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GDDs with two groups of unequal sizes and block size four

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ABSTRACT

We obtain several results towards the proof that the necessary conditions are sufficient for the existence of a GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2$. We also have some general results including the constructions for larger block sizes as well as when the first group size n_1 is not 1 or $\lambda_1 < \lambda_2$.

Keywords: Group divisible designs, block constructions, necessary and sufficient conditions

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1. Introduction

Group divisible designs (GDDs) are classical combinatorial designs studied for their applications as well as for their own sake. GDDs are inherently hard to construct, especially when the number of groups is less than the block size and group sizes are different.

For block size 3, a lot of work on GDDs is done. For example, Fu, Rodger and Sarvate [12, 13] obtained complete results on the existence of group divisible designs with block size 3 and m groups of size n. Colbourn, Hoffman and Rees [8] proved the sufficiency of the necessary conditions for the existence of GDDs denoted as GDD$(n, n, ..., n, u, 3; 0, 1)$, where one of the groups is of size $u \neq n$ (n is the size of other groups). The study of GDDs with unequal group sizes is more difficult in general. Pabhapote and Punnim [16] studied all triples of positive integers (n_1, n_2, λ) for which GDD$(n_1, n_2, 3; \lambda, 1)$ exists. They proved that the necessary conditions are sufficient for the existence of a GDD$(n_1, n_2, 3; \lambda, 1)$. Later, Pabhapote [15] proved the existence of a GDD$(n_1, n_2, 3; \lambda_1, \lambda_2)$ for all $n_1 \neq 2$ and

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$n_2 \neq 2$ in which $\lambda_1 \geq \lambda_2$. Chaiyasena, Hurd, Punim and Sarvate [6] obtained general results for GDDs with three groups of different sizes denoted by GDD$(n_1, n_2, n_3, 3; \lambda_1, \lambda_2)$. GDDs with block size four and different group sizes are not widely studied. Forbes showed that the necessary conditions for the existence of a GDD$(n, n, ..., n, u, 4; 0, 1)$ are sufficient for a range of different values of n and u [9, 10, 11]. Abel et al. studied the existence of a GDD$(n_1, n_2, 4; 0, 1)$ for a range of different values of n_1 and n_2 [1, 2, 3, 4, 5, 17]. Hunde and Woldemariam studied GDD$(n_1, n_2, 4; \lambda, \lambda_2)$ where $\lambda_2 = 1, 2$ [14].

The subject matter for this paper is to study GDDs with two groups of different sizes where the first group is of size 1 and the block size is 4. We begin with necessary definitions.

Definition 1.1. A Balanced Incomplete Block Design, BIBD(v, b, k, r, λ), is a pair $(V, \mathcal{B})$, where V is a v-set of elements and $\mathcal{B}$ is a collection of b k-subsets of V called blocks $(k < v)$ such that every element appears in r blocks, and every pair of distinct elements appears in exactly λ blocks. Unless we need to specify the values of b or r, we use the notation BIBD(v, k, λ).

Definition 1.2. A parallel class in a BIBD(v, k, λ) is a set of disjoint blocks of the BIBD whose union is V. A partition of the collection of all blocks into r parallel classes is called a resolution. A BIBD is said to be a resolvable BIBD, denoted RBIBD(v, k, λ) if it has a resolution.

A *pairwise balanced design* (PBD(v, K, λ)) is a block design of v points such that any two distinct points lie in λ blocks, where K is the set of cardinalities of blocks. If $\lambda = 1$, we denote it as PBD(v, K). A resolvable PBD(v, K) is denoted as RPBD(v, K). If all parallel classes of a RPBD(v, K) contain blocks of one size k, then it is said to be uniformly resolvable and is denoted as URD(v, K). The number of parallel classes in a URD(v, K) with blocks of size k is denoted as $r_k, k \in K$.

We need the following theorems for our work.

Theorem 1.3. [7] *A BIBD$(v, 3, \lambda)$ exists iff any of the following necessary conditions are satisfied:*

- *A BIBD$(v, 3, \lambda)$ where $\lambda \equiv 1, 5 \pmod 6$ exists iff $v \equiv 1, 3 \pmod 6$.*
- *A BIBD$(v, 3, \lambda)$ where $\lambda \equiv 2, 4 \pmod 6$ exists iff $v \equiv 0, 1 \pmod 3$.*
- *A BIBD$(v, 3, \lambda)$ where $\lambda \equiv 3 \pmod 6$ exists iff $v \equiv 1 \pmod 2$.*
- *A BIBD$(v, 3, \lambda)$ where $\lambda \equiv 0 \pmod 6$ exists iff $v \geq 3$.*

Theorem 1.4. [7] *A BIBD$(v, 4, \lambda)$ exists iff any of the following necessary conditions are satisfied:*

- $\lambda \equiv 1, 5 \pmod 6$ *and* $v \equiv 1, 4 \pmod{12}$*;*
- $\lambda \equiv 2, 4 \pmod 6$ *and* $v \equiv 1 \pmod 3$*;*
- $\lambda \equiv 3 \pmod 6$ *and* $v \equiv 0, 1 \pmod 4$*;*

- $\lambda \equiv 0 \pmod 6$ *and* $v \geq 4$.

Theorem 1.5. [18] *There exists a URD*$(v, \{3,4\})$ *with* $r_3, r_4 > 0$ *if and only if* $v \equiv 0 \pmod{12}$, r_4 *is odd and* $1 \leq r_4 \leq \frac{v}{3} - 1$, *except for* $v = 12$ *and* $r_4 = 3$.

Note that r_3 and r_4 are related. If x is an element in r_4 blocks of size 4, then x is appearing with $3r_4$ elements in the blocks of size 4. Therefore, x has to appear with the remaining $v - 1 - 3r_4$ elements in block of size 3, we have $r_3 = \frac{v-1-3r_4}{2}$.

