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Homogeneous mixed Bowtie systems

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ABSTRACT

A bowtie graph is the union of two edge disjoint 3-cycles which share a single vertex. A mixed bowtie is a partial orientation of a bowtie graph. In this paper, we consider decompositions of the complete mixed graph into mixed bowties consisting of a union of two isomorphic copies of mixed triples.

Keywords: mixed graph decomposition, bowtie graph, partial orientations of cycles

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1. Introduction

Graph decompositions have a long history in graph theory. The first book to survey the research in this area is Juraj Bosák's 1990 *Decompositions of Graphs* [2]. Recently (2024) *Graph Theory and Decomposition* appeared, including material on both graph decompositions and digraph decompositions [7]. Perhaps the best known graph decomposition is a Steiner triple system, which is equivalent to partitioning the edge set of a complete graph into the edge sets of a collection of 3-cycles. It is well-known that such a decomposition exists if and only if the order of the complete graph is 0 or 1 (mod 6) (see [8, Chapter 1]).

A *bowtie* graph consists of two 3-cycles which share exactly one vertex (Figure 1). A *bowtie system* of order v is a partitioning of the edge set of K_v into the edge sets of a collection of bowties. A bowtie system of order v exists if and only if $v \equiv 1$ or 9 (mod 12). The case $v \equiv 1 \pmod{12}$ was originally given in [6] and the case $v \equiv 9 \pmod{12}$ appeared in [1].

Related to graph and digraph decompositions are mixed graph decompositions. Formally, a *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices

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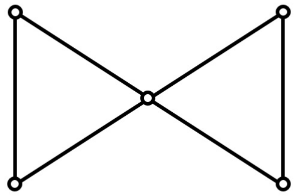

Fig. 1. The bowtie graph

with $|V|= v$, and C is a set of ordered pairs and unordered pairs of elements of V. The unordered pairs are *edges* and the ordered pairs are *arcs*. We denote the edge containing vertices v_1 and v_2 as $\{v_1, v_2\} = \{v_2, v_1\}$ and we denote the arc containing first vertex v_1 and second vertex v_2 as (v_1, v_2). Notice that for given vertices v_1 and v_2 we allow a mixed graph to have edge $\{v_1, v_2\}$ and arcs (v_1, v_2) and (v_2, v_1). We do not follow the convention of combining arcs (v_1, v_2) and (v_2, v_1) into the edge $\{v_1, v_2\}$ (as was done by Harary and Palmer when they first introduced mixed graphs [5]). The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where for each pair of distinct vertices $v_1, v_2 \in V$ we have $\{v_1, v_2\}, (v_1, v_2), (v_2, v_1) \in C$. Therefore there are twice as many arcs as edges in M_v. For m a mixed subgraph of M_v, an *m-decomposition* of M_v is a set $\{m_1, m_2, \ldots, m_k\}$ of edge and arc disjoint isomorphic copies of m such that the union of the edge sets and arc sets of the m_i are the edge set and arc set, respectively, of M_v. An m-decomposition of M_v is called an *m-system of order v*. Since M_v has twice as many arcs as edges, then a necessary condition for the existence of an m-system is that m has twice as many arcs as edges. The first such system to be studied concerned the partial orientations of a 3-cycle, given in Figure 2 and called *mixed triples*. In the notation of Figure 2, a T_i decomposition of M_v is a T_i *triple system* of order v. A T_i triple system of order v exists if and only if $v \equiv 1 \pmod 2$, except for $v \in \{3, 5\}$ when $i = 3$ [3].

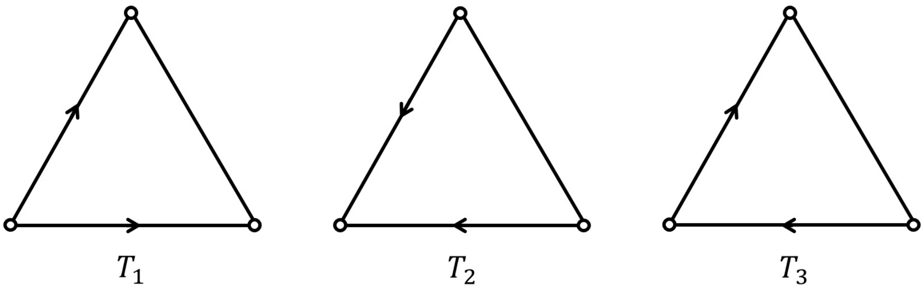

Fig. 2. Mixed triples

There are 25 partial orientations of a 6-cycle with with four arcs and two edges. A decomposition of M_v into such a partially oriented 6-cycle is a *mixed hexagon system*. Necessary and sufficient conditions for the existence of mixed hexagon systems are given in [4].

2. Homogeneous mixed Bowties

A partial orientation of a bowtie graph with four arcs and two edges such that the two resulting mixed triples are isomorphic is a *homogeneous mixed bowtie*. There are 12 homogeneous mixed bowties up to isomorphism. These are given in Figure 3. A decomposition of M_v into a collection of H_i^j mixed bowties is an H_i^j *homogeneous mixed bowtie system* (or simply an H_i^j *bowtie system*) of order v. The purpose of this paper is to give necessary and sufficient conditions for the existence of H_i^j homogeneous mixed bowtie systems for all values of i and j. We adopt the notation given in Figure 3.

