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On local antimagic chromatic number of the join of two special families of graphs | II

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ABSTRACT

Null graphs (respectively, 1-regular graphs) are the only regular graphs with local antimagic chromatic number 1 (respectively, undefined). In this paper, we proved that the join of 1-regular graph and a null graph has local antimagic chromatic number 3. As a by-product, we also obtained many families of (possibly disconnected or regular) bipartite and tripartite graph with local antimagic chromatic number 3.

Keywords: local antimagic chromatic number, null graphs, 1-regular graphs, join product

2020 Mathematics Subject Classification: 05C78, 05C69.

1. Introduction

Let $G = (V, E)$ be a connected graph of order p and size q. A bijection $f : E \to \{1, 2, \dots, q\}$ is called a *local antimagic labeling* if $f^+(u) \neq f^+(v)$ whenever $uv \in E$, where $f^+(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. The mapping f^+ which is also denoted by f_G^+ is called a *vertex labeling of* G *induced by* f, and the labels assigned to vertices are called *induced colors* under f. The *color number* of a local antimagic labeling f is the number of distinct induced colors under f, denoted by $c(f)$. Moreover, f is called a *local antimagic* $c(f)$*-coloring* and G is *local antimagic* $c(f)$*-colorable*. The *local antimagic chromatic number* $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G [1]. Let $G + H$ and mG denote the disjoint union of graphs G and H, and m copies of G, respectively. For integers $c < d$, let $[c, d] = \{n \in \mathbb{Z} \mid c \leq n \leq d\}$. Very few results on the local antimagic chromatic number of regular graphs are known (see [1, 6]). Throughout this paper, we let

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$V(aP_2 \vee O_m) = \{u_i, v_i, x_j \mid 1 \le i \le a, 1 \le j \le m\}$ and $E(aP_2 \vee O_m) = \{u_i x_j, v_i x_j, u_i v_i \mid 1 \le i \le a, 1 \le j \le m\}$. We also let $V(a(P_2 \vee O_m)) = \{u_i, v_i, x_{i,j} \mid 1 \le i \le a, 1 \le j \le m\}$ and $E(a(P_2 \vee O_m)) = \{u_i x_{i,j}, v_i x_{i,j}, u_i v_i \mid 1 \le i \le a, 1 \le j \le m\}$.

In [3], the author proved that all connected graphs except P_2 admit a local antimagic labeling. This implies that all graphs without a P_2 component admits a local antimagic labeling. Thus, $O_m, m \ge 1$ (respectively, $aP_2, a \ge 1$) is the only family of regular graphs with local antimagic chromatic number 1 (respectively, undefined). In [1], it was shown that $\chi_{la}(aP_2 \vee O_1) = 3$ for $a \ge 1$. In [9, 8], the authors proved that $\chi_{la}((2k+1)P_2 \vee O_m) = 3$ for all $k \ge 1, m \ge 2$. In this paper, we extend the ideas in [9, 8, 10, 11] to further prove that $\chi_{la}((2k)P_2 \vee O_m) = 3$ for all $k \ge 1$, $m \ge 2$. Moreover, we also obtain other families of bipartite and tripartite graphs with local antimagic chromatic number 3.

The following lemma in [12, Lemma 2.1] or [5, Lemma 2.3] is needed.

Lemma 1.1 ([5, Lemma 2.3]). *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the sets of vertices colored x and y, respectively, then G is a bipartite graph with bipartition (X, Y) and $|X| > |Y|$. Moreover, $x|X| = y|Y| = \frac{q(q+1)}{2}$.*

The contrapositive implies that a connected bipartite graph G with equal partite set size must have $\chi_{la}(G) \ge 3$. Note that if G is a disconnected bipartite graph with each component having equal partite set size, then we also have $\chi_{la}(G) \ge 3$.

2. Graphs of size $(4n+1) \times 2k$

Consider $(2k)(P_2 \vee O_{2n})$ of order $2k(2n+2)$ and size $2k(4n+1)$ for $k, n \ge 1$. We shall construct the following tables, which shows the label of each edge under a labeling f. First, we assume $k \ge 2$.