Definition 1.6. A group divisible design, GDD$(n_1, n_2, ..., n_m, k; \lambda_1, \lambda_2)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a v-set $(v = n_1 + n_2 + ... + n_m)$, $\mathcal{G}$ a partition of X into m subsets (called *groups*) of size $n_1, n_2, ..., n_m$ respectively and $\mathcal{B}$ is a collection of k-subsets of X (called *blocks*) such that

(i) the pairs of points within the same group, called *first associates* of each other, appear together in λ_1 blocks, and

(ii) the pairs of points not in the same group, called *second associates* of each other, appear together in λ_2 blocks.

If $n_1 = n_2 = \cdots = n_m = n$, it is denoted as a GDD$(n, m, k; \lambda_1, \lambda_2)$. To avoid confusion of the notation of GDDs with two groups of unequal sizes and block size 4, GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$, we specify the sizes of the two groups as n_1 and n_2, in the remaining of this paper unless $n_1 = 1$.

An important result for $k = 4$ and $\lambda_1 = 0$ and $\lambda_2 = \lambda$ [7] is as follows.

Theorem 1.7. *Necessary conditions for the existence of a GDD*$(n, m, 4; 0, \lambda)$ *given below*

(i) $m \geq 4$ *and*

(ii) $\lambda n(m-1) \equiv 0 \pmod 3$ *and*

(iii) $\lambda m(m-1)n^2 \equiv 0 \pmod{12}$

are sufficient where $(n, m, \lambda) \neq (2, 4, 1)$ *and* $(n, m, \lambda) \neq (6, 4, 1)$.

It is well known that the blocks of the BIBDs can be used as the building blocks of GDDs. For example, we will use the following construction repeatedly by combining the blocks of the two BIBDs.

Theorem 1.8. *If a BIBD*$(n_1 + n_2 + ... + n_m, k, \lambda_2)$ *and a BIBD*(n_i, k, λ_1) *exist for* $i = 1, 2, ...m$, *then GDD*$(n_1, n_2, ..., n_m, k; \lambda_1 + \lambda_2, \lambda_2)$ *exists.*

Another such construction given below is very useful to construct a GDD$(1, n, k; \lambda_1, \lambda_2)$, not only for $\lambda_1 \geq \lambda_2$, but also for $\lambda_1 < \lambda_2$. Even though keeping with the tradition of papers [14, 15, 16], we study in this note designs where $\lambda_1 \geq \lambda_2$.

Theorem 1.9. *If a BIBD*$(v = n, b, r, k, \lambda)$ *and a BIBD*$(v = n, k+1, \Lambda)$ *exist, then a GDD*$(n_1 = 1, n_2 = n, k+1; \lambda + u\Lambda, r)$ *exists where* u *is a nonnegative integer.*

Proof. Let $G_1 = \{x\}$ and $G_2 = \{1, \cdots, n\}$. The blocks of the required GDD($n_1 = 1, n_2 = n, k+1; \lambda + u\Lambda, r$) are obtained by combining u copies of the blocks of the BIBD($n, k+1, \Lambda$) on G_2, with the blocks obtained by taking union of G_1 with the blocks of the BIBD(n, b, r, k, λ) on G_2. □

We can have many corollaries from Theorem 1.9, for different values of k depending on the existence of BIBDs, but we are specifically interested in the following result.

Corollary 1.10. *A BIBD($v, b, r, 3, 1$) and a BIBD($v, 4, 2t$) where $v = 12s + 7$ and $r = 6s + 3$ exist, therefore, a GDD($n_1 = 1, n_2 = v, 4; \lambda_1 = 2t + 1, \lambda_2 = r$) exists where s and t are nonnegative integers.*

Corollary 1.10 gives specific examples for the case $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 1, 5 \pmod 6$ and $n \equiv 7 \pmod{12}$ of Theorem 6.8.

2. GDDs with two groups of unequal sizes

Using Theorem 1.4 and Theorem 1.8, we obtain the following families.

Theorem 2.1. *The following GDDs exist:*

(i) *A GDD($n_1, n_2, 4; \lambda+6s, \lambda$) where $\lambda \equiv 1, 5 \pmod 6$ exists for $n_1+n_2 \equiv 1, 4 \pmod{12}$ and $n_i \geq 4$ for $i = 1, 2$ and any nonnegative integer s.*

(ii) *A GDD($n_1, n_2, 4; \lambda + 6s, \lambda$) where $\lambda = 2, 4 \pmod 6$ exists for $n_1 + n_2 \equiv 1 \pmod 3$ and $n_i \geq 4$ for $i = 1, 2$ and any nonnegative integer s.*

(iii) *A GDD($n_1, n_2, 4; \lambda + 6s, \lambda$) where $\lambda = 3 \pmod 6$ exists for $n_1 + n_2 \equiv 0, 1 \pmod 4$ and $n_i \geq 4$ for $i = 1, 2$ and any nonnegative integer s.*

(iv) *A GDD($n_1, n_2, 4; \lambda + 6s, \lambda$) where $\lambda = 0 \pmod 6$ exists for $n_i \geq 4$ for $i = 1, 2$ and any nonnegative integer s.*

Theorem 2.2. (a) *If a URD($v, \{3, 4\}, \lambda$) exists with r_3 parallel classes of block size three, then a GDD($n_1 = 1, n_2 = v, 4; \lambda, \lambda_2 = r_3$) exists.*

(b) *If a URD($v, \{3, 4\}, \lambda$) exists with r_3 parallel classes, then a GDD($n_1 = 4, n_2 = v, 4; 4\lambda, r_3$) exists.*

Proof. (a) First we union the first group G_1 of size 1 with each block of size three from the URD($v, \{3, 4\}, \lambda$). The resulting blocks are then combined with the blocks of size four from the URD($v, \{3, 4\}, \lambda$). As a result, we have a GDD($n_1 = 1, n_2 = v, 4; \lambda, \lambda_2 = r_3$).