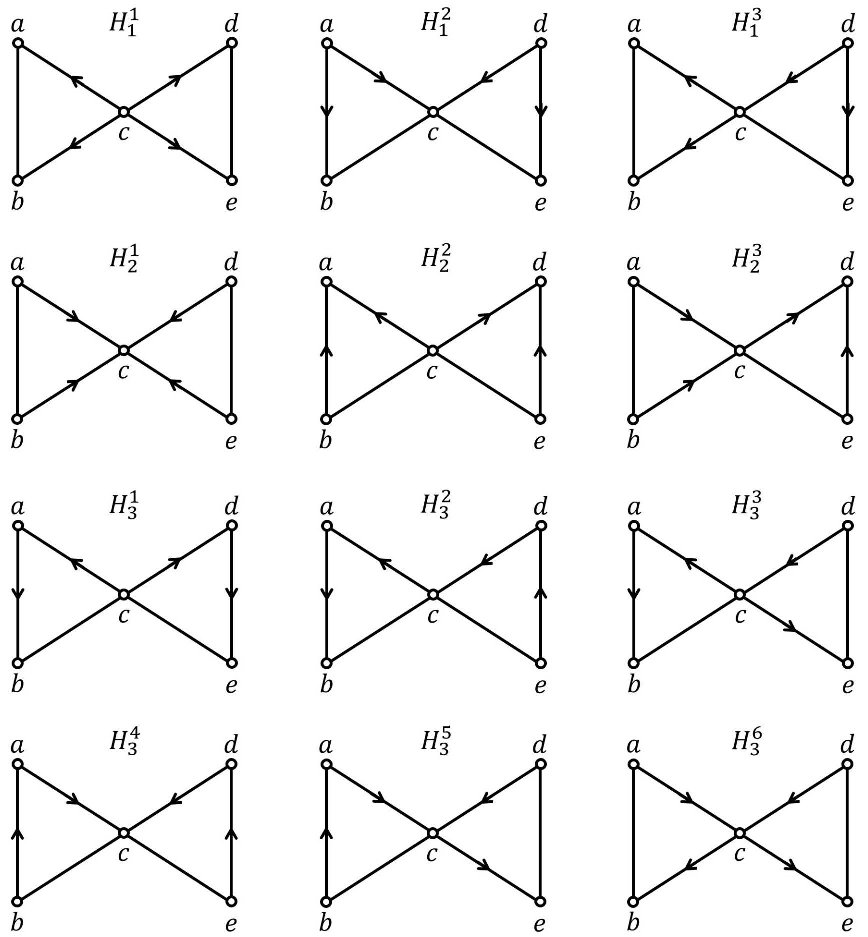

Fig. 3. The homogeneous mixed bowties. We denote mixed bowtie H_i^j, as shown, as $[a, b, c, d, e]_i^j$.

3. Some preliminary results

Since an H_i^j decomposition of M_v implies a T_i decomposition of M_v then a necessary condition for the existence of an H_i^j bowtie system of order v is $v \equiv 1 \pmod 2$. Another necessary condition for the existence of such a homogeneous mixed bowtie system is that the number of triples in the induced mixed triple system is even, so that v must be 1 (mod 4). This gives the following necessary condition.

Lemma 3.1. *If an H_i^j homogeneous mixed bowtie system of order v exists, then $v \equiv 1 \pmod 4$.*

Since a T_3 triple system of order 5 does not exist, then an H_3^j bowtie system of order 5 does not exist for any j.

Lemma 3.2. *If an H_i^j decomposition of M_5 exists, then every vertex of M_5 appears exactly once as a center vertex in the decomposition.*

Proof. If an H_i^j decomposition of M_5 exists, then there are five blocks and each vertex appears exactly once in each block. Now the total-degree of each vertex in M_5 is 12, so a given vertex cannot be the center vertex of three blocks since then it could not be in the other two blocks. A given vertex cannot be the center vertex of two blocks since then it could only appear in two other of the blocks and could not appear in all five blocks. A given vertex cannot be the center vertex of none of the blocks, since then in the union of the blocks it would only have total-degree 10. Therefore, any given vertex must appear exactly once as a center vertex. □

In mixed bowtie $[a, b, c, d, e]_1^2$, we call a and d *upper vertices*, b and e *lower vertices*, and c the *center vertex*. Notice that in $[a, b, c, d, e]_1^2$ the out-degree of each upper vertex is 2 and the total-degree (that is, the edge degree plus the in-degree plus the out-degree) of the center vertex is 4.

Lemma 3.3. *An H_1^2 decomposition of M_5 does not exist.*

The proof of Lemma 3.3 is lengthy and unelegant, so we give it in the appendix.

Lemma 3.4. *An H_1^3 decomposition of M_5 does not exist.*

Proof. Assume such a decomposition exists and let $[a, b, c, d, e]_1^3$ be some block of the decomposition. By Lemma 3.2 there must be some block of the decomposition of the form $[\alpha, \beta, d, \gamma, \delta]_1^3$. Since $[a, b, c, d, e]_1^3$ contains the arcs (d, c) and (d, e), and $[\alpha, \beta, d, \gamma, \delta]_1^3$ contains the arcs (d, α) and (d, β) then we have $\alpha, \beta \notin \{c, e\}$. So it must be that $\alpha, \beta \in \{a, b\}$. But then both $[a, b, c, d, e]_1^3$ and $[\alpha, \beta, d, \gamma, \delta]_1^3$ contain edge $\{a, b\}$, a contradiction. □

4. Results

We now show that the necessary conditions of Lemma 3.1 are sufficient, except in the case $v = 5$ for H_1^2, H_1^3, H_2^2, H_2^3, and H_3^j where $j \in \{1, 2, 3, 4, 5, 6\}$ for which we know that no such system exists. We give direct constructions for each system of the admissible orders by presenting a set of *base blocks*. That is, we take the vertex set of M_v to be $\{0, 1, 2, \ldots, v-1\}$ and give a set of isomorphic copies of H_i^j which, under the cyclic permutation $(0, 1, 2, \ldots, v-1)$, generate all copies of H_i^j in an H_i^j bowtie system.

