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Mixed extensions of generic finite games embedded into products of real projective spaces

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ABSTRACT

Finite games in normal form and their mixed extensions are a corner stone of noncooperative game theory. Often *generic* finite games and their mixed extensions are considered. But the properties which one expects in generic games and the existence of games with these properties are often treated only in passing. The paper considers strong properties and proves that generic games have these properties. The space of mixed strategy combinations is embedded in a natural way into a product of real projective spaces. All relevant hypersurfaces extend to this bigger space. The paper shows that for all games in the complement of a semialgebraic subset of codimension at least one all relevant hypersurfaces in the bigger space are smooth and maximally transversal. The proof uses the theorem of Sard and follows an argument of Khovanskii.

Keywords: Generic finite games, mixed extensions, transversality, projective spaces

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1. Introduction

Finite games in normal form and their mixed extensions are a corner stone of noncooperative game theory. Nash showed that the mixed extension of any finite game in normal form has Nash equilibria [9]. For special games, the set of Nash equilibria can be complicated. But for generic games it is finite and odd [5, 10, 11, 15]. These references and many others consider *generic* finite games and their mixed extensions.

The notion of a *generic* finite game is usually defined ad hoc and results on it are usually only proved as an aside. One fixes the finite set of players and for each player his

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finite set of pure strategies. Then one obtains an affine linear space $\mathcal{U}$ of tuples of possible utility functions. Generic games are then games with generic utility functions, and these are contained in the complement of a closed subset of measure zero. The properties which generic games are supposed to have, depend on the paper and the author.

In this paper we consider a strong version of genericity and prove that all games in the complement $\mathcal{U}-\mathcal{D}$ of a semialgebraic subset $\mathcal{D}$ of codimension at least one are generic.

We fix the number m of players $i\in\mathcal{A}=\{1,...,m\}$ and for each player i the finite set $S^i=\{s^i_0,...,s^i_{n_i}\}$ of n_i+1 pure strategies. $S=S^1\times...\times S^m$ is the set of pure strategy combinations. Then the finite game is determined by the tuple $U=(U^1,...,U^m):S\to\mathbb{R}^m$ of the utility functions $U^i:S\to\mathbb{R}$ of the players i. It is an element of the real vector space $\mathcal{U}=(\mathbb{R}^S)^m$. In the mixed extension $(\mathcal{A},G,V)$ of the finite game $(\mathcal{A},S,U)$, the set G is $G=G^1\times...\times G^m$, and the set G^i of mixed strategies of player i is the set of probability distributions over S^i, so it is a simplex of dimension n_i. The utility function $V^i:G\to\mathbb{R}$ is the multilinear extension of U^i (see formula (1)).

The multilinearity leads to a natural embedding $G^i\hookrightarrow\mathbb{P}^{n_i}\mathbb{R}$ of G^i into the real projective space $\mathbb{P}^{n_i}\mathbb{R}$ of dimension n_i and to an embedding of G into the product $\mathbb{P}^{\mathcal{A}}W:=\prod_{i\in\mathcal{A}}\mathbb{P}^{n_i}\mathbb{R}$ of real projective spaces. All the induced functions, which are relevant for understanding the game, its best reply maps and its Nash equilibria, extend from G to the space $\mathbb{P}^{\mathcal{A}}W$.

Our main result Theorem 3.1 says that there is a semialgebraic subset $\mathcal{D}\subset\mathcal{U}$ of codimension at least one such that for each $U\in\mathcal{U}-\mathcal{D}$, all natural hypersurfaces in G are smooth and extend to smooth hypersurfaces in $\mathbb{P}^{\mathcal{A}}W$ and that they are (in a precise sense) everywhere maximally transversal.

This implies immediately that for such a game the set of all Nash equilibria is finite and that each Nash equilibrium is regular in a strong sense.

It also provides a foundation for the argument of Wilson [15] and Rosenmüller [11] that the number of Nash equilibria of a generic game is odd. Both papers claim implicitly the existence of generic games with good properties without proof. The paper here provides such a proof.

Our proof follows an argument of Khovanskii [6] for a theorem on generic systems of Laurent polynomials with fixed Newton polyhedra. Like him, we have to deal with many charts, and we use Sard's theorem [12] that the subset of critical values of a C^∞-map between C^∞-manifolds has measure 0. The analogue of the toric compactification which Khovanskii constructs is in our situation the space $\mathbb{P}^{\mathcal{A}}W$.

The product $\mathbb{P}^{\mathcal{A}}W$ had been considered implicitly in [8] and explicitly in [13], but not in many other papers on games in normal form, although the embedding $G\hookrightarrow\mathbb{P}^{\mathcal{A}}W$ is natural.

We do not provide applications of Theorem 3.1 in this paper. But we expect that Theorem 3.1 will be useful and that applications will come.

In the case of two-player games, it is not so difficult to make precise what *generic* means and which properties such games should have [7, 14]. There one can refer essentially to linear algebra.

For games with three or more players, it is more difficult. Other proofs, that in a generic game the set of Nash equilibria is finite and each Nash equilibrium is regular, were given in [5](ch. 5), [4](Theorem 3) and [10](Theorem 2). Harsanyi used (as we do) Sard's theorem, Gül, Pearce and Stacchetti used a variant due to Stacchetti and Reinoza of Sard's theorem, and Ritzberger used a parametric transversality theorem.

Section 2 introduces more formally finite games in normal form and their mixed extensions, and it sets some notations. Section 3 formulates the main result Theorem 3.1. Section 4 gives some background material from differential topology. Section 4 proves Theorem 3.1.

2. The mixed extension of a finite game

This section introduces finite games, their mixed extensions, best reply maps and Nash equilibria, and it sets some notations.

Definition 2.1. (a) $(\mathcal{A},S,U)$ denotes a finite game. Here $m\in\mathbb{N}=\{1,2,3,...\}$, $\mathcal{A}:=\{1,...,m\}$ is the set of players, $S^i=\{s^i_0,...,s^i_{n_i}\}$ with $n_i\in\mathbb{N}$ is the set of pure strategies of player $i\in\mathcal{A}$, $S=S^1\times...\times S^m$ is the set of pure strategy combinations, $U^i:S\to\mathbb{R}$ is the utility function of player i, and $U=(U^1,...,U^m):S\to\mathbb{R}^m$. We denote $J^i:=\{1,...,n_i\}$, $J^i_0:=\{0\}\cup J^i$ and $J^{\mathcal{A}}_0:=\prod_{i=1}^m J^i_0$. The pure strategy combinations are given as tuples $(s^1_{j_1},...,s^m_{j_m})\in S$ with $\underline{j}=(j_1,...,j_m)\in J^{\mathcal{A}}_0$.

