

Directed graphs of integers with arcs determined by an arithmetic function

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ABSTRACT

We introduce a new arc in directed graphs of integers. Our arcs are determined by the values of the popular arithmetic functions such as the divisor function τ , the prime divisors counting functions ω and Ω , and the sum of digits function s_b , evaluated at the multiples N of a particular integer. Among other things, we determine the positive integers that have arcs to all except a finite number of positive integers. We also propose some possible research problems at the end of this article.

Keywords: arithmetic functions, graph labelings, directed graphs

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1. Introduction

A graph labeling is one of the most popular problems in the mixed area of number theory and graph theory concerning an assignment of integers to vertices and/or edges of a graph under some certain arithmetic or combinatorial conditions. For an up-to-date review of graph labeling, we refer the reader to Gallian [10]. Coprime graphs of integers, which is the graph whose vertex set is the set of integers and two vertices a and b are connected by an edge if and only if $\gcd(a, b) = 1$, provide an interesting connection between graph theory and number theory, and many of their properties have been continuously studied by several mathematicians; see for example in the work of Erdős [8], Erdős and Sárközy [9], Sárközy [22], Berliner et al. [4], Lee [16], and Berkove and Brilleslyper [3]. An interesting example of an open problem in graph labeling is the existence or non-existence of arbitrarily long directed paths where two vertices a and b are connected by a directed edge (or an arc)

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if a and b are primes, and b can be obtained by appending the decimal digits to the left or to the right of a . For more details about this problem, see for example, in [2, 11, 17]. In addition, a problem in TMO 2022 [25] is on a directed graph whose vertices $n, u \in \mathbb{Z}$ are connected if there exists a multiple N of n with $s_{10}(N) = u$, where $s_{10}(N)$ is the sum of the decimal digits of N . The study on graph labeling, coprime graphs, and a problem in TMO 2022 motivates us to introduce a new directed graph whose vertices are positive integers and the directed edges (or arcs) are determined by other arithmetic functions. For example, instead of considering only s_{10} , we will replace s_{10} by s_b where b is any integer larger than 1.

Let g be an arithmetic function and n, u positive integers. The set of positive integers is defined as $\mathbb{N} = 1, 2, 3, \dots$. We say that there exists a g -directed edge (or g -arc) from n to u if there exists a positive integer N such that N is divisible by n and $g(N) = u$. In this case, we also say that there exists a g -arc from n to u and write $n \xrightarrow{g} u$. If there is no confusion and the function g is understood, we sometimes drop the letter g and write only $n \rightarrow u$ and say that there exists an arc from n to u . We can think of this as a directed graph $G = (V, E_g)$ where $V = \mathbb{N}$ and there is an arc from n to u if $n \xrightarrow{g} u$.

We are motivated by some problems in graph labeling including those in [2, 11, 17, 25]. Our purpose is to obtain some basic properties of this new arc and find the integers n that have g -arcs to as many integers as possible. So we let $\text{Out}(g, n)$ be the set of all $u \in \mathbb{N}$ to which there exists a g -arc from n , that is,

$$\text{Out}(g, n) = \{u \in \mathbb{N} \mid n \xrightarrow{g} u\},$$

and we will determine the integers n such that $\text{Out}(g, n)$ is a cofinite subset of \mathbb{N} , i.e., $\mathbb{N} \setminus \text{Out}(g, n)$ is finite. The set $\text{In}(g, n)$ of $u \in \mathbb{N}$ such that $u \xrightarrow{g} n$ may be interesting too, but we postpone the investigation of these problems as a future project; see Questions 5 and 7 at the end of this article. Please see also Questions 6, 8, 9, 10 for some variations, motivations, and different problems.

To avoid some complications or trivialities, we restrict our attention to the g -arc where g is an arithmetic function that is surjective or has the range as a cofinite subset of $\mathbb{N} \cup \{0\}$. In particular, we study this arc when $g = s_b, \tau, \omega$, and Ω , where $s_b(n)$ is the sum of digits of n when n is written in base $b \geq 2$, $\tau(n)$ is the number of positive divisors of n , $\omega(n)$ is the number distinct prime divisors of n , and $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. Therefore the existence of our g -arc from n to u depends not only on the integers n and u , but also on the property of the arithmetic function g . Our study shows that, for each $n \in \mathbb{N}$, if $g = \omega$ and Ω , then $\text{Out}(g, n) = \{u \in \mathbb{N} \mid u \geq g(n)\}$. This is also true for $g = \tau$ if $n = p^k$ for some prime p and non-negative integer k . However, in the case $g = s_b$, this property generally does not hold and the solutions are related to Frobenius number problem. We refer the reader to [1, 14, 27, 26] for more details about Frobenius problems, and to [7, 12, 13, 15, 18, 19, 20, 21, 23, 24] for more details about s_b , happy functions, and other digital problems. We will also give more information about these references at the end of this article.