(b) Let $G_1 = \{1, 2, 3, 4\}$, disjoint from the second group G_2 of size v. Take four copies of the URD($v, \{3, 4\}, \lambda$) on G_2. In the i^{th} copy of the URD (for $i = 1, 2, 3, 4$), replace blocks of size 3 by the blocks of size 4 obtained by combining each block of size 3 with element i of G_1. These blocks along with 4λ copies of G_1 as blocks give the required design. □

Using Theorem 1.5, we have the following results.

Theorem 2.3. (a) *For $v \equiv 0 \pmod{12}, v > 12$, a GDD($n_1 = 4, n_2 = v, 4; 4, r_3$) exists* *for* $r_3 = \frac{v-1-3r_4}{2}$ *and* $1 \le r_4 \le \frac{v}{3} - 1$. *For* $v = 12$, *a GDD*$(n_1 = 4, n_2 = 12, 4; 4, 1)$ *exists.*

(b) *For* $n \equiv 0 \pmod{12}, n > 12$, *using URD with* $r_3 = 4$, *a GDD*$(n_1 = 1, n_2 = n, 4; 1, 4)$ *exists and a GDD*$(n_1 = 1, n_2 = n, 4; 6s+1, 6t+4)$ *exists.*

3. Some constructions for GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$

In this section, we present some constructions of GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$, using known designs as building blocks.

Theorem 3.1. *If a GDD*$(n = 3, m, 4; 0, \lambda)$ *exists and a BIBD*$(3m, 4, \mu)$ *exists, then a GDD*$(n_1 = 1, n_2 = 3m, 4; \mu + \lambda, \lambda)$ *exists.*

Proof. We divide $3m$ elements of the second group G_2 into m groups of size 3 each, and construct the blocks of a GDD$(n = 3, m, 4; 0, \lambda)$. These blocks together with the blocks of a BIBD$(3m, 4, \mu)$ on G_2 give the blocks of a GDD$(n = 3, m, 4; \mu, \mu + \lambda)$. Next, we form blocks of size 4 by taking union of $G_1 = \{x\}$ with each of the m groups to the three elements of G_2. Now, λ copies of such blocks along with the blocks of the GDD$(n = 3, m, 4; \mu, \mu + \lambda)$ formed previously give us a GDD$(n_1 = 1, n_2 = 3m, 4; \mu + \lambda, \lambda)$. □

As the necessary and sufficient conditions for the existence of a GDD$(n = 3, m, 4; 0, \lambda)$ are $m \ge 4$, $\lambda m(m-1) \equiv 0 \pmod 4$, and the necessary and sufficient conditions for the existence of a BIBD$(3m, 4, \mu)$ are (i) $\mu \equiv 3 \pmod 6$ and $m \equiv 0, 3 \pmod 4$; or (ii) $\mu \equiv 0 \pmod 6$ and any $m \ge 2$, we have the following corollary.

Corollary 3.2. *For any even* λ_2, *say* $2t$, *a GDD*$(n_1 = 1, n_2 = 3m, 4; \lambda_1 = 6s+2t+3, \lambda_2 = 2t)$ *for* $m \equiv 0, 3 \pmod 4$ *and a GDD*$(n_1 = 1, n_2 = 3m, 4; \lambda_1 = 6s+2t, \lambda_2 = 2t)$ *for* $m \ge 2$ *and any nonnegative integers* s *and* t *exists.*

Theorem 3.3. *If a GDD*$(n, m, 4; 0, 1)$ *exists, then a GDD*$(n_1 = 1, n_2 = nm, 4; n-2, \binom{n-1}{2})$ *exists.*

Proof. Let $G_1 = \{x\}$ and $G_2 = \{1, 2, \cdots, nm\}$. Let the groups of the GDD$(n, m, 4; 0, 1)$ be a partition $H_1, \cdots, H_m$ of G_2 where each H_i is of size n. Also, let T_i be the collection of all triples of H_i, $i = 1, \cdots, m$, i.e., $T_i = \{\{a, b, c\} | \{a, b, c\} \subseteq H_i\}$, and let $T = \cup_{i=1}^{m} T_i$.

Let B_1 be the collection of blocks of size 4 by adding x to each triple in T, and B_2 be the blocks of the GDD$(n, m, 4; 0, 1)$. Then $n-2$ copies of B_2 along with the blocks of B_1 form the required GDD$(n_1 = 1, n_2 = nm, 4; n-2, \binom{n}{2})$.

To check $\lambda_1 = n-2$, observe that if two elements are in a group H_i, then they occur in $n-2$ triples in T as the size of H_i is n. If two elements are in different groups, say H_i and $H_j (i \ne j)$, then in B_2 they occur only once, hence they occur $n-2$ times as we have $n-2$ copies of B_2.

To check $\lambda_2 = \binom{n-1}{2}$, consider a pair, say (a, x), if a is in H_i, then a occurs in $\binom{n-1}{2}$ subsets of H_i. Hence, the pair (a, x) occurs in $\binom{n-1}{2}$ blocks from B_1. □

From Theorem 1.7, a GDD$(4, m, 4; 0, 1)$ exists for $m \equiv 1 \pmod 3$, and we have:

Corollary 3.4. *A GDD*$(n_1 = 1, n_2 = 4m, 4; 2, 3)$ *exists for* $n_2 \equiv 4 \pmod{12}$.

4. Necessary conditions of GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$

Assuming a GDD(1, n, 4;λ_1, λ_2) exists, we count the replication number r_i of elements of the i^{th} group for i = 1, 2 and the required number of blocks b.