(b) $(\mathcal{A},G,V)$ denotes the *mixed extension* of the finite game in (a). Here

$$W^i:=\bigoplus_{j=0}^{n_i}\mathbb{R}\cdot s^i_j,\, W:=W^1\times...\times W^m,$$

$$A^i:=\left\{\sum_{j=0}^{n_i}\gamma^i_j s^i_j\in W^i\,|\sum_{j=0}^{n_i}\gamma^i_j=1,\right\},A:=A^1\times...\times A^m\subset W,$$

$$G^i:=\left\{\sum_{j=0}^{n_i}\gamma^i_j s^i_j\in A^i\,|\,\gamma^i_j\in[0,1]\right\},G:=G^1\times...\times G^m\subset A.$$

So, W^i and W are real vector spaces, $A^i\subset W^i$ and $A\subset W$ are affine linear subspaces of codimension 1 respectively m, $G^i\subset A^i$ is a simplex in A^i of the same dimension n_i as A^i, and $G\subset A$ is a product of simplices, so especially a convex polytope, and it has the same dimension $\sum_{i=1}^m n_i$ as A. The map $V^i_W:W\to\mathbb{R}^m$ is the multilinear extension of U^i,

$$V^i_W(g):=\sum_{(j_1,...,j_m)\in J^{\mathcal{A}}_0}\left(\prod_{k=1}^m\gamma^k_{j_k}\right)\cdot U^i(s^1_{j_1},...,s^m_{j_m}),\tag{1}$$

where, $g=(g^1,...,g^m)\in W$ with $g^k=\sum_{j=0}^{n_k}\gamma^k_j s^k_j$, $V^i_A:A\to\mathbb{R}$ is the restriction of V_W to A, and $V^i:G\to\mathbb{R}$ is the restriction of V^i_W to G. Then $V=(V^1,...,V^m):G\to\mathbb{R}^m$. An element $g\in G$ is called a *mixed strategy combination*. The support of an element

$$g^i=\sum_{j=0}^{n_i}\gamma^i_j s^i_j\in W^i\quad\text{is the set}\quad\text{supp}(g^i):=\{j\in J^i_0\,|\,\gamma^i_j\neq 0\}.$$

We also denote $G^{-i} := G^1 \times ... \times G^{i-1} \times G^{i+1} \times ... \times G^m$, and its elements $g^{-i} := (g^1, ..., g^{i-1}, g^{i+1}, ..., g^m) \in G^{-i}$. We follow the standard (slightly incorrect) convention and identify $G^i \times G^{-i}$ with G and (g^i, g^{-i}) with g.

(c) Fix $i \in \mathcal{A}$. The *best reply map* $r^i : G^{-i} \to \mathcal{P}(G^i)$ associates to each element $g^{-i} \in G^{-i}$ the set of its best replies in G^i,

$$r^i(g^{-i}) := \{g^i \in G^i \,|\, V^i(\widetilde{g}^i, g^{-i}) \leq V^i(g^i, g^{-i}) \text{ for any } \widetilde{g}^i \in G^i\}. \tag{2}$$

Its graph is the set

$$\mathrm{Gr}(r^i) := \bigcup_{g^{-i} \in G^{-i}} r^i(g^{-i}) \times \{g^{-i}\} \subset G^i \times G^{-i} = G.$$

A *Nash equilibrium* is an element of the set $\mathcal{N} := \bigcap_{i \in \mathcal{A}} \mathrm{Gr}(r^i)$. The set $\mathcal{N}$ is the set of Nash equilibria.

(d) Write $\underline{\gamma}^i = (\gamma^i_1, ..., \gamma^i_{n_i})$,

$$\begin{aligned} \underline{\gamma} &:= (\underline{\gamma}^1; ...; \underline{\gamma}^m) = (\gamma^1_1, ..., \gamma^1_{n_1}; ...; \gamma^m_1, ..., \gamma^m_{n_m}) \quad \text{and} \\ \underline{\gamma}^{-i} &:= (\underline{\gamma}^1; ...; \underline{\gamma}^{i-1}; \underline{\gamma}^{i+1}; ..., \underline{\gamma}^m). \end{aligned}$$

Fix a subset $\mathcal{B} \subset \mathcal{A}$. The monomials $\prod_{i \in \mathcal{B}} \gamma^i_{j_i}$ for $(j_i \,|\, i \in \mathcal{B}) \in \prod_{i \in \mathcal{B}} J^i_0$ in $\mathbb{R}[\gamma^i_0, \underline{\gamma}^i \,|\, i \in \mathcal{B}]$ (with $\prod_{i \in \emptyset}(..) := 1$) are called $\mathcal{B}$-*multilinear*. A polynomial which is a real linear combination of $\mathcal{B}$-multilinear monomials is also called $\mathcal{B}$-multilinear. Especially V^i_W is $\mathcal{A}$-multilinear. A polynomial in $\mathbb{R}[\gamma^i_0, \underline{\gamma}^i \,|\, i \in \mathcal{A}]$ is *multi affine linear* if each monomial in it with nonvanishing coefficient is $\mathcal{C}$-multilinear for a suitable set $\mathcal{C} \subset \mathcal{A}$.

The set $\mathcal{N}$ of Nash equilibria is not empty. This was first proved by Nash [9]. The following lemma is rather trivial, but worth to be noted.