We will propose some possible research problems on g -arc between integers at the end of this article too. Nevertheless, we do not claim that they are difficult or interesting.

They may be unimportant and may even be trivial. However, we would merely like to record them for ourselves and share them among interested readers. For more advanced graph-theoretic problems, see for example, in the books by Bollobás [5] and Bondy and Murty [6].

2. Main results

Recall that if $b \geq 2$ and $n \geq 1$ are integers and

$$n = (a_k a_{k-1} \cdots a_0)_b = \sum_{i=0}^k a_i b^i,$$

is the expansion of n in base b where $a_k \neq 0$ and $0 \leq a_i < b$ for all i , then we define the sum of digits of n in base b by $s_b(n) = a_1 + a_2 + \cdots + a_k$. So, for example, $s_{10}(123) = 6$ and $s_5(123) = s_5((443)_5) = 11$. We have the following result for s_b -arc.

Theorem 2.1. *Let $b \geq 2$, $n \geq 1$, and $d = \gcd(b - 1, n)$. Then the following statements hold.*

- (i) $\text{Out}(s_b, n) \subseteq \{u \in \mathbb{N} \mid u \equiv 0 \pmod{d}\}$.
- (ii) $\text{Out}(s_b, n)$ is cofinite if and only if $d = 1$.

Proof. In this proof, we write $n \rightarrow u$ instead of $n \xrightarrow{s_b} u$ for convenience. We first prove (i). So let $u \in \text{Out}(s_b, n)$. Then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $s_b(N) = u$. Since $b^k \equiv 1 \pmod{b - 1}$ for every $k \in \mathbb{N} \cup \{0\}$, we see that $s_b(N) \equiv N \pmod{b - 1}$. Since $d \mid b - 1$, we obtain $u = s_b(N) \equiv N \pmod{d}$. In addition, we have $d \mid n$ and $n \mid N$, and therefore $d \mid N$ and $u \equiv N \equiv 0 \pmod{d}$. This proves (i).

If $d > 1$, it follows immediately from (i) that $\text{Out}(s_b, n)$ is not cofinite. So to prove (ii), it is enough to consider the case that $d = 1$. If $n = 1$, then for any $u \in \mathbb{N}$, we can choose $N = \sum_{i=0}^{u-1} b^i$ to obtain $n \mid N$ and $s_b(N) = u$, and so $\text{Out}(s_b, n) = \mathbb{N}$. Next, let $n \geq 2$, $b = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, and $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} m$, where p_1, p_2, \dots, p_k are distinct primes, a_1, a_2, \dots, a_k, m are positive integers, n_1, n_2, \dots, n_k are non-negative integers, and $\gcd(m, b) = 1$. If $b = 2$ and $m = 1$, then n is a power of 2, and for any $u \in \mathbb{N}$, we can choose $N = n \sum_{i=0}^{u-1} 2^i$ to obtain $N \equiv 0 \pmod{n}$ and $s_2(N) = u$. This shows that if $b = 2$ and $m = 1$, then $\text{Out}(s_b, n) = \mathbb{N}$. Here we record the case that $\text{Out}(s_b, n) = \mathbb{N}$ for future reference. We have proved that

$$\text{Out}(s_b, n) = \mathbb{N} \text{ if } n = 1 \text{ or if } b = 2 \text{ and } m = 1. \tag{1}$$

So from this point on, we assume that $n \geq 2$ and if $b = 2$ then $m > 1$. Next, let

$$c = \max_{1 \leq j \leq k} n_j, \quad A = b^c \sum_{j=1}^n b^{j\phi(m)}, \quad B = b^c \left(\sum_{j=1}^{n-1} b^{j\phi(m)} + \left\lfloor \frac{b}{2} \right\rfloor b^{n\phi(m)-1} + \left\lceil \frac{b}{2} \right\rceil b^{(n+1)\phi(m)-1} \right),$$

where ϕ is the Euler totient function, $\lfloor x \rfloor$ is the largest integer not exceeding x and $\lceil x \rceil$ is the smallest integer larger than or equal to x . We assert that A and B have the following

properties:

$$A \equiv 0 \pmod{n}, \quad B \equiv 0 \pmod{n}, \quad \text{and} \quad \gcd(s_b(A), s_b(B)) = 1. \quad (2)$$

It is clear that A and B are divisible by $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. In addition, by Euler's theorem and the fact that $\gcd(b, m) = 1$, we obtain $b^{\phi(m)} \equiv 1 \pmod{m}$ and therefore

$$A \equiv nb^c \pmod{m}, \quad bB \equiv b^c \left(b(n-1) + \left\lfloor \frac{b}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil \right) \pmod{m}. \quad (3)$$

It is easy to verify that

$$\left\lfloor \frac{b}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil = b.$$