Lemma 4.1. *An* H_1^1 *homogeneous mixed bowtie system of order* v *exists if* $v \equiv 1$ (*mod* 4)*.*

Proof. Consider the set of mixed bowties:

$$\left\{ [2t-i,\, 2t+1+i,\, 0,\, 4t-i,\, 1+i]_1^1 \;\middle|\; i = 0,1,2,\ldots,t-1 \right\}.$$

This is a set of base blocks for an H_1^1 bowtie system of order $v = 4t+1$. □

Lemma 4.2. *An* H_1^2 *homogeneous mixed bowtie system of order* v *exists if* $v \equiv 1$ (*mod* 4)*,* $v \neq 5$*.*

Proof. Suppose $v = 4t+1$ where $t \geq 2$. Consider the set of mixed bowties:

$$\left\{ [4t-1-i,\, 4t-3-2i,\, 0,\, 2t+2+i,\, 3+2i]_1^2 \;\middle|\; \begin{array}{l} i = 0,1,\ldots,t-2, \\ i \neq \dfrac{2t-5}{3} \end{array} \right\}$$
$$\bigcup \left\{ [4t,\, 4t-1,\, 0,\, 2t+1,\, 1]_1^2 \;\middle|\; \frac{2t-5}{3} \notin \mathbb{N} \right\}$$
$$\bigcup \left\{ \begin{array}{l} \left[4t,\, 4t-1,\, 0,\, \dfrac{8t+1}{3},\, \dfrac{4t-1}{3}\right]_1^2, \\ \left[\dfrac{10t+2}{3},\, \dfrac{8t+1}{3},\, 0,\, 2t+1,\, 1\right]_1^2 \end{array} \;\middle|\; \frac{2t-5}{3} \in \mathbb{N} \right\}.$$

This is a set of base blocks for an H_1^2 bowtie system of order $v = 4t+1$ and $t \geq 2$. □

Lemma 4.3. *An* H_1^3 *homogeneous mixed bowtie system of order* v *exists if* $v \equiv 1$ (*mod* 4)*,* $v \neq 5$*.*

Proof. Suppose $v = 4t+1$ where $t \geq 2$. Consider the set of mixed bowties:

$$\left\{ [2t-1-i,\, 2t+2+i,\, 0,\, 2+i,\, 4+2i]_1^3 \;\middle|\; \begin{array}{l} i = 0,1,\ldots,t-2, \\ i \neq \dfrac{2t-5}{3} \end{array} \right\}$$
$$\cup \left\{ [2t,\, 2t+1,\, 0,\, 1,\, 2]_1^3 \;\middle|\; \frac{2t-5}{3} \notin \mathbb{N} \right\}$$
$$\cup \left\{ \begin{array}{l} \left[2t,\, 2t+1,\, 0,\, \dfrac{2t+1}{3},\, \dfrac{4t+2}{3}\right]_1^3, \\ \left[\dfrac{4t+2}{3},\, \dfrac{8t+1}{3},\, 0,\, 1,\, 2\right]_1^3 \end{array} \;\middle|\; \frac{2t-5}{3} \in \mathbb{N} \right\}.$$

This is a set of base blocks for an H_1^3 bowtie system of order $v = 4t+1$ and $t \geq 2$. □

The complete mixed graph M_v is self-converse, and the converse of H_1^j is H_2^j for $j = 1, 2, 3$, so an H_1^j decomposition of M_v exists if and only if an H_2^j decomposition of M_v exists for (respectively) $j = 1, 2, 3$. Lemmas 4.1, 4.2 and 4.3 therefore imply the sufficient conditions for the existence of an H_2^j decomposition of M_v, as follows.

Lemma 4.4. *An H_2^j homogeneous mixed bowtie system of order v exists if $v \equiv 1$ (mod 4) where $j \in \{1, 2, 3\}$, and $v \neq 5$ when $j \in \{2, 3\}$.*

Lemma 4.5. *An H_3^1 homogeneous mixed bowtie system of order v exists if $v \equiv 1$ (mod 4), $v \geq 9$.*

Proof. We consider several cases.