Theorem 4.1. *If a GDD(*1, n, 4; λ_1, λ_2*) exists, then*
(i) $r_1 = \frac{n\lambda_2}{3}$, *therefore,* $n\lambda_2 \equiv 0 \pmod 3$*;*
(ii) $r_2 = \frac{(n-1)\lambda_1+\lambda_2}{3}$, *therefore,* $(n-1)\lambda_1 + \lambda_2 \equiv 0 \pmod 3$*;*
(iii) $b = \frac{n(n-1)\lambda_1+2n\lambda_2}{12}$, *therefore,* $n(n-1)\lambda_1 + 2n\lambda_2 \equiv 0 \pmod{12}$.

Proof. (i). Let $G_1 = \{x\}$. Then x appears in r_1 blocks of the design, and there are 3 pairs containing x in each block. Hence, there are $3r_1$ pairs containing x. Second, there are n ways of choosing an element from the second group to form a pair with x, and it occurs in λ_2 blocks. That is, there are $n\lambda_2$ pairs containing x. Hence, $3r_1 = n\lambda_2$, and $r_1 = \frac{n\lambda_2}{3}$.

(ii). Let an element of the second group, say y, occurs in r_2 blocks. In each of these blocks, there are 3 pairs containing y. Hence, there are $3r_2$ pairs containing y. On the other hand, there are $(n-1)\lambda_1 + \lambda_2$ pairs in the design containing y. Therefore, $3r_2 = (n-1)\lambda_1 + \lambda_2$. Hence, $r_2 = \frac{(n-1)\lambda_1+\lambda_2}{3}$.

(iii). In a design with block size 4 and b blocks, there are $b\binom{4}{2}$=$6b$ pairs. On the other hand, there must be exactly $\binom{n}{2}\lambda_1$ pairs of type $(2, 0)$ and $n\lambda_2$ pairs of type $(1, 1)$ in the design. Therefore, $6b = \binom{n}{2}\lambda_1 + n\lambda_2$ and $b = \frac{n(n-1)\lambda_1+2n\lambda_2}{12}$. □

Theorem 4.2. *If a GDD(*1, n, 4;λ_1, λ_2*) exists, then*
(a) $b \geq \max\{r_1 + r_2 - \lambda_2, 2r_2 - \lambda_1\}$;
(b) $b \leq \frac{n(n-1)\lambda_1}{6}$*;*
(c) $\lambda_2 \leq \frac{(n-1)\lambda_1}{2}$, *i.e.* $\lambda_1 \geq \frac{2\lambda_2}{n-1}$*;*

Proof. (a) Consider the following two cases: (i) if x and y are elements chosen from different groups (say x is from the first group and y is from the second group), then there are r_1 blocks containing x and $r_2 - \lambda_2$ blocks containing y but not x. Hence, there are at least $r_1 + r_2 - \lambda_2$ blocks in the GDD.

(ii) When x and y are from the same group G_2, there are r_2 blocks containing x and $r_2 - \lambda_1$ blocks containing y but not x, which implies there are at least $2r_2 - \lambda_1$ blocks.

From (i) and (ii), the (b) follows.

(b) If a GDD(1, n, 4;λ_1, λ_2) exists, then every block contains at least 3 first associate pairs. This means $\lambda_1\binom{n}{2} \geq 3b$ (where $\lambda_1\binom{n}{2}$ is the number of first associate pairs in the design), and $b \leq \frac{n(n-1)\lambda_1}{6}$.

(c) From Theorem 4.1 and (b), $b = \frac{n(n-1)\lambda_1+2n\lambda_2}{12} \leq \frac{n(n-1)\lambda_1}{6}$, which is equivalent to $2n\lambda_2 \leq (n-1)n\lambda_1$, that is, $\lambda_2 \leq \frac{(n-1)\lambda_1}{2}$. □

Theorem 4.2 provides us one of the necessary conditions for the existence of a GDD$(1, n, 4; \lambda_1, \lambda_2)$: $\lambda_2 \leq \frac{(n-1)\lambda_1}{2}$. Furthermore, necessary conditions on n are given below where n is equal to the values specified in each entry and DNE means "does not exist".

5. Examples of GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$ for some small values of n

Let $G_1 = \{x\}$, $G_2 = \{1, 2, \cdots, n\}$. For $n = 4$, $\lambda_1 \geq \frac{2\lambda_2}{3}$ from the necessary condition in Theorem 4.2. From Table 1, we have two cases for $n = 4$: $\lambda_2 \equiv 0 \pmod 6$ and any λ_1; $\lambda_2 \equiv 3 \pmod 6$ and any λ_1. These two cases are equivalent to $\lambda_2 \equiv 0 \pmod 3$, i.e., $\lambda_2 = 3t$. Since $\lambda_1 \geq \frac{2\lambda_2}{3}$, we have $\lambda_1 \geq 2t$.

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	all $n \geq 3$	DNE	DNE	$0 \pmod 2$	DNE	DNE
1	$1, 4 \pmod{12}$	$0, 3 \pmod{12}$	DNE	$4, 7 \pmod{12}$	$0, 9 \pmod{12}$	DNE
2	$1 \pmod 3$	DNE	$0 \pmod 3$	$4 \pmod 6$	DNE	$0 \pmod 6$
3	$0, 1 \pmod 4$	DNE	DNE	$0, 3 \pmod 4$	DNE	DNE
4	$1 \pmod 3$	$0 \pmod 6$	DNE	$4 \pmod 6$	$0 \pmod 3$	DNE
5	$1, 4 \pmod{12}$	DNE	$0, 9 \pmod{12}$	$4, 7 \pmod{12}$	DNE	$0, 3 \pmod{12}$

Table 1. The necessary conditions on n for GDD$(1, n, 4; \lambda_1, \lambda_2)$

Example 5.1. Combining the blocks $\{1, 2, 3, x\}$, $\{1, 2, 4, x\}$, $\{1, 3, 4, x\}$, and $\{2, 3, 4, x\}$ with $\lambda_1 - 2$ copies of $\{1, 2, 3, 4\}$ gives us a GDD$(1, 4, 4; \lambda_1, 3)$. Similarly, union G_1 with the blocks of a BIBD$(4, 3, 2t)$ on G_2, and then combining these resulting blocks and $\lambda_1 - 2t$ copies of $\{1, 2, 3, 4\}$ give us a GDD$(1, 4, 4; \lambda_1, 3t)$ where $\lambda_1 \geq 2t$, as required by the necessary condition $\lambda_1 \geq \frac{2\lambda_2}{n-1}$ from Theorem 4.2.