From this, (3), and the fact that $m \mid n$ we obtain

$$A \equiv 0 \pmod{m} \quad \text{and} \quad bB \equiv nb^{c+1} \equiv 0 \pmod{m}. \quad (4)$$

Since $\gcd(b, m) = 1$, we obtain $B \equiv 0 \pmod{m}$. Therefore both A and B are divisible by $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and also by m , so they are divisible by n . It remains to show that $\gcd(s_b(A), s_b(B)) = 1$. First, it is clear that $s_b(A) = n$. Nevertheless, we have to be more careful in the calculation of $s_b(B)$ since the term $b^{(n-1)\phi(m)}$ and $\lfloor b/2 \rfloor b^{n\phi(m)-1}$ may correspond to the same position in the b -adic expansion of B . Recall that (1) is already proved and we assume that if $b = 2$, then $m > 1$. Since $\gcd(m, b) = 1$, we see that if $b = 2$, then $m \geq 3$. This implies that either $b \geq 3$ or $\phi(m) > 1$. If $b \geq 3$, then

$$1 + \left\lfloor \frac{b}{2} \right\rfloor \leq 1 + \frac{b}{2} < b,$$

and so

$$s_b(B) = n - 1 + \left\lfloor \frac{b}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil = n - 1 + b.$$

If $\phi(m) > 1$, then

$$(n-1)\phi(m) < n\phi(m) - 1,$$

and so

$$s_b(B) = n - 1 + \left\lfloor \frac{b}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil = n - 1 + b.$$

In any case, $s_b(B) = n - 1 + b$, and so $\gcd(s_b(A), s_b(B)) = \gcd(n, n - 1 + b) = d = 1$. This proves (4).

Recall that the Frobenius number of coprime positive integers a and ℓ is

$$(a-1)(\ell-1) - 1,$$

that is, if $a, \ell \in \mathbb{N}$ and $\gcd(a, \ell) = 1$, then for every positive integer $n \geq (a-1)(\ell-1)$, there exist $x, y \in \mathbb{N} \cup \{0\}$, such that $n = ax + \ell y$. Let $a = s_b(A)$, $\ell = s_b(B)$. Then a and ℓ are coprime. We assert that there is an arc from n to each positive integer $u \geq (a-1)(\ell-1)$.

We first write such an integer $u = ax + \ell y$ for some $x, y \in \mathbb{N} \cup \{0\}$. Then we construct N as the concatenation of A and B as

$$N = (AAA \cdots ABBB \cdots B)_b, \tag{5}$$

where the number of A in (5) is x and the number of B in (5) is y . Since both $A, B \equiv 0 \pmod{n}$, we have $N \equiv 0 \pmod{n}$. In addition,

$$s_b(N) = xs_b(A) + ys_b(B) = ax + \ell y = u.$$

This proves our assertion. In other words,

$$\text{Out}(s_b, n) \text{ contains every integer } u \geq (a - 1)(\ell - 1).$$

Hence $\text{Out}(s_b, n)$ is a cofinite subset of \mathbb{N} , as required. This completes the proof. \square

Remark 2.2. According to the proof of Theorem 2.1, if $\gcd(b - 1, n) = 1$, then

$$\{u \in \mathbb{N} \mid u \geq (n - 1)(n - 2 + b)\} \subseteq \text{Out}(s_b, n).$$

However, $(n - 1)(n - 2 + b)$ is not the least upper bound for the largest excluded integer.

Example 2.3. Let $b = 10$. If $n = 2^k$ for some $k \in \mathbb{N}$, then $n \mid 10^k$ and so

$$s_{10} \left(\sum_{i=1}^n 10^{i(k+1)-1} \right) = u,$$

for all $u \in \mathbb{N}$. This ensures that $\text{Out}(s_{10}, 2^k) = \mathbb{N}$. For the case $n = 14$, notice that $s_{10}(14) = 5$ and $s_{10}(42) = 6$. By following the constructive proof in Theorem 2.1, we obtain that any integer $u \geq 20$ is in $\text{Out}(s_{10}, 14)$ as shown in Table 1.

u	N
5	14
6	42
20	14,141,414
21	14,141,442
22	14,144,242
23	14,424,242
24	42,424,242
25	1,414,414,414
26	1,414,414,442
\vdots	\vdots

Table 1. The table show integers u in $\text{Out}(s_{10}, 14)$ and multiples N of 14 such that $s_{10}(N) = u$.

Although the Frobenius number formula for two coprime positive integers that we used in the proof of Theorem 2.1 is well known and is not difficult to prove, finding a general Frobenius number formula for three integers a_1, a_2 and, a_3 with $\gcd(a_1, a_2, a_3) = 1$ is not easy. In fact, it was unknown for quite some time until Tripathi [27, 26] studied this problem in his thesis and published the results in 2017.