Lemma 2.2. *Let $(\mathcal{A}, G, V)$ be the mixed extension of a finite game $(\mathcal{A}, S, U)$.*

(a) *The tuple $\underline{\gamma}^i$ is a tuple of (affine linear) coordinates on A^i, because in A^i we have $\gamma^i_0 = 1 - \sum_{j=1}^{n_i} \gamma^i_j$. The tuple $\underline{\gamma} = (\underline{\gamma}^1; ...; \underline{\gamma}^m)$ is a tuple of (affine linear) coordinates on A. The map V^i_A is a multi affine linear polynomial in $\underline{\gamma}$. It has the shape*

$$V^i_A(g) = \kappa^i(\underline{\gamma}^{-i}) + \sum_{j=1}^{n_i} \gamma^i_j \cdot \lambda^i_j(\underline{\gamma}^{-i}) \quad \text{for} \quad g \in A, \tag{3}$$

where κ^i and all λ^i_j are unique multi affine linear polynomials in $\underline{\gamma}^{-i}$. Define additionally

$$\lambda^i_0 := 0 \quad \text{for } i \in \mathcal{A}. \tag{4}$$

(b) *An element $g = (g^i, g^{-i}) \in G$ is in $\mathrm{Gr}(r^i)$ if and only if the following holds.*

$$\lambda^i_j(\underline{\gamma}^{-i}) - \lambda^i_k(\underline{\gamma}^{-i}) = 0 \qquad \text{if } j, k \in \mathrm{supp}(g^i), \tag{5}$$

$$\lambda^i_j(\underline{\gamma}^{-i}) - \lambda^i_k(\underline{\gamma}^{-i}) \geq 0 \qquad \text{if } j \in \mathrm{supp}(g^i), k \notin \mathrm{supp}(g^i). \tag{6}$$

Proof. (a) Part (a) holds because A^i is the affine hyperplane in W^i defined by $\gamma_0^i = 1 - \sum_{j=1}^{n_i} \gamma_j^i$.

(b) If $g \in \mathrm{Gr}(r^i)$ then a change of $g^i = \sum_{j=0}^{n_i} \gamma_j^i s_j^i$ may not increase $V_A^i(g)$ in (3). Therefore $\lambda_j^i(\underline{\gamma}^{-i})$ for $j \in \mathrm{supp}(g^i)$ must be the maximum of all $\lambda_k^i(\underline{\gamma}^{-i})$. This includes the case $j = 0$ if $0 \in \mathrm{supp}(g^i)$, and it includes the case $k = 0$ if $0 \notin \mathrm{supp}(g^i)$. □

3. Compactification of A and generic games

Consider as in Section 2 a finite set $\mathcal{A} = \{1, ..., m\}$ of players and for each player $i \in \mathcal{A}$ a finite set $S^i = \{s_0^i, ..., s_{n_i}^i\}$ with $n_i \in \mathbb{N}$ of pure strategies. Let $\mathcal{U}^i = \mathbb{R}^S$ be the set of all possible utility functions $U^i : S \to \mathbb{R}$. The set of all possible utility maps $U = (U^1, ..., U^m)$ is then $\mathcal{U} := \prod_{i=1}^m \mathcal{U}^i \cong (\mathbb{R}^S)^m$.

Consider a fixed map U, the tuple of all hyperplanes in A which bound $G \subset A$ and the subvarieties $(\lambda_j^i - \lambda_k^i)^{-1}(0) \subset A$ for $i \in \mathcal{A}$ and $j, k \in J_0^i$ with $j < k$. By Lemma 2.2 (b), the graphs $\mathrm{Gr}(r^i)$ of the best reply maps and the set $\mathcal{N}$ of Nash equilibria are determined by the geometry of these hyperplanes and these subvarieties.

The hyperplanes which bound G are smooth and transversal. Theorem 3.1 below will imply that for generic $U \in \mathcal{U}$ also the subvarieties $(\lambda_j^i - \lambda_k^i)^{-1}(0)$ are smooth hypersurfaces in A and that they and the hyperplanes in A which bound G are as transversal as possible.

The fact that V_W^i is $\mathcal{A}$-multilinear motivates to consider the natural compactification of A^i to the real projective space $\mathbb{P}W^i$, which is the set $(W^i - \{0\})/\mathbb{R}^*$ of lines through 0 in W^i, and the natural compactification of A to the product of real projective spaces

$$\mathbb{P}^{\mathcal{A}}W := \prod_{i=1}^{m} \mathbb{P}W^i. \tag{7}$$

We denote $\mathbb{P}^{-i}W := \prod_{j \in \mathcal{A} - \{i\}} \mathbb{P}W^i$, and we identify (following the slightly incorrect convention in Definition 2.1 (b)) $\mathbb{P}W^i \times \mathbb{P}^{-i}W$ with $\mathbb{P}^{\mathcal{A}}W$. Under the projection

$$pr_W : \prod_{i=1}^{m} (W^i - \{0\}) \to \mathbb{P}^{\mathcal{A}}W, \tag{8}$$

the affine linear space $A \subset W$ embeds as a Zariski open subset into $\mathbb{P}^{\mathcal{A}}W$. The complement is the union of m hyperplanes

$$H^{i,\infty} := (\mathbb{P}W^i - A^i) \times \mathbb{P}^{-i}W \subset \mathbb{P}W^i \times \mathbb{P}^{-i}W = \mathbb{P}^{\mathcal{A}}W, \tag{9}$$

for $i \in \mathcal{A}$. For a subset $B^i \subset A^i$, let $\overline{B^i}^{Zar}$ denote its Zariski closure in $\mathbb{P}W^i$, that is the smallest real algebraic subvariety in $\mathbb{P}W^i$ which contains B^i. The Zariski closure in $\mathbb{P}^{\mathcal{A}}W$ of a subset $B \subset \mathbb{P}^{\mathcal{A}}W$ is denoted by $\overline{B}^{Zar}$. For $i \in \mathcal{A}$ and $j \in J_0^i$ denote by

$$H^{i,j} := \overline{\{g^i \in A^i \,|\, \gamma_j^i = 0\}}^{Zar} \times \mathbb{P}^{-i}W \subset \mathbb{P}W^i \times \mathbb{P}^{-i}W = \mathbb{P}^{\mathcal{A}}W, \tag{10}$$

the Zariski closures in $\mathbb{P}^{\mathcal{A}}W$ of the hyperplanes in A which bound G. Define

$$J_0^{i,2} := \{(j,k) \in J_0^i \times J_0^i \,|\, j < k\}.$$

For $i \in \mathcal{A}$ and $(j,k) \in J_0^{i,2}$ consider the difference $\lambda_j^i - \lambda_k^i$ as a function on A (so lift it as a function from $\prod_{l \neq i} A^l$ to A), and consider the Zariski closure in $\mathbb{P}^{\mathcal{A}}W$

$$H^{i,(j,k)} := \overline{(\lambda_j^i - \lambda_k^i)^{-1}(0)}^{Zar} \subset \mathbb{P}^{\mathcal{A}}W. \tag{11}$$