Example 5.1 gives:

Corollary 5.2. *The necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD$(1, 4, 4; \lambda_1, \lambda_2)$.*

For $n = 5$, we have only one case $\lambda_1 \equiv 3 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$ from Table 1. Let $\lambda_2 = 6s$, we have $\lambda_1 \geq 3s$ by the necessary condition $\lambda_1 \geq \frac{2\lambda_2}{n-1}$ from Theorem 4.2.

Example 5.3. Joining G_1 with every 3-subset of $G_2 = \{1, 2, 3, 4, 5\}$ gives us a GDD$(1, 5, 4; 3, 6)$. Furthermore, combining s copies of the blocks of a GDD$(1, 5, 4; 3, 6)$ with $3t - 3s$ (where $t \geq s$) copies of every 4-subset of G_2 results in a GDD$(1, 5, 4; 3t, 6s)$, again notice that the necessary condition $\lambda_1 \geq \frac{2\lambda_2}{n-1}$ is satisfied.

Example 5.3 gives:

Corollary 5.4. *The necessary conditions (given in Table 1 and Theorem 4.2) are suffi-*

cient for the existence of a GDD$(1, 5, 4; \lambda_1, \lambda_2)$.

Example 5.5. A GDD(1, 6, 4; 4, 1) has the blocks $\{1, 2, 3, x\}$, $\{4, 5, 6, x\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, and $\{2, 3, 5, 6\}$. In addition, for $\lambda_1 \equiv 4 \pmod 6 = 6t + 4$ where t is a nonnegative integer, by combining the blocks of t copies of a BIBD(6, 4, 6) and the blocks of a GDD(1, 6, 4; 4, 1), we have a GDD$(1, 6, 4; \lambda_1, 1)$.

Example 5.6. A GDD(1, 6, 4; 4, 7) has the blocks $\{1, 2, 3, 4\}$, $\{3, 4, 5, 6\}$, $\{1, 2, 5, 6\}$ $\{1, 2, 3, x\}$, $\{1, 2, 4, x\}$, $\{3, 4, 5, x\}$, $\{3, 4, 6, x\}$, $\{1, 5, 6, x\}$, $\{2, 5, 6, x\}$, $\{1, 3, 5, x\}$, $\{1, 3, 6, x\}$, $\{1, 4, 5, x\}$, $\{1, 4, 6, x\}$, $\{2, 3, 5, x\}$, $\{2, 3, 6, x\}$, $\{2, 4, 5, x\}$ and $\{2, 4, 6, x\}$. In addition, for $\lambda_1 \equiv 4 \pmod 6 = 6t + 4$ where t is a nonnegative integer, by combining the blocks of t copies of a BIBD(6, 4, 6) and the blocks of a GDD(1, 6, 4; 4, 7), we have a GDD$(1, 6, 4; \lambda_1, 7)$.

Example 5.7. A BIBD(7, 4, 4) can be considered as a GDD(1, 6, 4; 4, 4). The blocks of a GDD(1, 6, 4; 4, 10) are obtained by taking the union of G_1 with each block of BIBD(6, 3, 4) on G_2.

Note that if $n = 6$ and $\lambda_1 = 4$, one of the necessary conditions $\lambda_2 \leq \frac{(n-1)\lambda_1}{2}$ implies that $\lambda_2 \leq 10$. From the necessary conditions in Table 1, if $\lambda_1 \equiv 4 \pmod 6$, then $\lambda_2 \equiv 1 \pmod 6$ or $\lambda_2 \equiv 4 \pmod 6$. This implies the necessary conditions of a GDD$(1, 6, 4; 4, \lambda_2)$ are sufficient from Examples 5.5, 5.6 and 5.7.

Example 5.8. Since a BIBD(7, 4, 6) is also a GDD(1, 6, 4; 6, 6), by combining the blocks of a BIBD(7, 4, 6) and the blocks of a GDD(1, 6, 4; 4, 7), we have a GDD(1, 6, 4; 10, 13). In addition, for $\lambda_1 \equiv 4 \pmod 6 = 6t + 4$ where t is a nonnegative integer, by combining the blocks of t copies of a BIBD(6, 4, 6) and the blocks of a GDD(1, 6, 4; 4, 7), we have a GDD$(1, 6, 4; \lambda_1, 13)$.

In general, we have many cases to consider for $n = 6$:
(1) $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$;
(2) $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 3 \pmod 6$;
(3) $\lambda_1 \equiv 2 \pmod 6$ and $\lambda_2 \equiv 2 \pmod 6$;
(4) $\lambda_1 \equiv 2 \pmod 6$ and $\lambda_2 \equiv 5 \pmod 6$;
(5) $\lambda_1 \equiv 4 \pmod 6$ and $\lambda_2 \equiv 1 \pmod 6$;
(6) $\lambda_1 \equiv 4 \pmod 6$ and $\lambda_2 \equiv 4 \pmod 6$.
We restrict our attention to $\lambda_1 \geq \lambda_2$ in these cases.