From Theorem 2.1, the following natural questions may occur: is the set $\text{Out}(s_b, n)$ always infinite? In addition if $\gcd(b-1, n) = 1$, we see from (1) that $\text{Out}(s_b, n) = \mathbb{N}$ when $n = 1$ or when $b = 2$ and $m = 1$. Can we find a simple necessary and sufficient condition for $\text{Out}(s_b, n) = \mathbb{N}$? We have a complete answer to these questions in the next two theorems.

Theorem 2.4. *For integers $b \geq 2$ and $n \geq 1$, $\text{Out}(s_b, n)$ is an infinite set. More precisely, $\text{Out}(s_b, n)$ contains every multiple of $s_b(n)$.*

Proof. If $u = ks_b(n)$ for some positive integer k , then the integer $N = (nn \cdots n)_b$ constructed by the concatenation of k terms of n satisfies $n \mid N$ and

$$s_b(N) = ks_b(n) = u.$$

So every multiple of $s_b(n)$ is contained in $\text{Out}(s_b, n)$, as required. \square

Theorem 2.5. *Let $b \geq 2$ and $n \geq 1$ be integers. Then $\text{Out}(s_b, n) = \mathbb{N}$ if and only if $\gcd(b-1, n) = 1$ and every prime divisor of n is also a divisor of b .*

Remark 2.6. If $n = 1$, then $\gcd(b-1, n) = 1$ and n has no prime divisor, so it is included in the case $\text{Out}(s_b, n) = \mathbb{N}$. Nevertheless, for clarity, if one prefer, the above theorem can be rewritten as $\text{Out}(s_b, n) = \mathbb{N}$ if and only if (i) $n = 1$ or (ii) $n \geq 2$, $\gcd(b-1, n) = 1$, and every prime dividing n is a divisor of b .

Proof. We first observe that if $\gcd(b-1, n) > 1$, then we obtain by Theorem 2.1 that $\text{Out}(s_b, n) \neq \mathbb{N}$. Next, we consider the case that there exists a prime p dividing n and $p \nmid b$. Suppose for a contradiction that $\text{Out}(s_b, n) = \mathbb{N}$. So in particular, $1 \in \text{Out}(s_b, n)$. Then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $s_b(N) = 1$. But $s_b(N) = 1$ implies that $N = b^\ell$ for some $\ell \in \mathbb{N} \cup \{0\}$. Since $p \mid n$ and $n \mid N$, we have $p \mid b^\ell$, which implies $p \mid b$, a contradiction. Hence $\text{Out}(s_b, n) \neq \mathbb{N}$, as required.

For the converse, assume that $\gcd(b-1, n) = 1$ and every prime divisor of n is also a divisor of b . Since we already proved this result when $n = 1$ in (1), we can suppose that $n \geq 2$. Let $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where p_1, p_2, \dots, p_k are distinct primes and b_1, b_2, \dots, b_k are positive integers. Since every prime divisor of n is also a divisor of b , we can write $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where n_1, n_2, \dots, n_k are non-negative integers. To show that $\text{Out}(s_b, n) = \mathbb{N}$, let $u \in \mathbb{N}$ be given. Let $n_0 = \max_{1 \leq i \leq k} n_i$ and $N = b^{n_0} \sum_{i=0}^{u-1} b^i$. Then $n \mid N$ and $s_b(N) = u$. So $n \rightarrow u$, as required. This completes the proof. \square

Another question that may occur from Theorem 2.1 is in the subset relation in (i). By Theorem 2.5, we obtain that if $\gcd(b-1, n) = 1$ and every prime divisor of n is a divisor of

b , then the subset relation in (i) of Theorem 2.1 becomes an equality. If $\gcd(b - 1, n) > 1$, is the subset relation a strict subset or an equality? The next two examples show that it may be an equality or a strict subset relation.

Example 2.7. Let $b = 10$ and $n = 33$. In this case, we have $d = \gcd(b - 1, n) = 3$ and we will show that $u = 3$ is not an element of $\text{Out}(s_b, n)$. Suppose by way of contradiction that there exists an arc from n to 3. Then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $s_{10}(N) = 3$. Since $11 \mid n$, we see that $11 \mid N$. Since $s_{10}(N) = 3$, it is easy to see that N must be in the following form:

$$N = 3 \times 10^a, \text{ or } N = (2 \times 10^a) + 10^b, \text{ or } N = 10^a + 10^b + 10^c,$$

where $a, b, c \in \mathbb{N} \cup \{0\}$. Reducing these modulo 11, we obtain

$$N \equiv 3(-1)^a, 2(-1)^a + (-1)^b, (-1)^a + (-1)^b + (-1)^c \equiv 1, -1, 3, -3 \pmod{11},$$

which contradicts the fact that $11 \mid N$. Hence $\text{Out}(s_b, n) \neq \{u \in \mathbb{N} \mid u \equiv 0 \pmod{d}\}$.