Case 1(a). Suppose $v = 9$. Consider the following set of mixed bowties:

$$\left\{[1,\ 5,\ 0,\ 8,\ 2]_3^1,\ [6,\ 8,\ 0,\ 7,\ 3]_3^1\right\}.$$

Case 1(b). Suppose $v \equiv 1$ (mod 8), say $v = 8t + 1$ where $t \geq 2$. Consider the set of mixed bowties:

$$\begin{aligned} &\left\{[6t-1-i,\ 7t+2+i,\ 0,\ 8t-1-i,\ t+2+i]_3^1 \;\middle|\; i = 0, 1, \ldots, t-2,\ i \neq \frac{t-3}{2}\right\} \\ &\cup\ \left\{[1+i,\ 5t-i,\ 0,\ 7t-i,\ 2t+1+i]_3^1 \mid i = 0, 1, \ldots, t-1\right\} \\ &\cup\ \left\{[6t,\ 7t+1,\ 0,\ 8t,\ t+1]_3^1 \;\middle|\; \frac{t-3}{2} \notin \mathbb{N}\right\} \\ &\cup\ \left\{\begin{array}{l} \left[\frac{11t+1}{2},\ \frac{15t+1}{2},\ 0,\ 8t,\ t+1\right]_3^1, \\ \left[6t,\ 7t+1,\ 0,\ \frac{15t+1}{2},\ \frac{3t+1}{2}\right]_3^1 \end{array} \;\middle|\; \frac{t-3}{2} \in \mathbb{N}\right\}. \end{aligned}$$

Case 2(a). Suppose $v = 21$. Consider the set of bowties:

$$\left\{[17,\ 16,\ 0,\ 12,\ 7]_3^1,\quad [6,\ 4,\ 0,\ 8,\ 12]_3^1,\quad [9,\ 6,\ 0,\ 3,\ 18]_3^1,\quad [5,\ 19,\ 0,\ 7,\ 20]_3^1,\quad [2,\ 13,\ 0,\ 1,\ 11]_3^1\right\}.$$

Case 2(b). Suppose $v \equiv 5$ (mod 16), say $v = 16t+5$. Consider the set of mixed bowties:

$$\begin{aligned} &\left\{[2t+4+2i,\ 2t+2+i,\ 0,\ 6t+3+2i,\ 4t+2+i]_3^1 \mid i = 0, 1, \ldots, 2t-4\right\} \\ &\cup \left\{[6t+2+4i,\ 8t+4+2i,\ 0,\ 6t+4+4i,\ 8t+3+2i]_3^1 \mid i = 0, 1, \ldots, t-2\right\} \\ &\cup \left\{[2t+5+2i,\ 14t+6+i,\ 0,\ 4t+5+2i,\ 15t+6+i]_3^1 \mid i = 0, 1, \ldots, t-3\right\} \\ &\cup \left\{[2,\ 10t+3,\ 0,\ 10t-2,\ 10t+2]_3^1,\ [12t+5,\ 12t+4,\ 0,\ 6t+1,\ 16t+4]_3^1,\right. \\ &\qquad \left.[10t+2,\ 6t+1,\ 0,\ 1,\ 10t+1]_3^1\right\} \\ &\cup \left\{[2t+1,\ 14t+4,\ 0,\ 4t+3,\ 15t+5]_3^1,\ [2t+3,\ 14t+5,\ 0,\ 4t+1,\ 15t+4]_3^1\right\} \\ &\cup \left\{[6t-2,\ 4t-1,\ 0,\ 10t-1,\ 6t]_3^1,\ [6t,\ 4t,\ 0,\ 10t-3,\ 6t-1]_3^1\right\} \end{aligned}$$

Case 3(a). Suppose $v = 13$. Consider the set of mixed bowties:

$$\{[2,\ 11,\ 0,\ 4,\ 12]_3^1,\quad [1,\ 7,\ 0,\ 3,\ 8]_3^1,\quad [11,\ 10,\ 0,\ 7,\ 4]_3^1\}.$$

Case 3(b). Suppose $v \equiv 13 \pmod{16}$, say $v = 16t + 13$. Consider the set of mixed bowties:

$$\begin{aligned}
&\left\{[2t+5+2i,\ 2t+3+i,\ 0,\ 6t+6+2i,\ 4t+4+i]_3^1 \mid i=0,1,\ldots,2t-2\right\}\\
&\cup\left\{[2t+2+2i,\ 14t+11+i,\ 0,\ 4t+4+2i,\ 15t+12+i]_3^1 \mid i=0,1,\ldots,t\right\}\\
&\cup\left\{[2+2i,\ 10t+5-2i,\ 0,\ 2t+3-2i,\ 8t+8+2i]_3^1 \mid i=0,1,\ldots,t-1\right\}\\
&\cup\left\{[1,\ 10t+7,\ 0,\ 10t+4,\ 6t+3]_3^1,\ \ [6t+3,\ 4t+2,\ 0,\ 10t+7,\ 6t+4]_3^1,\right.\\
&\qquad\left.[12t+11,\ 12t+10,\ 0,\ 3,\ 10t+8]_3^1\right\}.
\end{aligned}$$

In each case, the given set is a set of base blocks for an H_3^1 bowtie system. Therefore an H_3^1 bowtie system exists of order $v \equiv 1 \pmod 4$, $v \geq 9$. □

The complete mixed graph M_v is self-converse, and the converse of H_3^1 is H_3^4, so an H_3^1 decomposition of M_v exists if and only if an H_3^4 decomposition of M_v exists. Lemma 4.5 therefore implies the sufficient conditions for the existence of an H_3^4 decomposition of M_v, as follows.

Lemma 4.6. *An H_3^4 homogeneous mixed bowtie system of order v exists if $v \equiv 1 \pmod 4$, $v \geq 9$.*

Lemma 4.7. *An H_3^2 homogeneous mixed bowtie system of order v exists if $v \equiv 1 \pmod 4$, $v \geq 9$.*

Proof. We consider several cases.