In Section 6, existence is shown along with general values of n for three cases ((1), (3) and (6)), except for cases (2), (4) and (5). These cases are taken care of below:

Case 2. Let $\lambda_1 = 6s$ and $\lambda_2 = 6t+3$ where $s \geq t$. Since we assume $\lambda_1 \geq \lambda_2$, $6s \geq 6t+3$, i.e. $s \geq \lceil t + \frac{1}{2} \rceil = t + 1$. Using Example 5.5, we have a GDD(1, 6, 4; 4, 1). Combining the blocks of a GDD(1, 6, 4; 4, 1) with the blocks of a BIBD$(7, 4, 6t + 2)$ on $G_1 \cup G_2$ (where G_1 is the first group and G_2 is the second group) and $s - t - 1$ copies of the blocks of a

BIBD$(6,4,6)$ on G_2 give us the blocks of a GDD$(1,6,4;\lambda_1=6s,\lambda_2=6t+3)$.

Case 4. Let $\lambda_1 = 6s+2$ and $\lambda_2 = 6t+5$ where $s \geq t$. Union the first group G_1 with the blocks of a BIBD$(6,3,2)$ on the second group G_2 give us the blocks of a GDD$(1,6,4;2,5)$. Combining these blocks with the blocks of a BIBD$(7,4,6t)$ on $G_1 \cup G_2$, we get a GDD$(1,6,4;6t+2,6t+5)$. The blocks of a GDD$(1,6,4;6t+2,6t+5)$ combined with the blocks of a BIBD$(6,4,6s-6t)$ on G_2 give us the blocks of a GDD$(1,6,4;\lambda_1 = 6s+2,\lambda_2=6t+5)$.

Case 5. Let $\lambda_1 = 6s+4$ and $\lambda_2 = 6t+1$ where $s \geq t$. Note that a BIBD$(7,4,6t+1)$ cannot be used since it does not exist. Combining the blocks of a GDD$(1,6,4;4,1)$ (using Example 5.5) with the blocks of a BIBD$(7,4,6t)$ on $G_1 \cup G_2$ (where G_1 is the first group and G_2 is the second group) and the blocks of a BIBD$(6,4,6s-6t)$ on G_2 give us the blocks of a GDD$(1,6,4;\lambda_1=6s+4,\lambda_2=6t+1)$.

We have the following corollary for $n=6$ and $\lambda_1 \geq \lambda_2$.

Corollary 5.9. *The necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD*$(1,6,4;\lambda_1,\lambda_2)$ *where* $\lambda_1 \geq \lambda_2$.

Example 5.10. Suppose $\lambda_1 \equiv 1 \pmod 2$, $\lambda_2 \equiv 3 \pmod 6 = 3\lambda$ and $\lambda_1 \geq \lambda$. Union the first group G_1 with the blocks of a BIBD$(7,3,\lambda)$ on the second group G_2, and combining the resulting blocks and $\frac{\lambda_1-\lambda_2}{2}$ copies of the blocks of BIBD$(7,4,2)$ gives us a GDD$(1,7,4;\lambda_1 \equiv 1 \pmod 2, \lambda_2 \equiv 3 \pmod 6)$. Using the same construction we can also obtain a GDD$(1,7,4;\lambda_1 \equiv 0 \pmod 6, \lambda_2 \equiv 0 \pmod 6)$.

Corollary 5.11. *The necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD*$(1,7,4;\lambda_1,\lambda_2)$ *where* $\lambda_1 \geq \lambda_2$.

One may continue to construct smaller designs as given in this section, but in the next section, we give some general results.

6. GDD$(n_1=1, n_2=n, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2$

Theorem 6.1. *If a GDD*$(n_1=1,n_2=n,k;\lambda_1,\lambda_2)$ *and a GDD*$(n,m,k;\ 0,\lambda_1)$ *exist, a GDD*$(n_1=1,n_2=nm,k;\lambda_1,\lambda_2)$ *exists.*

Proof. Let $G_1=\{x\}$ and $G_2=\{1,2,\cdots,nm\}$. Form a partition of $S_1,\cdots,S_m$ of G_2 into m groups of size n. Let H_i be a GDD$(n_1=1,n_2=n,k;\lambda_1,\lambda_2$ on G_1 and S_i, $i=1,\cdots,m$. Let S be a GDD$(n,m,k;0,\lambda_1)$ where m groups are $S_1,\cdots,S_m$. The blocks of S together with the blocks of H_i, $i=1,\cdots,m$ give us the blocks of the required GDD$(n_1=1,n_2=nm,k;\lambda_1,\lambda_2)$ on G_1 and G_2. Given any pair of distinct elements from G_2, either both elements are in the same S_i or they are in different S_i's $(i=1,\cdots,m)$. Note that any pair of distinct elements in the same S_i occurs in the blocks of H_i λ_1 times, and any pair of distinct elements from different S_i's occurs exactly λ_1 times in the blocks of S. A pair (x,y) where $x \in G_1$ and $y \in G_2$ occurs λ_2 times in H_j if y is in S_j $(j=1,\cdots,m)$. □

Example 6.2. Since a GDD$(n_1 = 1, n_2 = 7, 4; 1, 3)$ exists and a GDD$(n = 7, m, 4; 0, 1)$ exists for $m \equiv 1, 4 \pmod{12}$, a GDD$(n_1 = 1, n_2 = 7m, 4; 1, 3)$ exists for $m \equiv 1, 4 \pmod{12}$. In particular, a GDD$(n_1 = 1, n_2 = 91, 4; 1, 3)$ exists. Note that $\lambda_1 < \lambda_2$.

We get an important corollary to Theorem 6.1.