There is a situation where $\gcd(b - 1, n) > 1$ and the subset relation in (i) of Theorem 2.1 is in fact an equality too as shown in the next example.

Example 2.8. Let $b = 10$ and $n = 3$. In this case, we have $d = \gcd(b - 1, n) = 3$. To show that the subset relation in Theorem 2.1 (i) can be replaced by an equality, let $u \equiv 0 \pmod{3}$. Then $u = 3q$ for some $q \geq 1$. Then we can choose $N = 3 \sum_{i=0}^{q-1} 10^i$ to obtain $N \equiv 0 \pmod{n}$ and $s_{10}(N) = 3q = u$, as required.

From this, it is interesting to determine exactly when the subset relation in Theorem 2.1 is strict or is an equality. See Question 1 and the list of other questions at the end of this article.

Recall that for each $m \in \mathbb{N}$, the number of positive divisors of m is denoted by $\tau(m)$ and the arithmetic function τ is called the divisor function. It is well known that τ is a multiplicative function, that is $\tau(1) = 1$ and if $m, n \in \mathbb{N}$ and $\gcd(m, n) = 1$, then $\tau(mn) = \tau(m)\tau(n)$. Next, we study some properties of the τ -arc and $\text{Out}(\tau, n)$.

Theorem 2.9. *Let n be a positive integer. Then the following statements hold.*

- (i) $\text{Out}(\tau, n) \subseteq \{u \in \mathbb{N} \mid u \geq \tau(n)\}$.
- (ii) $\text{Out}(\tau, n) = \mathbb{N}$ if and only if $n = 1$.
- (iii) $\text{Out}(\tau, n)$ is a cofinite subset of \mathbb{N} if and only if $n = p^k$ for some prime p and non-negative integer k .
- (iv) $\text{Out}(\tau, n)$ is always an infinite set.

Proof. For (i), let $u \in \text{Out}(\tau, n)$. Then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $\tau(N) = u$. Since $n \mid N$, every divisor of n is also divisor of N . Therefore $\tau(n) \leq \tau(N) = u$, as required.

For (ii), if $n = 1$ and u is a positive integer, then we can choose $N = p^{u-1}$ to obtain $n \mid N$ and $\tau(N) = u$, and so $\text{Out}(\tau, n) = \mathbb{N}$. We observe that $\tau(n) = 1$ if and only if $n = 1$. So if $\text{Out}(\tau, n) = \mathbb{N}$, then we obtain from (i) that $\tau(n) = 1$, which implies $n = 1$.

For (iii), assume that $\text{Out}(\tau, n)$ is cofinite. Since the number of primes is infinite, there exists a prime number $u \in \text{Out}(\tau, n)$. So there is $N \in \mathbb{N}$ such that $n \mid N$ and $\tau(N) = u$. Let $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where a_1, a_2, \dots, a_k are positive integers and p_1, p_2, \dots, p_k are distinct primes. By the well known formula for $\tau(N)$, we obtain

$$u = \tau(N) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1). \quad (6)$$

Since u is a prime, the only positive divisors of u is 1 and u . Therefore (6) implies that $k = 1$ and $a_1 = u - 1$. So $N = p_1^{u-1}$. Since $n \mid N$, we see that $n = p_1^k$ for some non-negative integer $k \leq u - 1$. Next, let $n = p^k$ where p is a prime and k is a non-negative integer. If $k = 0$, then the result follows from (ii). So suppose $k \geq 1$. We assert that

$$\text{Out}(\tau, n) = \{u \in \mathbb{N} \mid u \geq k + 1\}. \quad (7)$$

By (i), we see that $\text{Out}(\tau, n)$ is a subset of the set on the right-hand side of (7). Next, $u \in \mathbb{N}$ and $u \geq k + 1$. Let $N = p^{u-1}$. Then $n \mid N$ and $\tau(N) = u$. So $u \in \text{Out}(\tau, n)$ and (7) is proved. Therefore $\text{Out}(\tau, n)$ is cofinite. This proves (iii).

Next, we prove (iv). If $n = 1$, then (iv) follows from (ii). So let $n > 1$ and write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where a_1, a_2, \dots, a_k are positive integers and p_1, p_2, \dots, p_k are distinct primes. We assert that the integer u defined by

$$u = u_\ell = 2^\ell (a_1 + 1)(a_2 + 1) \cdots (a_k + 1),$$

is an element of $\text{Out}(\tau, n)$ for every $\ell \in \mathbb{N}$. Let q_1, q_2, \dots, q_ℓ be distinct primes and different from p_1, p_2, \dots, p_k . Let $m = \prod_{i=1}^\ell q_i$ and $N = mn$. Then $n \mid N$, and since $(m, n) = 1$ and τ is multiplicative, we also obtain

$$\begin{aligned} \tau(N) &= \tau(m)\tau(n) \\ &= 2^\ell (a_1 + 1)(a_2 + 1) \cdots (a_k + 1) \\ &= u_\ell. \end{aligned}$$

So $u_\ell \in \text{Out}(\tau, n)$. Since ℓ is arbitrary, $\text{Out}(\tau, n)$ is infinite. This completes the proof. \square

Remark 2.10. From Theorem 2.9,

$$\text{Out}(\tau, p^k) = \{u \in \mathbb{N} \mid u \geq k + 1\},$$

where p is prime and k is a non-negative integer.