The notion *everywhere transversal* in Theorem 3.1 is defined in Definition 4.4 (b). In Theorem 3.1, a subset of the set of all hyperplanes in (9) and (10) and all subvarieties in (11) is considered. Such a subset is characterized by its set of indices, namely sets $T^i \subset J_0^i \cup \{\infty\}$ and sets $R^i \subset J_0^{i,2}$ for $i \in \mathcal{A}$ define a subset

$$\bigcup_{i \in \mathcal{A}} \left(\{H^{i,j} \,|\, j \in T^i\} \cup \{H^{i,(j,k)} \,|\, (j,k) \in R^i\} \right). \tag{12}$$

A set $R^i \subset J_0^{i,2}$ defines a graph with vertex set J_0^i and set R^i of edges. The set in (12) is called *good* if the graph (J_0^i, R^i) is a union of trees.

The reason for the introduction of this notion of a *good* set is that in the case of $|\text{supp}(g^i)| \geq 3$ there is some redundancy in the Eqs. (5). Equality for all pairs (j,k) with $j,k \in \text{supp}(g^i)$ is implied by equality for a set of pairs (j,k) such that the graph with vertex set $\text{supp}(g^i)$ and edge set this set of pairs is a tree.

Theorem 3.1. *Let $\mathcal{A} = \{1, ..., m\}$ and S be as in Section 2 and as above. There is a semialgebraic subset $\mathcal{D} \subset \mathcal{U}$ of codimension at least one (equivalently, it is of Lebesgue measure 0) such that for any tuple $U \in \mathcal{U} - \mathcal{D}$ of utility functions the following holds. The hyperplanes $H^{i,j}, i \in \mathcal{A}, j \in J_0^i \cup \{\infty\}$, and the subvarieties $H^{i,(j,k)}, i \in \mathcal{A}, (j,k) \in N_0^{i,2}$, are smooth hypersurfaces in $\mathbb{P}^{\mathcal{A}}W$. Any good subset of them is everywhere transversal.*

The proof of Theorem 3.1 will be given in Section 4. Before, Section 4 will recall some basic notions and facts from differential topology.

4. Transversality of submanifolds

This section recalls two basic facts from differential topology, Sard's theorem and the implicit function theorem. It defines transversality of submanifolds and it formulates a useful lemma.

Definition 4.1. Let M and N be C^∞-manifolds, and let $F : M \to N$ be a C^∞-map. A point $p \in M$ is a *regular point* of F if the linear map $dF|_{T_pM} \colon T_pM \to T_{F(p)}N$ is surjective. A point $p \in M$ is a *critical point* if it is not a regular point. A point $q \in N$ is a *regular value* if either $F^{-1}(q) = \emptyset$ or every point $p \in F^{-1}(q)$ is a regular point. A point $q \in N$ is a *critical value* if it is not a regular value.

The following theorem is famous. It is also crucial in the proof of Theorem 3.1.

Theorem 4.2. *(Sard's theorem [12], see also [3](Theorem 2.1)) Let $F : M \to N$ be a C^∞-map between C^∞-manifolds. The subset of N of critical values of F has Lebesgue measure 0.*

In our situation, the setting is semialgebraic. Therefore the subset of critical values is then a semialgebraic subset of N of Lebesgue measure 0. This means that it has everywhere smaller dimension than N. A variant of Sard's theorem in the semialgebraic setting says precisely this [2](ch. 9.5) [1](2.5.12).

The implicit function theorem says how a map $F : M \to N$ looks near a regular point $p \in M$. It is a trivial fibration with smooth fibers.

Theorem 4.3. *(Implicit function theorem, e.g. [3](Theorem 1.3)) Let $F : M \to N$ be a C^∞-map between C^∞-manifolds, and let $p \in M$ be a regular point of F. Then* $\dim M \geq \dim N$, *and there are open neighborhoods $U_1 \subset M$ of p and $U_2 \subset N$ of $F(p)$ with $U_2 \supset F(U_1)$, open balls $B_1 \subset \mathbb{R}^{\dim M}$ around 0 and $B_2 \subset \mathbb{R}^{\dim N}$ around 0 and C^∞-diffeomorphisms $\varphi_1 : B_1 \to U_1$ and $\varphi_2 : B_2 \to U_2$ with the following property. The map $\varphi_2^{-1} \circ F \circ \varphi_1 : B_1 \to B_2$ is the standard projection in (13),*

$$\begin{array}{ccc} U_1 & \xrightarrow{F} & U_2 \\ \uparrow \varphi_1 & & \uparrow \varphi_2 \\ B_1 & \xrightarrow{\varphi_2^{-1}\circ F\circ\varphi_1} & B_2 \\ (x_1, ..., x_{\dim M}) & \mapsto & (x_1, ..., x_{\dim N}) \end{array} \tag{13}$$

Definition 4.4. (a) Let M be a C^∞-manifold, let $p \in M$ and let $H \subset M$ be a subset with $p \in H$ and which is in a neighborhood of p a C^∞-submanifold of M. A *defining map* F for the pair (H, p) is a function $F : U \to \mathbb{R}^n$ with $U \subset M$ an open neigborhood of p such that F is regular on each point of U and $H \cap U = F^{-1}(F(p))$. Then $n = \dim M - \dim H \cap U$.

Remark: Defining maps for (H, p) exist and are related by local diffeomorphisms.

(b) Let M be a C^∞-manifold, and let $H_1, ..., H_a$ be C^∞-submanifolds. Now it will be defined when they are *transversal at a point* $p \in M$.

(i) They are transversal at $p \in M - \cup_{i=1}^a H_i$.

(ii) They are transversal at $p \in \cap_{i=1}^a H_i$ if for some (or for any, that is equivalent) tuple of defining maps $F_i : U \to \mathbb{R}^{n_i}$ of (H_i, p) $(i \in \{1, ..., a\})$ with joint definition domain U the map $(F_1, ..., F_a) : U \to \mathbb{R}^{n_1+...+n_a}$ is regular at p.