Corollary 6.3. *If $m \equiv 1 \pmod 3$ and λ_1 is even, then a GDD$(n, m, 4;\ 0, \lambda_1)$ exists. Therefore, as the necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD$(n_1 = 1, n_2 = 6, 4; \lambda_1, \lambda_2)$ for any even $\lambda_1 \geq \lambda_2$ by Corollary 5.9, then the same holds for a GDD$(n_1 = 1, n_2 = 6m, 4; \lambda_1, \lambda_2)$. In particular, the necessary conditions are sufficient for*

(i) $\lambda_2 \equiv 1 \pmod 6$ *and* $\lambda_1 \equiv 4 \pmod 6$ *and* $n \equiv 6 \pmod{36}$*;*

(ii) $\lambda_2 \equiv 5 \pmod 6$ *and* $\lambda_1 \equiv 2 \pmod 6$ *and* $n \equiv 6 \pmod{36}$*;*

Similarly, as the necessary conditions are sufficient for a GDD$(n_1 = 1, n_2 = 7, 4; \lambda_1, \lambda_2)$ for $\lambda_1 \geq \lambda_2$ by Corollary 5.11, the necessary conditions are sufficient for

(iii) $\lambda_2 \equiv 3 \pmod 6$ *and* $\lambda_1 \equiv 1, 5 \pmod 6$ *and except for* $n \equiv 7 \pmod{84}$.

Using Theorem 1.4 and Theorem 1.8, we have the following Lemmas 6.4 and 6.5.

Lemma 6.4. *Necessary conditions (given in Table 1 and Theorem 4.2)are sufficient for the existence of a GDD$(1, n, 4; \lambda_1, \lambda_2)$ where $\lambda_1 \equiv \lambda_2 \pmod 6$ and $\lambda_1 \geq \lambda_2$ (the six cases in the main diagonal in Table 1).*

Lemma 6.5. *Necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD$(1, n, 4; \lambda_1, \lambda_2)$ where $\lambda_2 \equiv 0 \pmod 6$ and $\lambda_1 \geq \lambda_2$ (the first column in Table 1).*

For the two cases in the third column in Table 1, note that Lemma 6.4 implies that the necessary conditions are sufficient for the existence of a GDD$(1, n, 4; \lambda_1 = 6s + 2, \lambda_2 = 6t+2)$ where $s \geq t$. Next, let $n = 3t$ where $t \equiv 0, 3 \pmod 4$ (i.e. $n \equiv 0, 9 \pmod{12}$). The union of the blocks of a BIBD$(3t + 1, 4, 2)$ and the blocks of a BIBD$(3t, 4, 3)$ on $G_1 \cup G_2$ (both exist by Theorem 1.4 for $t \equiv 0, 3 \pmod 4$) gives us a GDD$(1, 3t, 4; 5, 2)$. The blocks of these blocks combined with the blocks of t copies a BIBD$(3t+1, 4, 6)$ and $s-t$ copies of a BIBD$(3t, 4, 6)$ on $G_1 \cup G_2$ give us the blocks of a GDD$(1, n, 4; \lambda_1 = 6s + 5, \lambda_2 = 6t + 2)$ where $s \geq t$. We have the following lemma.

Lemma 6.6. *Necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD$(1, n, 4; \lambda_1, \lambda_2)$ where $\lambda_2 \equiv 2 \pmod 6$ and $\lambda_1 \geq \lambda_2$ (the third column in Table 1).*

In the fifth column of Table 1, let $\lambda_2 = 6s+4$, $\lambda_1 = 6t+1$ and $\lambda_1 \geq \lambda_2$ (i.e. $6(t-s)-3 \geq 0$). If $n \equiv 0, 9 \pmod{12}$, then a BIBD$(n+1, 4, 6s+4)$ exists and a BIBD$(n, 4, 6(t-s)-3)$ exists. By Theorem 1.8, a GDD$(1, n, 4; 6s+4+6(t-s)-3 = 6t+1, 6s+4)$ exists. Therefore, we have the following lemma.

Lemma 6.7. *Necessary conditions (given in Table 1 and Theorem 4.2) are sufficient for the existence of a GDD*$(1, n, 4; \lambda_1, \lambda_2)$ *where* $\lambda_2 \equiv 4 \pmod 6$ *and* $\lambda_1 \geq \lambda_2$ *(the fifth column in Table 1).*

Similarly, in each of the following cases in Columns 2, 4 and 6 in Table 1, since a BIBD$(n+1, 4, \lambda_2)$ exists and a BIBD$(n, 4, \lambda_1-\lambda_2)$ exists, by Theorem 1.8, a GDD$(1, n, 4; \lambda_1, \lambda_2)$ exists where $\lambda_1 \geq \lambda_2$:

- $\lambda_2 \equiv 1 \pmod 6$ and $\lambda_1 \equiv 4 \pmod 6$ and $n \equiv 0 \pmod{12}$;
- $\lambda_2 \equiv 5 \pmod 6$ and $\lambda_1 \equiv 2 \pmod 6$ and $n \equiv 0 \pmod{12}$;
- $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 4$;
- $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 1, 5 \pmod 6$ and $n \equiv 4 \pmod{12}$;
- $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 2, 4 \pmod 6$ and $n \equiv 4 \pmod{12}$.

We have the following theorem from the above lemmas and Corollary 6.3.