Considering (i) of Theorem 2.9, it is natural to ask whether the subset relation can be replaced by an equality. The next theorem shows that it is an equality if and only if $n = 1$ or n is a prime power.

Theorem 2.11. *Let n be a positive integer. Then $\text{Out}(\tau, n) = \{u \in \mathbb{N} \mid u \geq \tau(n)\}$ if and only if $n = p^k$ for some prime p and non-negative integer k .*

Proof. If $n = 1$, then the result follows from Theorem 2.9. If $n = p^k$ where p is a prime and $k \in \mathbb{N}$, then we already proved it in (7). So the converse of this theorem holds. Next, suppose $n \neq 1$ and n is not a prime power. Then $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where $k \geq 2$, n_1, n_2, \dots, n_k are positive integers and p_1, p_2, \dots, p_k are distinct primes. We only need to find $u \in \mathbb{N}$ such that $u \geq \tau(n)$ and $u \notin \text{Out}(\tau, n)$. Suppose, by way of contradiction, that $u = \tau(n) + 1$ is an element of $\text{Out}(\tau, n)$. Then there exists $q \in \mathbb{N}$ such that $\tau(nq) = u$. Clearly $q \geq 2$. So if $\gcd(q, n) = 1$, then $u = \tau(n)\tau(q) \geq 2\tau(n) > \tau(n) + 1$, which is not the case. So $\gcd(q, n) > 1$. Let $d = \gcd(q, n)$. Since $d > 1$ and $d \mid n$, we write $d = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ where d_1, d_2, \dots, d_k are non-negative integers and there is at least one $j = 1, 2, \dots, k$ such that $d_j \geq 1$. Since $nd \mid nq$, we see that $\tau(nd) \leq \tau(nq)$. Therefore

$$\begin{aligned} u = \tau(nq) &\geq \tau(nd) = \prod_{i=1}^k (n_i + d_i + 1) \\ &= \prod_{i=1}^k (n_i + 1) \prod_{i=1}^k \left(1 + \frac{d_i}{n_i + 1}\right) \\ &= \tau(n) \prod_{i=1}^k \left(1 + \frac{d_i}{n_i + 1}\right) \\ &\geq \tau(n) \left(1 + \frac{d_j}{n_j + 1}\right) \\ &\geq \tau(n) \left(1 + \frac{1}{n_j + 1}\right) \\ &> \tau(n) \left(1 + \frac{1}{\tau(n)}\right) \\ &= \tau(n) + 1, \end{aligned}$$

which is a contradiction. Hence $u = \tau(n) + 1$ is not an element of $\text{Out}(\tau, n)$. This completes the proof. \square

Next, we give some results on $\text{Out}(\omega, n)$ and $\text{Out}(\Omega, n)$.

Theorem 2.12. *The set $\text{Out}(\omega, n)$ is cofinite for every $n \in \mathbb{N}$. More precisely, we have*

$$\text{Out}(\omega, n) = \{u \in \mathbb{N} \mid u \geq \omega(n)\}. \tag{8}$$

Consequently, $\text{Out}(\omega, n) = \mathbb{N}$ if and only if $n = p^k$ for some prime p and integer $k \geq 0$.

Proof. To prove (8), let $u \in \text{Out}(\omega, n)$. Then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $\omega(N) = u$. Since $n \mid N$, we have $u = \omega(N) \geq \omega(n)$. Next, suppose $u \geq \omega(n)$. If $u = \omega(n)$, then we can choose $N = n$ to obtain $n \mid N$ and $\omega(N) = u$. So suppose $u = \omega(n) + \ell$ for some $\ell \in \mathbb{N}$. Since the number of primes is infinite, there are primes $p_1 > p_2 > \cdots > p_\ell$ that are not the divisors of n . Let $N = np_1 p_2 \cdots p_\ell$. Then $n \mid N$ and $\omega(N) = \omega(n) + \ell = u$. This proves (8). If $n = p^k$ where p is a prime and k is a non-negative integer, then $\omega(n) = 0$ or $\omega(n) = 1$, so we obtain from (8) that $\text{Out}(\omega, n) = \mathbb{N}$. If $n \neq 1$ and n is not a prime

power, then $\omega(n) \geq 2$, and we obtain from (8) that $\text{Out}(\omega, n)$ is not \mathbb{N} . So the proof is complete. \square

Theorem 2.13. *For each $n \in \mathbb{N}$, we have*

$$\text{Out}(\Omega, n) = \{u \in \mathbb{N} \mid u \geq \Omega(n)\}. \quad (9)$$

Consequently, $\text{Out}(\Omega, n)$ is cofinite for every $n \in \mathbb{N}$, and $\text{Out}(\Omega, n) = \mathbb{N}$ if and only if $n = 1$ or n is a prime.