(iii) They are transversal at $p \in \cup_{i=1}^a H_i$ if the subset $\{H_i \,|\, p \in H_i\}$ of the set of manifolds $H_1, ..., H_a$ is transversal at p (this was defined in part (ii)).

Finally, they are *transversal* (or *everywhere transversal*) if they are transversal at each point of M.

Part (a) of the following lemma states an obvious consequence on the intersection of a family of transversal submanifolds. Part (b) gives a useful criterion for proving transversality of a family of submanifolds.

Lemma 4.5. *Let M be a C^∞-manifold, and let $H_1, ..., H_a$ be C^∞-submanifolds.*

(a) *If they are transversal then either* $\cap_{i=1}^a H_i = \emptyset$ *or this intersection is a submanifold of* M *of dimension* $\dim M - \sum_{i=1}^{a} \operatorname{codim} H_i$. *Especially, in the second case, this number is non-negative.*

(b) *Suppose that for some* $b \in \{1,..,a-1\}$ *the submanifolds* $H_1, ..., H_b$ *are transversal at a point* $p \in \cap_{i=1}^a H_i$. *The intersection* $L := \cap_{i=1}^b H_i$ *is by part (a) in a suitable open neighborhood of* p *a submanifold of* M. *The following two conditions are equivalent:*

(i) $H_1, ..., H_a$ *are transversal at* p.

(ii) *For* $j \in \{b+1, ..., a\}$ *there are defining maps* $F_j : U_j \to \mathbb{R}^{n_j}$ *on open neighborhoods* $U_j \subset M$ *of* p *such that for* $U := \bigcap_{j=b+1}^a U_j$ *the intersection* $U \cap L$ *is a submanifold of* U *and the map* $(F_{b+1}, ..., F_a)|_{U\cap L}: U \cap L \to \mathbb{R}^{n_{b+1}+...+n_a}$ *is regular at* p.

Proof. (a) Part (a) follows immediately from the definition of transversality and the implicit function theorem.

(b) By the implicit function theorem we can choose an open neighborhood U of p in M, coordinates $\underline{x} = (x_1, ..., x_m)$ on U with $\underline{x}(p) = 0$ and defining maps $F_i := U \to \mathbb{R}^{n_i}$ of (H_i, p) for $i \in \{1, ..., a\}$ such that $(F_1, ..., F_b) = (x_1, ..., x_{\sum_{i=1}^b n_i})$. Condition (i) is equivalent to

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}(p)\right)_{\substack{i\,\in\,\{1,\,...,\,a\}\\ j\,\in\,\{1,\,...,\,\dim M\}}} = \sum_{i=1}^{a} n_i. \tag{14}$$

Condition (ii) is equivalent to

$$\operatorname{rank}\left(\frac{\partial F_i|_{U\cap L}}{\partial x_j}(p)\right)_{\substack{i\,\in\,\{b+1,\,...,\,a\}\\ j\,\in\,\{1+\sum_{i=1}^b n_i,\,...,\,\dim M\}}} = \sum_{i=b+1}^{a} n_i. \tag{15}$$

The matrix in (14) has a block triangular shape $\left(\begin{array}{c|c} \mathbf{1} & 0 \\ \hline * & * \end{array}\right)$ with the matrix in (15) in the lower right place. Therefore (14) and (15) are equivalent. □

Remark 4.6. (i) In condition (ii) in Lemma 4.5 (b), one can replace the existence of the defining maps by demanding that the condition holds for all choices of defining maps.

(ii) In the proof in Section 4 of the transversality in Theorem 3.1, Lemma 4.5 (b) will be useful. The hyperplanes $H^{i,j}$ in a given set in (12) take the role of the submanifolds $H_1, ..., H_b$. The reason is that they are fixed if one moves $U \in \mathcal{U}$, while the other subvarieties $H^{i,(j,k)}$ move if one moves $U \in \mathcal{U}$.

(iii) In Lemma 4.5 (b) one cannot replace condition (ii) by the simpler condition (ii)': $L \cap H_{b+1}, ..., L \cap H_a$ *are submanifolds and are transversal at* p. The reason is that $L \cap H_j$ for some $j \in \{b+1, ..., a\}$ can be a submanifold of the correct dimension although L and H_j are not transversal.

Proof of Theorem 3.1.

This section is devoted to the proof of Theorem 3.1.

First we discuss affine charts on $\mathbb{P}W^i$ and on $\mathbb{P}^{\mathcal{A}}W$. The linear coordinates $(\gamma^i_0, ..., \gamma^i_{n_i})$ on W^i induce $n_i + 1$ affine charts of the projective space $\mathbb{P}W^i$. But none of them contains the whole set G^i. The following linear coordinates $\underline{\widetilde{\gamma}}^i := (\widetilde{\gamma}^i_0, ..., \widetilde{\gamma}^i_{n_i})$ on W^i are equally natural, and one of the affine charts which they induce on $\mathbb{P}W^i$ will turn out to be A^i,

$$\widetilde{\gamma}^i_j := \gamma^i_j \text{ for } j \in J^i, \quad \widetilde{\gamma}^i_0 := \sum_{j=0}^{n_i} \gamma^i_j. \tag{16}$$

From now on we use the induced homogeneous coordinates $(\widetilde{\gamma}^{i_0} : ... : \widetilde{\gamma}^i_{n_i})$ on $\mathbb{P}W^i$ if not said otherwise. The $n_i + 1$ induced affine charts of $\mathbb{P}W^i$ are

$$A^i_j := \{(\widetilde{\gamma}^i_0 : ... : \widetilde{\gamma}^i_{n_i}) \in \mathbb{P}W^i \,|\, \underline{\widetilde{\gamma}}^i \in W^i, \widetilde{\gamma}^i_j = 1\} \subset \mathbb{P}W^i \quad \text{for } j \in J^i_0. \tag{17}$$