Table 1. Each column sum is $12kn + 9k + 1$

i	1	2	3	$\cdots$	k-1	k
$f(u_i x_{i,2n-1})$	4k(n-1) + 8k+1	4k(n-1) + 9k+1	4k(n-1) + 9k+2	$\cdots$	4k(n-1) + 10k-2	4k(n-1) + 10k-1
$f(u_i x_{i,2n})$	4k(n-1) + 6k	4k(n-1) + 6k+1	4k(n-1) + 6k+2	$\cdots$	4k(n-1) + 7k-2	4k(n-1) + 7k-1
$f(u_i v_i)$	4k(n-1) + 7k	4k(n-1) + 6k-1	4k(n-1) + 6k-3	$\cdots$	4k(n-1) + 4k+5	4k(n-1) + 4k+3

i	k+1	k+2	$\cdots$	2k-2	2k-1	2k
$f(u_i x_{i,2n-1})$	4k(n-1) + 8k+2	4k(n-1) + 8k+3	$\cdots$	4k(n-1) + 9k-1	4k(n-1) + 9k	4k(n-1)+10k
$f(u_i x_{i,2n})$	4k(n-1) + 7k+1	4k(n-1) + 7k+2	$\cdots$	4k(n-1) + 8k-2	4k(n-1) + 8k-1	4k(n-1) + 8k
$f(u_i v_i)$	4k(n-1) + 6k-2	4k(n-1) + 6k-4	$\cdots$	4k(n-1) + 4k+4	4k(n-1) + 4k+2	4k(n-1) + 3k+1

Table 2. Each column sum is $12kn - 3k + 2$

i	1	2	3	$\cdots$	k-1	k
$f(u_iv_i)$	4k(n-1) + 7k	4k(n-1) + 6k-1	4k(n-1) + 6k-3	$\cdots$	4k(n-1) + 4k+5	4k(n-1) + 4k+3
$f(v_ix_{i,2n-1})$	4k(n-1) + 1	4k(n-1) + k+1	4k(n-1) + k+2	$\cdots$	4k(n-1) + 2k-2	4k(n-1) + 2k-1
$f(v_ix_{i,2n})$	4k(n-1)+2k+1	4k(n-1) + 2k+2	4k(n-1) + 2k+3	$\cdots$	4k(n-1)+ 3k-1	4k(n-1) + 3k

i	k+1	k+2	$\cdots$	2k-2	2k-1	2k
$f(u_iv_i)$	4k(n-1) + 6k-2	4k(n-1) + 6k-4	$\cdots$	4k(n-1) + 4k+4	4k(n-1) + 4k+2	4k(n-1) + 3k+1
$f(v_ix_{i,2n-1})$	4k(n-1) + 2	4k(n-1) + 3	$\cdots$	4k(n-1) + k-1	4k(n-1) + k	4k(n-1) + 2k
$f(v_ix_{i,2n})$	4k(n-1)+3k+2	4k(n-1) + 3k+3	$\cdots$	4k(n-1)+ 4k-1	4k(n-1) + 4k	4k(n-1) + 4k+1

For $j \in [1, n-1]$,

Table 3. Each column sum is $8kn + 4k + 1$

i	1	2	$\cdots$	2k-1	2k
$f(u_ix_{i,2j-1})$	2k(4n+3-2j) -2k+1	2k(4n+3-2j) -2k+2	$\cdots$	2k(4n+3-2j)-1	2k(4n+3-2j)
$f(u_ix_{i,2j})$	2k(2j-1)+2k	2k(2j-1)+2k-1	$\cdots$	2k(2j-1)+2	2k(2j-1)+1

Table 4. Each column sum is $8kn + 1$

i	1	2	$\cdots$	2k-1	2k
$f(v_ix_{i,2j-1})$	4k(j-1) + 1	4k(j-1) + 2	$\cdots$	4k(j-1) + 2k-1	4k(j-1) + 2k
$f(v_ix_{i,2j})$	2k(4n+2-2j)	2k(4n+2-2j) -1	$\cdots$	2k(4n+2-2j) -2k+2	2k(4n+2-2j) -2k+1

When $n = 1$, we only use Tables 1 and 2. For $n \geq 2$ and $k = 1$, we only use the first and the last columns of each table. For clarity, the table is given below, where $j \in [1, n-1]$.

i	1	2
$f(u_ix_{i,2j-1})$	8n+5-4j	8n+6-4j
$f(u_ix_{i,2j})$	4j	4j-1
$f(u_ix_{i,2n-1})$	4n+5	4n+6
$f(u_ix_{i,2n})$	4n+2	4n+4
$f(u_iv_i)$	4n+3	4n
$f(v_ix_{i,2j-1})$	4j-3	4j-2
$f(v_ix_{i,2j})$	8n+4-4j	8n+3-4j
$f(v_ix_{i,2n-1})$	4n-3	4n-2
$f(v_ix_{i,2n})$	4n-1	4n+1

We now have the following observations that hold for $(n, k) \neq (1, 1)$.

(A) Each integer in $[1, 2k(4n + 1)]$ serves as an edge label once.