Proof. The proof is similar to that of Theorem 2.12, so we skip some details. If $u \in \text{Out}(\Omega, n)$, then there exists $N \in \mathbb{N}$ such that $n \mid N$ and $\Omega(N) = u$, and so $u = \Omega(N) \geq \Omega(n)$. Next, let $u \geq \Omega(n)$. If $u = \Omega(n)$, then we can choose $N = n$; if $u = \Omega(n) + \ell$ for some $\ell \in \mathbb{N}$, then we choose $N = np_1p_2 \cdots p_\ell$ where p_1, p_2, \dots, p_ℓ are distinct primes that does not divide n , which leads to the conclusion that $u \in \text{Out}(\Omega, n)$. This proves (9). We observe that $\Omega(n) \in \{0, 1\}$ if and only if $n = 1$ or n is a prime. Then the other part of this theorem follows in the same way as in the proof of Theorem 2.12. So the proof is complete. \square

3. Comments and some open questions

In this section, we give a list of some open problems. However, we do not claim that they are difficult or interesting. They may be unimportant or may even be trivial. However, we would merely like to record them for ourselves and share them among interested readers.

Question 3.1. *Let $b \geq 2$, $n \geq 1$, and $d = \gcd(b - 1, n)$. Can one find a necessary and sufficient condition for the equality*

$$\text{Out}(s_b, n) = \{u \in \mathbb{N} \mid u \equiv 0 \pmod{d}\} ? \quad (10)$$

When $d = 1$, we obtain such a condition in Theorem 2.5, but when $d > 1$, the problem is open. By Theorems 2.1 and 2.4, it is easy to see that if d is divisible by $s_b(n)$, then (10) holds. Is the divisibility $s_b(n) \mid d$ also a necessary condition for (10)?

Question 3.2. *By Examples 2.7 and 2.8, we know that (10) may or may not hold. In the case that $d > 1$ and $\text{Out}(s_b, n)$ is not equal to the set on the right-hand side of (10), can one completely determine the elements of $\text{Out}(s_b, n)$?*

Question 3.3. *We obtain in Theorem 2.4 that $\text{Out}(s_b, n)$ contains every multiple of $s_b(n)$. Does $\text{Out}(s_b, n)$ contain an infinite number of positive integers that are not divisible by $s_b(n)$? When $d = 1$ and $s_b(n) \geq 2$, the answer is yes. Can we determine a simple necessary and sufficient condition for the infinitely many elements in $A \cap \text{Out}(s_b, n)$ where A is the set of positive integers that are not divisible by $s_b(n)$? In that case, does $\text{Out}(s_b, n)$ has a natural density?*

Question 3.4. For a cofinite proper subset A of \mathbb{N} , we call the largest integer in $\mathbb{N} \setminus A$ the Frobenius number of A . By Theorems 2.1 and 2.5, we know that $\text{Out}(s_b, n)$ is a cofinite proper subset of \mathbb{N} when $\gcd(b-1, n) = 1$ and there exists a prime p that divides n but does not divide b . Can one determine the Frobenius number of $\text{Out}(s_b, n)$ in this case? The reader can find more information on numerical sets and Frobenius numbers, for example in the book by Alfonsín [1] and in the introduction of the article by Guhl et al. [14].

Question 3.5. We obtain some basic properties of $\text{Out}(g, n)$ where $g = s_b, \tau, \omega, \Omega$ but we do not have any result on the set of $u \in \mathbb{N}$ that there exists an arc from u to n . So for each arithmetic function g and $n \in \mathbb{N}$, let

$$\text{In}(g, n) = \left\{ u \in \mathbb{N} \mid u \xrightarrow{g} n \right\}.$$

What are the properties of $\text{In}(g, n)$? Is $\text{In}(g, n)$ infinite? Can one determine the set of all positive integers n such that $\text{In}(g, n) = \mathbb{N}$ or a cofinite subset of \mathbb{N} ? Does such an n exist?

Question 3.6. We can extend the g -arc from the multiple of n to an arithmetic progression $r \pmod{n}$ by defining that there exists a (g, r) -arc from n to u if there exists $N \in \mathbb{N}$ such that $N \equiv r \pmod{n}$ and $g(N) = u$. Let

$$\text{Out}(g, n, r) = \{ u \in \mathbb{N} \mid \exists \text{ a } (g, r)\text{-arc from } n \text{ to } u \},$$

and

$$\text{In}(g, n, r) = \{ u \in \mathbb{N} \mid \exists \text{ a } (g, r)\text{-arc from } u \text{ to } n \}.$$

So if $r = 0$, then this is the g -arc that we study in this article. When $0 < r < n$, what are the results analogous to our theorems?