A^i_j comes equipped with natural coordinates $\underline{\gamma}^{i,j} = (\gamma^{i,j}_0, ..., \gamma^{i,j}_{j-1}, \gamma^{i,j}_{j+1}, ..., \gamma^{i,j}_{n_i})$ from the isomorphism

$$\mathbb{R}^{n_i} \to A^i_j, \quad \underline{\gamma}^{i,j} \mapsto (\gamma^{i,j}_0 : ... : \gamma^{i,j}_{j-1} : 1 : \gamma^{i,j}_{j+1} : ... : \gamma^{i,j}_{n_i}),$$

and we have a natural embedding

$$\alpha^{i,j} : A^i_j \hookrightarrow W^i, \quad \underline{\gamma}^{i,j} \mapsto (\gamma^{i,j}_0, ..., \gamma^{i,j}_{j-1}, 1, \gamma^{i,j}_{j+1}, ..., \gamma^{i,j}_{n_i}),$$

with $\mathrm{pr}^i_W \circ \alpha^{i,j} = \mathrm{id}$ where $\mathrm{pr}^i_W : W^i - \{0\} \to \mathbb{P}W^i$ is the natural projection. The image $\alpha^{i,0}(A^i_0) \subset W^i$ coincides with the set $A^i \subset W^i$ in Definition 2.1 (b), and the identification $\underline{\gamma}^{i,0} = \underline{\gamma}^i$ identifies A^i_0 with A^i.

All possible products over $i \in \mathcal{A}$ of these charts give the affine charts

$$A_{\underline{j}} := \prod_{i=1}^m A^i_{j_i} \cong \mathbb{R}^{\sum_{i=1}^m n_i} \quad \text{for } \underline{j} = (j_1, ..., , j_m) \in J^{\mathcal{A}}_0 = \prod_{i=1}^m J^i_0, \tag{18}$$

of $\mathbb{P}^{\mathcal{A}}W$ with coordinates $\underline{\gamma}^{\underline{j}} = (\underline{\gamma}^{1,j_1}; ...; \underline{\gamma}^{m,j_m})$. The embeddings $\alpha^{i,j_i} : A^i_{j_i} \hookrightarrow W^i$ combine to an embedding

$$\alpha^{\underline{j}} : A_{\underline{j}} \hookrightarrow W$$

with $\alpha^{\underline{j}} \circ \mathrm{pr}_W = \mathrm{id}$ on $A_{\underline{j}}$. The image $\alpha^{\underline{0}}(A_{\underline{0}}) \subset W$ coincides with the set $A \subset W$ in Definition 2.1 (b), and the identification $\underline{\gamma}^{\underline{0}} = \underline{\gamma}$ identifies $A_{\underline{0}}$ with A.

Now we discuss the hyperplanes $H^{i,j}$ and the subvarieties $H^{i,(j,k)}$ from Section 3 with respect to the new linear coordinates $\underline{\widetilde{\gamma}} = (\underline{\widetilde{\gamma}}^1; ...; \underline{\widetilde{\gamma}}^m)$ on W. We define for $i \in \mathcal{A}$

$$\widetilde{H}^{i,j} := H^{i,j} \text{ for } j \in J^i, \quad \widetilde{H}^{i,0} := H^{i,\infty}, \quad \widetilde{H}^{i,\infty} := H^{i,0}.$$

We have for $i\in\mathcal{A}$

$$\widetilde{H}^{i,j}=H^{i,j}=\mathrm{pr}_W\left(\{\widetilde{\underline{\gamma}}\in\prod_{k=1}^m(W^k-\{0\})\,|\,\widetilde{\gamma}^i_j=0\}\right)\quad\text{for } j\in J^i, \tag{19}$$

$$\widetilde{H}^{i,0}=H^{i,\infty}=\mathrm{pr}_W\left(\{\widetilde{\underline{\gamma}}\in\prod_{k=1}^m(W^k-\{0\})\,|\,\widetilde{\gamma}^i_0=0\}\right), \tag{20}$$

$$\widetilde{H}^{i,\infty}=H^{i,0}=\mathrm{pr}_W\left(\{\widetilde{\underline{\gamma}}\in\prod_{k=1}^m(W^k-\{0\})\,|\,\widetilde{\gamma}^i_0-\sum_{j=1}^{n_i}\widetilde{\gamma}^i_j=0\}\right). \tag{21}$$

Obviously for each chart $A_{\underline{j}}$, the complement is

$$\mathbb{P}^{\mathcal{A}}W-A_{\underline{j}}=\bigcup_{i=1}^m\widetilde{H}^{i,j_i}. \tag{22}$$

For each of the hyperplanes $\widetilde{H}^{i,j}\cap A_{\underline{j}}$ with $j\in J^i_0-\{j_i\}$, in the chart $A_{\underline{j}}$ a defining map (in the sense of Definition 4.4 (a)) is $\widetilde{\gamma}^i_j\circ\alpha^{\underline{j}}$. For each of the hyperplanes $\widetilde{H}^{i,\infty}\cap A_{\underline{j}}$ in the chart $A_{\underline{j}}$ a defining map is $\left(\widetilde{\gamma}^i_0-\sum_{j=1}^{n_i}\widetilde{\gamma}^i_j\right)\circ\alpha^{\underline{j}}$.

Obviously, each subset of the set $\{\widetilde{H}^{i,j}\,|\,i\in\mathcal{A},j\in J^i_0\cup\{\infty\}\}$ of all these hyperplanes is transversal everywhere in $\mathbb{P}^{\mathcal{A}}W$.

The map $V^i_W:W\to\mathbb{R}$ is an $\mathcal{A}$-multilinear map also in the new linear coordinates $\widetilde{\underline{\gamma}}$. Write it as a sum

$$V^i_W(\widetilde{\underline{\gamma}})=\widetilde{\gamma}^i_0\cdot K^i(\widetilde{\underline{\gamma}}^{-i})+\sum_{j=1}^{n_i}\widetilde{\gamma}^i_j\cdot\Lambda^i_j(\widetilde{\underline{\gamma}}^{-i})\quad\text{for}\quad\widetilde{\underline{\gamma}}=(\widetilde{\underline{\gamma}}^1,...,\widetilde{\underline{\gamma}}^m)\in W. \tag{23}$$