Theorem 6.8. *The necessary conditions are sufficient for the existence of a GDD*$(1, n, 4; \lambda_1, \lambda_2)$*, where* $\lambda_1 \geq \lambda_2$*, except possibly for the following cases:*

- $\lambda_2 \equiv 1 \pmod 6$ *and* $\lambda_1 \equiv 4 \pmod 6$ *and* $n \equiv 6 \pmod{12}$ *except for* $n \equiv 6 \pmod{36}$*;*
- $\lambda_2 \equiv 5 \pmod 6$ *and* $\lambda_1 \equiv 2 \pmod 6$ *and* $n \equiv 6 \pmod{12}$ *except for* $n \equiv 6 \pmod{36}$*;*
- $\lambda_2 \equiv 3 \pmod 6$ *and* $\lambda_1 \equiv 0 \pmod 6$ *and* $n \equiv 2 \pmod 4$*;*
- $\lambda_2 \equiv 3 \pmod 6$ *and* $\lambda_1 \equiv 1, 5 \pmod 6$ *and* $n \equiv 7 \pmod{12}$ *except for* $n \equiv 7 \pmod{84}$*;*
- $\lambda_2 \equiv 3 \pmod 6$ *and* $\lambda_1 \equiv 2, 4 \pmod 6$ *and* $n \equiv 10 \pmod{12}$*.*

Recall that Corollary 1.10 gives specific examples for the case $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 1, 5 \pmod 6$ and $n \equiv 7 \pmod{12}$ of Theorem 6.8.

7. Summary

In this paper, we studied GDD$(n_1 = 1, n_2 = n, 4; \lambda_1, \lambda_2)$, and proved the sufficiency of the necessary conditions (given in Table 1 and Theorem 4.2)for the existence of the GDDs for many cases where $\lambda_1 \geq \lambda_2$. We also had some general results and provided constructions for larger block sizes as well as when the first group size n_1 is not 1 or $\lambda_1 < \lambda_2$. A natural next step is to complete the open cases listed in Theorem 6.8 and systematically study the existence when $\lambda_1 < \lambda_2$.

References

[1] R. J. R. Abel, T. Britz, Y. A. Bunjamin, and D. Combe. Group divisible designs with block size 4 and group sizes 2 and 5. *Journal of Combinatorial Designs*, 30:367–383, 6, 2022. https://doi.org/10.1002/jcd.21830.

[2] R. J. R. Abel, T. Britz, Y. A. Bunjamin, and D. Combe. Group divisible designs with block size 4 where the group sizes are congruent to 2 mod 3. *Discrete Mathematics*, 345:112740, 3, 2022. https://doi.org/10.1016/j.disc.2021.112740.

[3] R. J. R. Abel, T. Britz, Y. A. Bunjamin, and D. Combe. Group divisible designs with block size 4 and group sizes 4 and 7. *Journal of Combinatorial Designs*, 32:328–346, 6, 2024. https://doi.org/10.1002/jcd.21932.

[4] R. J. R. Abel, T. Britz, Y. A. Bunjamin, D. Combe, and T. Feng. Group divisible designs with block size 4 where the group sizes are congruent to 1 mod 3. *Discrete Mathematics*, 346:113277, 4, 2023. https://doi.org/10.1016/j.disc.2022.113277.

[5] R. J. R. Abel, Y. A. Bunjamin, and D. Combe. The 4-gdds of type 3562. *Discrete Mathematics*, 345:112983, 11, 2022. https://doi.org/10.1016/j.disc.2022.112983.

[6] A. Chaiyasena, S. P. Hurd, N. Punnim, and D. G. Sarvate. Group divisible designs with two association classes. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 82:179–198, 2012.

[7] C. J. Colbourn and D. H. D. (eds). *Handbook of Combinatorial Designs (2nd edition)*. Chapman and Hall, CRC Press, Boca Raton, 2007.

[8] C. J. Colbourn, D. G. Hoffman, and R. Rees. A new class of group divisible designs with block size three. *Journal of Combinatorial Theory, Series A*, 59:73–89, 1992. https://doi.org/10.1016/0097-3165(92)90099-G.

[9] A. D. Forbes. Group divisible designs with block size 4 and type $g^u m^1$ - ii. *Journal of Combinatorial Designs*, 27:311–349, 5, 2019. https://doi.org/10.1002/jcd.21650.

[10] A. D. Forbes. Group divisible designs with block size 4 and type $g^u m^1$ - iii. *Journal of Combinatorial Designs*, 27:643–672, 11, 2019. https://doi.org/10.1002/jcd.21668.

[11] A. D. Forbes and K. A. Forbes. Group divisible designs with block size 4 and type $g^u m^1$. *Journal of Combinatorial Designs*, 26:519–539, 11, 2018. https://doi.org/10.1002/jcd.21607.

[12] H. L. Fu and C. A. Rodger. Group divisible designs with two associate classes: n = 2 or m = 2. *Journal of Combinatorial Theory, Series A*, 83:94–117, 1998.

[13] H. L. Fu, C. A. Rodger, and D. G. Sarvate. The existence of group divisible designs with first and second associates having block size three. *Ars Combinatoria*, 54:33–50, 2000.

[14] N. Hunde and D. M. Woldemariam. A construction of gdds with block size four with unequal group size. Working paper.

[15] N. Pabhapote. Group divisible designs with two associate classes and with two unequal groups. *International Journal of Pure and Applied Mathematics*, 81(1):191–198, 2012.

[16] N. Pabhapote and N. Punnim. Group divisible designs with two associate classes and $\lambda 2=1$. *International Journal of Mathematics and Mathematical Sciences*, 2011(1):148580, 2011. https://doi.org/10.1155/2011/148580.

[17] C. Wang, R. J. R. Abel, T. Britz, Y. A. Bunjamin, and D. Combe. Group divisible designs with block size 4 and group sizes 4 and 10 and some other 4-gdds. *Discrete Mathematics*, 348:114254, 2, 2025. https://doi.org/10.1016/j.disc.2024.114254.

[18] H. Wei and G. Ge. Uniformly resolvable designs with block sizes 3 and 4. *Discrete Mathematics*, 339(3):1069–1085, 2016. https://doi.org/10.1016/j.disc.2015.10.042.

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