Question 3.7. We may consider the g -arc from n to ∞ too. A sequence $(a_m)_{m \geq 1}$ is a g -arc from n to ∞ if $a_1 = n$, $a_m \rightarrow \infty$ as $m \rightarrow \infty$, and there exists a g -arc from a_m to a_{m+1} for every $m \geq 1$. Nevertheless, without any restriction on the number of steps, this problem may not be interesting. We say that there exists a k -bounded g -arc from n to u if there exists $N \in \mathbb{N}$ such that $N \equiv 0 \pmod{n}$, $N \leq kn$, and $g(N) = u$. A sequence $(a_m)_{m \geq 1}$ is a k -bounded g -arc from n to ∞ if $a_1 = n$, $a_m \rightarrow \infty$ as $m \rightarrow \infty$, and there exists a k -bounded g -arc from a_m to a_{m+1} for every $m \in \mathbb{N}$. What are the results corresponding to our theorems for k -bounded g -arc? If n and k are given, does there exist a k -bounded g -arc from n to ∞ when $g = s_b, \tau, \omega, \Omega$, or other arithmetic functions?

Question 3.8. We say that n and u are g -friends if there exists a g -arc from n to u and there exists a g -arc from u to n . For each $g = s_b, \tau, \omega, \Omega$, can we determine all pairs (n, u) which are g -friends?

Question 3.9. A triangle in a directed graph $G = (V, E)$ is a triple (v_1, v_2, v_3) such that there exist directed edges from v_1 to v_2 , v_2 to v_3 , and v_3 to v_1 . If $n \geq 3$, an n -polygon in G is an n -tuple (v_1, v_2, \dots, v_n) such that there are directed edges connecting from v_n to

v_1 and from v_i to v_{i+1} for each $i = 1, 2, \dots, n-1$. Suppose g is an arithmetic function, $V = \mathbb{N}$, and there exists a directed edge from n to u if there exists a g -arc from n to u . Are there infinitely many triangles or n -polygons in G ?

Question 3.10. Let $e \geq 1$ and $b \geq 2$ be integers, and let $S_{e,b} : \mathbb{N} \rightarrow \mathbb{N}$ be defined as follows: if $n = (a_k a_{k-1} \cdots a_0)_b = \sum_{0 \leq i \leq k} a_i b^i$ is the b -adic expansion of n where $a_k \neq 0$ and $0 \leq a_i < b$ for all i , then

$$S_{e,b}(n) = a_k^e + a_{k-1}^e + \cdots + a_0^e.$$

The function $S_{e,b}$ is called an (e, b) -happy function, and it has been studied by many mathematicians, see for example in Guy's book [15, Chapter E34], and in the articles by El-Sedy and Siksek [23], Grundman and Teeple [13], Gilmer [12], Chase [7], Noppakaew et al. [19], Phoopha et al. [21], and Subwattanachai and Pongsriiam [24]. In particular, if $e = 1$, then $S_{e,b} = s_b$ is the sum of digits function. So the study of $S_{e,b}$ -arc may lead to an interesting generalization of s_b -arc. What are the corresponding results to our theorems if we replace s_b by $S_{e,b}$?

4. Conclusion

We are motivated by various problems in graph labeling, so we introduce a new directed graph whose vertices are integers and the arcs are determined by arithmetic functions. We determine the set of all positive integers that connected to all but a finite number of positive integers.

5. Comments and Acknowledgments

We already mentioned the work of various mathematicians on graph labeling and coprime graphs [8, 9, 22, 4, 16, 3], which inspired us to write this article. Another inspiration came from the last problem in Thailand Mathematical Olympiad TMO 2022 [25] in which the first author participated as a teacher. After TMO 2022 ended, the first author sent a question to the third author who had heard of Gelfond's theorem on sum of digits function $s_b(n)$, which contains a hint to an answer to the problem, and we decided to extend this problem without the use of Gelfond's theorem. The interested reader can find more information about Gelfond's theorem, for example, in Morgenbesser's diploma thesis [18], references therein, and many other articles in the literature.

After some modification of languages, the contestants in TMO 2022 were asked about the cofiniteness of $\text{Out}(s_{10}, n)$, which are covered and extended to $\text{Out}(s_b, n)$ for any $b \geq 2$ and $n \geq 1$ in (ii) of Theorem 2.1 in this article. We would like to thank and show the support to the organizers, teachers, students, and sponsors of TMO by writing this article. We hope that this will motivate some students to learn more about mathematics.

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