Here $K^i(\widetilde{\underline{\gamma}}^{-i})$ and $\Lambda^i_j(\widetilde{\underline{\gamma}}^{-i})$ are $\mathcal{A}-\{i\}$-multilinear in $\widetilde{\underline{\gamma}}^{-i}$. Define additionally

$$\Lambda^i_0:=0. \tag{24}$$

We have for $i\in\mathcal{A}$ and $(j,k)\in J^{i,2}_0$

$$H^{i,(j,k)}=\mathrm{pr}_W\left(\{\widetilde{\underline{\gamma}}\in\prod_{k=1}^m(W^k-\{0\})\,|\,(\Lambda^i_j-\Lambda^i_k)(\widetilde{\underline{\gamma}})=0\}\right). \tag{25}$$

The intersection $H^{i,(j,k)}\cap A_{\underline{l}}$ of the subvariety $H^{i,(j,k)}\subset\mathbb{P}^{\mathcal{A}}W$ with the affine chart $A_{\underline{l}}$ for $\underline{l}=(l_1,...,l_m)\in J^{\mathcal{A}}_0$ is the zero set of the function $(\Lambda^i_j-\Lambda^i_k)\circ\alpha^{\underline{l}}$.

Especially, the identifications $\underline{\gamma}^{\underline{0}}=\underline{\gamma}$ and $A_{\underline{0}}=A$ yield the identification $\Lambda^i_j\circ\alpha^{\underline{0}}=\lambda^i_j$. This connects the description of $H^{i,(j,k)}$ in (25) with the one in (11).

Now the main point will be to prove that there is a subset $\mathcal{D}\subset\mathcal{U}$ of Lebesgue measure 0 with the properties in Theorem 3.1. Later we will argue that it is semialgebraic. Then $\mathcal{D}$ having Lebesgue measure 0 is equivalent to $\mathcal{D}$ having codimension at least one in $\mathcal{U}$.

Choose some $\underline{l}\in J^{\mathcal{A}}_0$ and consider the chart $A_{\underline{l}}$ of $\mathbb{P}^{\mathcal{A}}W$. Choose for each $i\in\mathcal{A}$ a set $T^i\subset J^i_0\cup\{\infty\}$ with $l_i\notin T^i$ (we exclude the case l_i because $\widetilde{H}^{i,l_i}\subset\mathbb{P}^{\mathcal{A}}W-A_{\underline{l}}$ by (22)) and a set $R^i \subset J_0^{i,2}$ such that (J_0^i, R^i) is a union of trees and such that (for all i together) $\bigcup_{i=1}^m R^i \neq \emptyset$. Though $T^i = \emptyset$ (for some or all $i \in \mathcal{A}$) and $R^i = \emptyset$ (for some, but not all $i \in \mathcal{A}$) are allowed. Now we go into Lemma 4.5 (b) with

$$M = A_{\underline{l}} \quad \text{and} \quad \{H_1, ..., H_b\} = \{\widetilde{H}^{i,j} \cap A_{\underline{l}} \,|\, i \in \mathcal{A}, j \in T^i\},$$

(which is empty if all $T^i = \emptyset$) and

$$L = \bigcap_{i \in \mathcal{A}, j \in T^i} \widetilde{H}^{i,j} \cap A_{\underline{l}}, \tag{26}$$

which is an affine linear subspace of $A_{\underline{l}}$ of codimension $\sum_{i=1}^m |T^i|$. We can choose a part of the coordinates $\underline{\gamma}^{\underline{l}}$ on $A_{\underline{l}}$ which forms affine linear coordinates on L. We choose such a part and call it $\underline{\gamma}^{\underline{l},L}$.

For a moment, fix one map $U \in \mathcal{U}$ and consider the map

$$\begin{aligned} F^{(U)} : L \to N \quad &\text{with } N := \mathbb{R}^{\sum_{i \in \mathcal{A}} |R^i|} \\ \underline{\gamma}^{\underline{l}} \mapsto &\left(((\Lambda_j^i - \Lambda_k^i) \circ \alpha^{\underline{l}})(\underline{\gamma}^{\underline{l}}) \,|\, i \in \mathcal{A}, (j,k) \in R^i \right). \end{aligned} \tag{27}$$

Each entry $(\Lambda_j^i - \Lambda_k^i) \circ \alpha^{\underline{l}})|_L$ of this map $F^{(U)}$ is a multi affine linear function in those coordinates of $\underline{\gamma}^{\underline{l},L}$ which are not in $\underline{\gamma}^{i,\underline{l}}$.

By Sard's theorem (Theorem 4.2) the set of critical values of $F^{(U)}$ has Lebesgue measure 0 in N.

Now we consider simultaneously all $U \in \mathcal{U}$. We claim that the set of $U \in \mathcal{U}$ such that $0 \in N$ is a critical value of $F^{(U)}$ has Lebesgue measure 0 in $\mathcal{U}$. The argument for this will connect some natural coordinate system on $\mathcal{U}$ with the coefficients of the monomials in the entries of $F^{(U)}$.

The space of possible maps $F^{(U)}$ can be identified with the product

$$\mathcal{U}^{(I)} := \prod_{i \in \mathcal{A}, (j,k) \in R^i} \mathbb{R}[\underline{\gamma}^{\underline{l},L} \backslash \underline{\gamma}^{i,\underline{l}}]_{\mathbf{mal}}, \tag{28}$$

where $\mathbb{R}[\underline{\gamma}^{\underline{l},L} \backslash \underline{\gamma}^{i,\underline{l}}]_{\mathbf{mal}}$ denotes the space of multi affine linear polynomials in those coordinates in $\underline{\gamma}^{\underline{l},L}$ which are not in $\underline{\gamma}^{i,\underline{l}}$ (here **mal** stands for **m**_ulti_ **a**_ffine_ **l**_inear_). The space of possible constant summands of the entries of $F^{(U)}$ can be identified with the space $N = \mathbb{R}^{\sum_{i \in \mathcal{A}} |R^i|}$.