Case 1. Suppose $v \equiv 1 \pmod 8$, say $v = 8t + 1$. Consider the set of bowties:

$$\left\{\begin{matrix}[8t-i,\ t+1+i,\ 0,\ 7t-2i,\ t-i]_3^2,\\ [7t-i,\ 2t+1+i,\ 0,\ 3t+2+2i,\ 3t+1+i]_3^2\end{matrix}\ \middle|\ i=0,1,2,\ldots,t-1\right\}.$$

Case 2(a). Suppose $v = 21$. Consider the set of bowties:

$$\left\{\begin{matrix}[12,\ 7,\ 0,\ 1,\ 5]_3^2,\ \ [9,\ 6,\ 0,\ 2,\ 17]_3^2,\ \ [5,\ 19,\ 0,\ 6,\ 3]_3^2,\\ [2,\ 13,\ 0,\ 8,\ 1]_3^2,\ \ [8,\ 12,\ 0,\ 11,\ 10]_3^2\end{matrix}\right\}.$$

Case 2(b). Suppose $v \equiv 5 \pmod{16}$, say $v = 16t+5$, $t \geq 2$. Consider the set of bowties:

$$\begin{aligned}
&\left\{[2t+4+2i,\ 2t+2+i,\ 0,\ 2t+1+i,\ 12t+3-i]_3^2 \,\middle|\, i=0,1,\ldots,2t-2\right\}\\
&\cup\left\{[6t+2+4i,\ 8t+4+2i,\ 0,\ 14t+6+2i,\ 8t+2-2i]_3^2 \,\middle|\, i=0,1,\ldots,t-2,\ i\neq\frac{t}{3}\right\}\\
&\cup\left\{[2t+5+2i,\ 14t+6+i,\ 0,\ 5t+4+i,\ t-1-i]_3^2 \,\middle|\, i=0,1,\ldots,t-3\right\}\\
&\cup\left\{[2,\ 10t+3,\ 0,\ 16t+1,\ 6t+3]_3^2 \,\middle|\, \frac{t}{3}\notin\mathbb{N}\right\}\\
&\cup\left\{\begin{matrix}[7t+\frac{t}{3}+2,\ 8t+\frac{2t}{3}+4,\ 0,\ 16t+1,\ 6t+3]_3^2,\\ [2,\ 10t+3,\ 0,\ 14t+\frac{2t}{3}+6,\ 7t+\frac{t}{3}+2]_3^2\end{matrix}\ \middle|\ \frac{t}{3}\in\mathbb{N}\right\}
\end{aligned}$$

$$\cup\left\{[12t+5,\ 12t+4,\ 0,\ 6t+2,\ 1]_3^2,\ \ [10t+2,\ 6t+1,\ 0,\ 6t+5,\ 6t+4]_3^2\ \right\}$$
$$\cup\left\{[2t+1,\ 14t+4,\ 0,\ 5t+3,\ t]_3^2,\ \ [2t+3,\ 14t+5,\ 0,\ 5t+2,\ t+1]_3^2\ \right\}.$$

Case 3(a). Suppose $v = 13$. Consider the set of bowties:

$$\left\{[11,\ 10,\ 0,\ 3,\ 9]_3^2,\ [2,\ 11,\ 0,\ 5,\ 1]_3^2,\ [1,\ 7,\ 0,\ 8,\ 5]_3^2\right\}.$$

Case 3(b). Suppose $v \equiv 13 \pmod{16}$, say $v = 16t+13$. Consider the set of bowties:

$$\left\{[2t+5+2i,\ 2t+3+i,\ 0,\ 2t+2+i,\ 12t+9-i]_3^2 \,\middle|\, i=0,1,\dots,2t-1\right\}$$
$$\cup\left\{[4t+8+2i,\ 15t+14+i,\ 0,\ 4t+6+i,\ 2t-i]_3^2 \,\middle|\, i=0,1,\dots,t-2\right\}$$
$$\cup\left\{[2+2i,\ 10t+5-2i,\ 0,\ 10t+8-4i,\ 8t+5-2i]_3^2 \,\middle|\, i=0,1,\dots,t-1\right\}$$
$$\cup\left\{[4t+4,\ 15t+12,\ 0,\ 4t+5,\ 2t+1]_3^2,\ [4t+6,\ 15t+13,\ 0,\ 4t+4,\ 2t+2]_3^2\right\}$$
$$\cup\left\{[12t+11,\ 12t+10,\ 0,\ 4t+3,\ 10t+9]_3^2,\ [1,\ 10t+7,\ 0,\ 6t+8,\ 6t+5]_3^2\right\}.$$

In each case, the given set is a set of base blocks for an H_3^2 bowtie system. Therefore an H_3^2 bowtie system exists of order $v \equiv 1 \pmod 4$, $v \geq 9$. □

Lemma 4.8. *An H_3^3 homogeneous mixed bowtie system of order v exists if $v \equiv 1 \pmod 4$, $v \geq 9$.*

Proof. We consider several cases.

