l,m)) = 2^n + 4\sigma(P_{n-2}) + l + m$. □

Definition 3.7. The class of fuses is defined as follows: the vertices of a Fuse $F_l(k)$ $(l \geq 1$ and $k \geq 2)$are $v_0 v_1, v_2, \cdots, v_{n-1}$ with $n = l + k + 1$, so that the first $l + 1$ vertices form a path from $v_0 v_1, v_2, \cdots, v_l$, and the remaining vertices $v_{l+1}, v_{l+2}, \cdots, v_{n-1}$ are independent and adjacent only to v_l. The path sometimes called the wick, while the remaining vertices are sometimes called the sparks. For example, $F_1(k)$ is the star $K_{1,k+1}$ on $k + 2$ vertices.

Corollary 3.8. *For the Fuse graph $F_l(k)$ $(l \geq 2)$, the covering cover pebbling number is equal to*

$$2^l + 4\sigma(P_{l-2}) + k + 1 - \begin{cases} 1 & \text{if } |P_A^{1l}| = 0(\text{ mod } 2), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P_{l+2} : v_0v_1 \cdots v_lv_{l+1}$. Clearly, $I = \{v_l\}$ and $J = \{v_1\}$, so $s_1 = 1$, $s_l = k$ and $I \cup J = \{v_1, v_l\}$. Then $P_A^{1l} : v_2 \cdots v_{l-1}$. Now we are going to apply Theorem 3.3.

Case 1. $|P_A^{1l}| = 0 (\text{mod } 2)$.

$$T_{11} = 2 + 2^l - 1,$$
$$T_{12} = 4\sigma(P_{l-2}),$$

and

$$T_{13} = k - 1.$$

Thus $\sigma(F_l(k)) = 2^l + 4\sigma(P_{l-2}) + k$.

Case 2. $|P_A^{1l}| \neq 0 (\text{mod } 2)$.

$$T_{11} = 2 + 2^l, \quad T_{12} = 4\sigma(P_{l-2}) \quad \text{and} \quad T_{13} = k - 1.$$

Thus $\sigma(F_l(k)) = 2^l + 4\sigma(P_{l-2}) + k + 1$. □

References

[1] S. Arumugam and S. Ramachandran. Covering set in graphs. In *Invitation to Graph Theory*. Scitech publications (India) pvt. ltd., 2008.

[2] F. R. K. Chung. Pebbling in hypercubes. *SIAM Journal on Discrete Mathematics*, 2:467–472, 1989. https://doi.org/10.1137/0402041.

[3] D. Herscovici, B. Hester, and G. Hurlbert. Generalizations of graham's pebbling conjeture. *Discrete Mathematics*, 312:2286–2293, 2012. https://doi.org/10.1016/j.disc.2012.03.032.

[4] A. Lourdusamy, S. S. Jayaseelan, and T. Mathivanan. Covering cover pebbling number for even cyclic lollipop. *General Mathmatics Notes*, 5(2):24–41, 2011.

[5] A. Lourdusamy, S. S. Jayaseelan, and T. Mathivanan. Covering cover pebbling number for odd cyclic lollipop. *International Journal of Pure and Applied Sciences and Technology*, 5(2):1–40, 2011.

[6] A. Lourdusamy, S. S. Jayaseelan, and T. Mathivanan. Covering cover pebbling number for sun. *International Journal of Pure and Applied Sciences and Technology*, 5(2):109–118, 2011.

[7] A. Lourdusamy and T. Mathivanan. Covering cover pebbling number for the square of a cycle. *Journal of Prime Research in Mathematics*, 8:102–105, 2012.

[8] A. Lourdusamy and T. Mathivanan. Covering cover pebbling number for the square of a path. *Gulf Journal of Mathematics*, 2(2):83–87, 2014. http://dx.doi.org/10.56947/gjom.v2i2.199.

[9] A. Lourdusamy and T. Mathivanan. The domination cover pebbling number for some cyclic graphs and path graphs. *Ars Combinatoria*, 138:51–61, 2018.

[10] A. Lourdusamy and A. P. Tharani. Covering cover pebbling number of hypercube and diameter d graphs. *Journal of the Korea soceity of Mathematical Education Series B*, 15(2):121–134, 2008.

[11] A. Lourdusamy and A. P. Tharani. Covering cover pebbling number. *Utilitas Mathematica*, 78:41–54, 2009.