There are natural linear maps $p_{(I)}$ and $p_{(II)}$ defined by

$$p_{(I)} : \mathcal{U} \to \mathcal{U}^{(I)}, p_{(I)}(U) = F^{(U)}, \tag{29}$$

$$p_{(II)} : \mathcal{U}^{(I)} \to N, p_{(II)}(F) = F(0). \tag{30}$$

Because of the hypothesis that (J_0^i, R^i) is for any $i \in \mathcal{A}$ a union of trees, the maps $p_{(I)}$ and $p_{(II)}$ are surjective. To see this, recall that the $\mathcal{A}$-multilinear maps V_W^i are sums of all monomials in the variables $\widetilde{\gamma}_j^i$ with coefficients, which form a natural linear coordinate system of the $\mathbb{R}$-vector space $\mathcal{U}$. For example, the coefficient of the monomial

$\widetilde{\gamma}^i_j \prod_{a\in\mathcal{A}-\{i\}} \widetilde{\gamma}^a_{l_a}$ in V^i_W is the coefficient of the constant term in the multi affine linear function $\Lambda^i_j \circ \alpha^{\underline{l}}$. The entries of $F^{(U)}$ for a fixed U are the multi affine linear functions $((\Lambda^i_j - \Lambda^i_k)\circ\alpha^{\underline{l}})|_L$ for $i\in\mathcal{A},(j,k)\in R^i$. The condition that (J^i_0,R^i) is a union of trees takes care that sufficiently many of the coefficients of the monomials in the maps V^i_W turn up in the maps $p_{(I)}$ and $p_{(II)}$, so that these maps are surjective.

For $F\in\ker(p_{(II)})\subset\mathcal{U}^{(I)}$ denote by $C(F)\subset N$ the set of critical values of $F:L\to N$. It has Lebesgue measure 0 in N by Sard's theorem (Theorem 4.2). For each map $F-c: L\to N$ with $c\in N$ the value $0\in N$ is a critical value only if $c\in C(F)$. Therefore 0 is a critical value of $F^{(U)}:L\to N$ for $U\in\mathcal{U}$ only if

$$U\in \bigcup_{F\in\ker(p_{(II)})}\ \bigcup_{c\in C(F)} p^{-1}_{(I)}(F-c), \tag{31}$$

which is a set of Lebesgue measure 0 in $\mathcal{U}$. (see e.g. Theorem 2.8 (Fubini) in [3]).

There are only finitely many charts $A_{\underline{l}}$ and finitely many choices of sets T^i and R^i as above. Each leads only to a set of Lebesgue measure 0 in $\mathcal{U}$. Also their union $\mathcal{D}$ has Lebesgue measure 0 in $\mathcal{U}$. If $U\in\mathcal{U}-\mathcal{D}$, then for each map $F^{(U)}:L\to N$ as above, the value $0\in N$ is a regular value.

For an arbitrary chart $A_{\underline{l}}$, the special choice $T^a=\emptyset$ for all $a\in\mathcal{A}$, $R^i=\{(j,k)\}$ for one $i\in\mathcal{A}$, and $R^a=\emptyset$ for all $a\neq i$ shows that $H^{i,(j,k)}\cap A_{\underline{l}}$ is a smooth hypersurface for $U\in\mathcal{U}-\mathcal{D}$. It also shows that the map $(\Lambda^i_j-\Lambda^i_k)\circ\alpha_{\underline{l}}$ is a defining map for the pair $(H^{i,(j,k)},p)$ at each point $p\in H^{i,(j,k)}\cap A_{\underline{l}}$ (in the sense of Definition 4.4 (a)).

For an arbitrary chart $A_{\underline{l}}$ and all choices of the sets T^i and R^i, the construction of $\mathcal{D}$ above together with Lemma 4.5 (b) and Definition 4.4 (b) show that for $U\in\mathcal{U}-\mathcal{D}$ any good subset of the smooth hypersurfaces $H^{i,j}$ and $H^{i,(j,k)}$ is transversal on $A_{\underline{l}}$. Considering all charts together, we obtain all statements of Theorem 3.1 except that $\mathcal{D}$ is semialgebraic.

$\mathcal{D}$ is semialgebraic because of the following. For a chart $A_{\underline{l}}$ and sets T^i and R^i as above, the maps $F^{(U)}:L\to N$ unite to an algebraic map

$$\mathcal{F}:\mathcal{U}\times L\to\mathcal{U}\times N,\quad (U,x)\mapsto(U,F^{(U)}(x)). \tag{32}$$

The set of its critical points in $\mathcal{U}\times L$ with critical value $(U,0)\in\mathcal{U}\times\{0\}\subset\mathcal{U}\times N$ is an algebraic subset of $\mathcal{U}\times L$. Its image under the projection to $\mathcal{U}$ is semialgebraic. Also the union $\mathcal{D}$ of these sets over all choices of charts $A_{\underline{l}}$ and sets T^i and R^i is semialgebraic. □

Remark 4.7. This proof of Theorem 3.1 is inspired by the proof of Khovanskii of a theorem on generic systems of Laurent polynomials with fixed Newton polyhedra. He considers complex coefficients, we consider real coefficients. But apart from that, our situation and proof can be seen as a special case of his situation and proof. It is the main theorem in §2 in [6]. Though, he uses, but does not formulate the Lemma 4.5. The analogue of the toric compactification which he has to construct is in our situation the space $\mathbb{P}^{\mathcal{A}}W$.

The proof of Theorem 3.1 in this Section 4 implies the following corollary. The regularity which it expresses is most useful in the case of points (e.g. Nash equilibria) in the standard affine chart $A_{\underline{0}}$ and written with the standard defining maps in this chart.

Corollary 4.8. *Consider the situation in Theorem 3.1. Let U be the utility map of a game with $U \in \mathcal{U} - \mathcal{D}$ (so it is generic). Let $A_{\underline{l}} \subset \mathbb{P}^{\mathcal{A}}W$ be any one of the affine charts of $\mathbb{P}^{\mathcal{A}}W$. Let $\underline{\gamma}^{\underline{l}}$ be any point in $A_{\underline{l}}$. Consider any set of smooth hypersurfaces as in (12) which is good and which contains the point $\underline{\gamma}^{\underline{l}}$. Then the Jacobian of the defining maps in this chart (which were used in the proof above) for these hypersurfaces is nondegenerate.*

Proof. By Theorem 3.1 the subvarieties in a set as in (12) are smooth hypersurfaces and everywhere transversal. The transversality in any affine chart $A_{\underline{l}}$ was shown by proving that the point $\underline{\gamma}^{\underline{l}}$ is a regular point of the tuple of defining maps in this chart of the hypersurfaces. □

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