Case 1(a). Suppose $v = 9$. Consider the set of bowties:

$$\{[6,\ 8,\ 0,\ 2,\ 5]_3^3,\ \ [4,\ 5,\ 0,\ 1,\ 3]_3^3\}.$$

Case 1(b). Suppose $v \equiv 1 \pmod 8$, say $v = 8t+1$ where $t \geq 2$. Consider the set of bowties:

$$\left\{[6t-1-i,\ 7t+2+i,\ 0,\ 2+i,\ t+4+2i]_3^3 \,\middle|\, i=0,1,\dots,t-2\right\}$$
$$\cup\left\{[6t,\ 7t+1,\ 0,\ 1,\ t+2]_3^3 \,\middle|\, \frac{2t-2}{3}\not\in\mathbb{N}\right\}$$
$$\cup\left\{[1+i,\ 5t-i,\ 0,\ t+1+i,\ 3t+2+2i]_3^3 \,\middle|\, i=0,1,\dots,t-1,\ i\neq\frac{2t-2}{3}\right\}$$
$$\cup\left\{\left[\frac{2t+1}{3},\ \frac{13t+2}{3},\ 0,\ 1,\ t+2\right]_3^3,\ \left[6t,\ 7t+1,\ 0,\ \frac{5t+1}{3},\ \frac{13t+2}{3}\right]_3^3 \,\middle|\, \frac{2t-2}{3}\in\mathbb{N}\right\}.$$

Case 2(a). Suppose $v = 21$. Consider the set of bowties:

$$\left\{\begin{matrix}[17,\ 16,\ 0,\ 14,\ 13]_3^3, & [12,\ 7,\ 0,\ 15,\ 19]_3^3, & [5,\ 19,\ 0,\ 12,\ 18]_3^3,\\ [1,\ 11,\ 0,\ 13,\ 4]_3^3, & [3,\ 18,\ 0,\ 19,\ 11]_3^3 & \end{matrix}\right\}.$$

Case 2(b). Suppose $v \equiv 5 \pmod{16}$, say $v = 16t+5$ where $t \geq 2$. Consider the set of bowties:

$$\big\{[2t+4+2i,\ 2t+2+i,\ 0,\ 10t+2-2i,\ 14t+4-i]_3^3 \;\big|\; i=0,1,\ldots,2t-2\big\}$$
$$\cup\ \big\{[6t+2+4i,\ 8t+4+2i,\ 0,\ 10t+1-4i,\ 2t-1-2i]_3^3 \;\big|\; i=0,1,\ldots,t-2\big\}$$
$$\cup\ \big\{[2t+5+2i,\ 14t+6+i,\ 0,\ 12t-2i,\ 11t+1-i]_3^3 \;\big|\; i=0,1,\ldots,t-3\big\}$$
$$\cup\ \big\{[2,\ 10t+3,\ 0,\ 6t+7,\ 4]_3^3,\ [12t+5,\ 12t+4,\ 0,\ 10t+4,\ 10t+3]_3^3,\ [10t+2,\ 6t+1,\ 0,\ 16t+4,\ 10t]_3^3\big\}$$
$$\cup\ \big\{[2t+1,\ 14t+4,\ 0,\ 12t+2,\ 11t+2]_3^3,\ [2t+3,\ 14t+5,\ 0,\ 12t+4,\ 11t+3]_3^3\big\}$$

Case 3(a). Suppose $v = 13$. Consider the set of bowties:

$$\{[7,\ 4,\ 0,\ 2,\ 12]_3^3,\ \ [2,\ 11,\ 0,\ 9,\ 8]_3^3,\ \ [1,\ 7,\ 0,\ 10,\ 5]_3^3\}.$$

Case 3(b). Suppose $v \equiv 13 \pmod{16}$, say $v = 16t + 13$. Consider the set of bowties:

$$\big\{[2t+5+2i,\ 2t+3+i,\ 0,\ 10t+7-2i,\ 14t+11-i]_3^3 \;\big|\; i=0,1,\ldots,2t-1\big\}$$
$$\cup\ \big\{[4t+4+2i,\ 15t+12+i,\ 0,\ 14t+11-2i,\ 12t+9-i]_3^3 \;\big|\; i=0,1,\ldots,t\big\}$$
$$\cup\ \left\{[2+2i,\ 10t+5-2i,\ 0,\ 14t+10+2i,\ 6t+5+4i]_3^3 \;\middle|\; i=0,1,\ldots,t-1,\ i\neq\frac{2t}{3}\right\}$$
$$\cup\ \big\{[12t+11,\ 12t+10,\ 0,\ 16t+10,\ 10t+5]_3^3\big\}$$
$$\cup\ \left\{[1,\ 10t+7,\ 0,\ 6t+6,\ 12t+10]_3^3 \;\middle|\; \frac{t}{3}\notin\mathbb{N}\right\}$$
$$\cup\ \left\{\begin{array}{c}\left[1,\ 10t+7,\ 0,\ 15t+\frac{t}{3}+10,\ 8t+\frac{2t}{3}+5\right]_3^3, \\ \left[t+\frac{t}{3}+2,\ 8t+\frac{2t}{3}+5,\ 0,\ 6t+6,\ 12t+10\right]_3^3\end{array} \;\middle|\; \frac{t}{3}\in\mathbb{N}\right\}$$

In each case, the given set is a set of base blocks for an H_3^3 bowtie system. Therefore an H_3^3 bowtie system exists of order $v \equiv 1 \pmod 4$ and $v \geq 9$. □

The complete mixed graph M_v is self-converse, and the converse of H_3^3 is H_3^5, so an H_3^3 decomposition of M_v exists if and only if an H_3^5 decomposition of M_v exists. Lemma 4.8 therefore implies sufficient conditions for the existence of an H_3^5 decomposition of M_v, as follows.

Lemma 4.9. *An H_3^5 homogeneous mixed bowtie system of order v exists if $v \equiv 1 \pmod 4$, $v \geq 9$.*

Lemma 4.10. *An H_3^6 homogeneous mixed bowtie system of order v exists if $v \equiv 1 \pmod 4$, $v \geq 9$.*

Proof. We consider several cases.

Case 1(a). Suppose $v = 9$. Consider the set of bowties:

$$\{[2,\ 5,\ 0,\ 1,\ 3],\quad [8,\ 4,\ 0,\ 3,\ 2]\}.$$

Case 1(b). Suppose $v \equiv 1 \pmod 8$, say $v = 8t + 1$ where $t \geq 2$. Consider the set of bowties:

$$\begin{aligned}&\big\{[2+i,\ t+4+2i,\ 0,\ 8t-1-i,\ 5t-3-2i]_3^6 \mid i = 0,1,\ldots,t-2\big\}\\ \cup\ &\big\{[t+2+i,\ 3t+4+2i,\ 0,\ 2t+2+i,\ t+3+2i]_3^6 \mid i = 0,1,\ldots,t-2\big\}\\ \cup\ &\big\{[t+1,\ 3t+2,\ 0,\ 8t,\ 5t-1]_3^6,\ [1,\ t+2,\ 0,\ 2t+1,\ t+1]_3^6\big\}\end{aligned}$$

Case 2(a). If $v = 21$ then consider the set of bowties:

$$\left\{\begin{array}{l}[4,\ 20,\ 0,\ 9,\ 16]_3^6,\ [15,\ 19,\ 0,\ 12,\ 18]_3^6,\ [18,\ 15,\ 0,\ 16,\ 14]_3^6,\\ [14,\ 13,\ 0,\ 19,\ 11]_3^6,\ [20,\ 10,\ 0,\ 13,\ 4]_3^6\end{array}\right\}.$$

Case 2(b). Suppose $v \equiv 5 \pmod{16}$, say $v = 16t+5$, $t \geq 2$. Consider the set of bowties:

$$\begin{aligned}&\big\{[14t+1-2i,\ 16t+3-i,\ 0,\ 10t+2-2i,\ 14t+4-i]_3^6 \mid i = 0,1,\ldots,2t-2\big\}\\ \cup\ &\big\{[10t+3-4i,\ 2t+2-2i,\ 0,\ 10t+1-4i,\ 2t-1-2i]_3^6 \mid i = 0,1,\ldots,t-2\big\}\\ \cup\ &\big\{[14t-2i,\ 12t+1-i,\ 0,\ 12t-2i,\ 11t+1-i]_3^6 \mid i = 0,1,\ldots,t-3\big\}\\ \cup\ &\big\{[16t+3,\ 10t+1,\ 0,\ 6t+7,\ 4]_3^6,\ [4t,\ 16t+4,\ 0,\ 10t+4,\ 10t+3]_3^6,\\ &\quad [6t+3,\ 12t+4,\ 0,\ 16t+4,\ 10t]_3^6\big\}\\ \cup\ &\big\{[14t+4,\ 12t+3,\ 0,\ 12t+2,\ 11t+2]_3^6,\ [14t+2,\ 12t+2,\ 0,\ 12t+4,\ 11t+3]_3^6\big\}.\end{aligned}$$

Case 3. Suppose $v \equiv 13 \pmod{16}$, say $v = 16t + 13$. Consider the set of bowties:

$$\begin{aligned}&\big\{[14t+8-2i,\ 16t+11-i,\ 0,\ 10t+7-2i,\ 14t+11-i]_3^6 \mid i = 0,1,\ldots,2t-1\big\}\\ \cup\ &\big\{[14t+9-2i,\ 12t+8-i,\ 0,\ 12t+7-2i,\ 11t+7-i]_3^6 \mid i = 0,1,\ldots,t-1\big\}\\ \cup\ &\big\{[16t+11-2i,\ 10t+3-4i,\ 0,\ 14t+10+2i,\ 6t+5+4i]_3^6 \mid i = 0,1,\ldots,t-1\big\}\\ \cup\ &\big\{[14t+11,\ 12t+9,\ 0,\ 6t+6,\ 12t+10]_3^6,\ [4t+2,\ 16t+12,\ 0,\ 12t+9,\ 11t+8]_3^6,\\ &\quad [16t+12,\ 10t+6,\ 0,\ 16t+10,\ 10t+5]_3^6\big\}.\end{aligned}$$

In each case, the given set is a set of base blocks for an H_3^6 bowtie system. Therefore an H_3^6 bowtie system exists of order $v \equiv 1 \pmod 4$ and $v \geq 9$. □

We now have the necessary and sufficient conditions for the existence of a homogeneous mixed bowtie system for each possible homogeneous mixed bowtie. Based on the lemmas of Section 3 and this section, we have the following theorem.

Theorem 4.11. *The necessary and sufficient conditions for the existence of an H_i^j homogeneous mixed bowtie system of order v are as follows:*

(1) *For $(i, j) \in \{(1, 1), (2, 1)\}$, an H_i^j homogeneous mixed bowtie system of order v exists if and only if $v \equiv 1 \pmod 4$, and*

(2) *for $(i, j) \in \{(1, 2), (1, 3), (2, 2), (2, 3)\} \cup \{(3, j) \mid j = 1, 2, 3, 4, 5, 6\}$, an H_i^j homogeneous mixed bowtie system of order v exists if and only if $v \equiv 1 \pmod 4$ and $v \neq 5$.*

5. Appendix. Proof of Lemma 3.3

Lemma 3.3. An H_1^2 decomposition of M_5 does not exist.

Proof. Assume such a decomposition exists. By Lemma 3.2, each vertex of M_5 must be the center of exactly one block of the decomposition. Also, in H_1^2 each upper vertex is of out-degree two, whereas each lower vertex and the center vertex of H_1^2 are of out-degree zero. Since the out-degree of each vertex of M_5 is 4, then each vertex of M_5 must be an upper vertex of exactly two blocks in a decomposition. We now hack through various collections of H_1^2 blocks to show that no hypothesized decomposition exists.

The vertex labels are arbitrary so we take them as 0, 1, 2, 3, 4. Without loss of generality, one of the blocks is $B = [3,1,0,4,2]_1^2$. As just explained, vertex 4 must be an upper vertex of another block. The only possible such blocks that share no edges or arcs with block B are $B_a = [0,2,1,4,3]_1^2$, $B_b = [0,2,3,4,1]_1^2$, and $B_c = [2,0,3,4,1]_1^2$. Similarly, vertex 3 must be an upper vertex of another block and the only possible such blocks sharing no edges or arcs with block B are $B_d = [3,4,2,0,1]_1^2$, $B_e = [3,2,4,0,1]_1^2$, and $B_f = [3,2,4,1,0]_1^2$. Now B_a and B_d share arc $(0,1)$, B_a and B_e share arc $(0,1)$, and blocks B_b and B_d share arc $(0,2)$. So an H_1^2 decomposition of M_5 which contains block B, must contain one of the following collections of three blocks: (i) B, B_a, B_f, (ii) B, B_b, B_e, (iii) B, B_b, B_f, (iv) B, B_c, B_d, (v) B, B_c, B_e, and (vi) B, B_c, B_f. We now show that none of these six necessary cases can be completed to give a decomposition.

(i) In B, B_a, B_f, vertex 0 is an upper vertex once, so there must be another block in a decomposition in which 0 is an upper vertex. The only possible such blocks that share no edges or arcs with blocks B, B_a, B_f are $B_g = [0,3,4,2,1]_1^2$ and $B_h = [0,4,3,1,2]_1^2$. However, block B_f has 4 as the center vertex and by Lemma 3.2 only one block of a decomposition of M_5 can have 4 as the center vertex. Hence, block B_g cannot be a block of the decomposition, and the blocks must be B, B_a, B_f, B_k. Then, also be Lemma 3.2, the fifth block of the decomposition must have center vertex 2. But the arcs $(0,2)$, $(1,2)$, $(3,2)$, and $(4,2)$ appear in blocks B_a, B_h, B_f, and B, respectively, so there is no choice for the upper vertices of a block with center vertex 2 which is consistent with blocks B, B_a, B_f. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_a, B_f.

(ii) In B, B_b, B_e, each of vertices 0, 3, and 4 appear twice as upper vertices. In the remaining two blocks of a decomposition of M_5, 1 and 2 must be the upper vertices, so that the center vertex of these two blocks must be either 0, 3, or 4. However, 0 is the center vertex of B, 3 is the center vertex of B_b, and 4 is the center vertex of B_e. So, by Lemma 3.2, neither of the two blocks needed in the decomposition exist. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_b, B_e.

(iii) In B, B_b, B_f, vertex 1 is an upper vertex once, so there must be another block in a decomposition in which 1 is an upper vertex. But there is no such block that shares no edges or arcs with blocks B, B_b, B_f. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_b, B_f.

(iv) In B, B_c, B_d, vertex 0 is an upper vertex once, so there must be another block in a decomposition in which 0 is an upper vertex. The only possible such block that shares no edges or arcs with blocks B, B_c, B_d is $B_i = [2,1,4,0,3]_1^2$ (notice 3 cannot be the center vertex by Lemma 3.2, since 3 is the center vertex of B_c). In blocks B, B_c, B_d, B_i the center vertices are 0, 3, 2, and 4, respectively, and 1 has not appeared as an upper vertex. So in the fifth block of a decomposition, 1 must appear twice as an upper vertex and as the center vertex, which is not possible. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_c, B_d.

(v) In B, B_c, B_e, vertex 2 is an upper vertex once, so there must be another block in a decomposition in which 2 is an upper vertex. Since the centers of these blocks are 0, 3, 4, then by Lemma 3.2 the block with 2 as an upper vertex must have center vertex 1. However, there are no such blocks which share no edges or arcs with B, B_c, B_e. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_c, B_e.

(vi) In B, B_c, B_f, vertex 2 is an upper vertex once, so there must be another block in a decomposition in which 2 is an upper vertex. Since the centers of these blocks are 0, 3, and 4 then by Lemma 3.2 the block with 2 as an upper vertex must have center 1. However, there are no such blocks which share no edges or arcs with B, B_c, B_f. Therefore, no H_1^2 decomposition of M_5 exists containing blocks B, B_c, B_f.

Since an H_1^2 decomposition of M_5 must contain one of the six trios of blocks considered, we have a contradiction. That is, no H_1^2 decomposition of M_5 exists. □

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