(B) From the above tables, we have $f^+(u_i) = (n-1)(8kn + 4k + 1) + 12kn + 9k + 1 = 8kn^2 + 8kn + 5k + n$ and $f^+(v_i) = (n-1)(8kn+1) + 12kn - 3k + 2 = 8kn^2 + 4kn - 3k + n + 1$ for each i. Clearly, $f^+(u_i) > f^+(v_i)$.

(C) Suppose $n = 1$. For each $1 \le i \le k$, $f(u_i x_{i,2n-1}) + f(u_{2k+1-i} x_{2k+1-i,2n-1}) = 8kn + 10k + 1$, $f(u_i x_{i,2n}) + f(u_{2k+1-i} x_{2k+1-i,2n}) = 8kn + 6k$, whereas $f(v_i x_{i,2n-1}) + f(v_{2k+1-i} x_{2k+1-i,2n-1}) = 8kn - 6k + 1$, $f(v_i x_{i,2n}) + f(v_{2k+1-i} x_{2k+1-i,2n}) = 8kn - 2k + 2$. Thus, sum of rows $f(u_i x_{i,2n-1})$ and $f(v_i x_{i,2n-1})$ entries is $k(8kn + 10k + 1) + k(8kn - 6k + 1) = k(16kn + 4k + 2)$. Similarly, sum of rows $f(u_i x_{i,2n})$ and $f(v_i x_{i,2n})$ entries is $k(8kn + 6k) + k(8kn - 2k + 2) = k(16kn + 4k + 2)$.

(D) Suppose $n \ge 2$ and $1 \le j \le n - 1$. For each $1 \le i \le k$, $f(u_i x_{i,2j-1}) + f(u_{2k+1-i} x_{2k+1-i,2j-1}) = 4k(4n + 3 - 2j) - 2k + 1$, $f(u_i x_{i,2j}) + f(u_{2k+1-i} x_{2k+1-i,2j}) = 4k(2j - 1) + 2k + 1$, whereas $f(v_i x_{i,2j-1}) + f(v_{2k+1-i} x_{2k+1-i,2j-1}) = 8k(j - 1) + 2k + 1$, $f(v_i x_{i,2j}) + f(v_{2k+1-i} x_{2k+1-i,2j}) = 4k(4n + 2 - 2j) - 2k + 1$. Together with Observation (C), we conclude that for $1 \le j \le n$, sum of rows $f(u_i x_{i,2j-1})$ and $f(v_i x_{i,2j-1})$ entries is $k[4k(4n+3-2j)-2k+1]+k[8k(j-1)+2k+1] = k(16kn+4k+2)$, and sum of rows $f(u_i x_{i,2j})$ and $f(v_i x_{i,2j})$ entries is $k[4k(2j-1)+2k+1]+k[4k(4n+2-2j)-2k+1] = k(16kn+4k+2)$.

(E) Suppose $k = rs$, $r \ge 2$, $s \ge 1$. We can divide the $2k$ columns into $2r$ blocks of s column(s) with the b-th block containing $(b - 1)s + 1, (b - 1)s + 2, \dots, bs$ columns. From Observations (C) and (D), we can conclude that sum of row $f(u_i x_{i,j})$ and $f(v_i x_{i,j})$ entries in block b-th and $(2r + 1 - b)$-th collectively is a constant $s(16kn + 4k + 2)$ for $1 \le j \le 2n, 1 \le b \le r, s \ge 1$.

(F) For each $1 \le i \le k$ and $1 \le j \le 2n$, $f(u_i x_{i,j}) + f(v_{2k+1-i} x_{2k+1-i,j}) = f(v_i x_{i,j}) + f(u_{2k+1-i} x_{2k+1-i,j}) = 8kn + 2k + 1$.

Theorem 2.1. *For $k, n \ge 1$, $\chi_{la}((2k)P_2 \vee O_{2n}) = 3$.*

Proof. Note that $\chi_{la}((2k)P_2 \vee O_{2n}) \ge \chi((2k)P_2 \vee O_{2n}) = 3$. A local antimagic 3-coloring of $2P_2 \vee O_2$ is given in the figure below with induced vertex labels $14, 19, 22$.

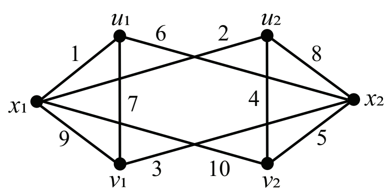

Fig. 1. Graph $2P_2 \vee O_2$

Suppose $(n, k) \ne (1, 1)$. Consider $G = 2k(P_2 \vee O_{2n})$. Since G has size $2k(4n + 1)$, we can now define a bijection $f : E(G) \to [1, 2k(4n + 1)]$ according to the tables above. For each $j \in [1, 2n]$, merging the vertices in $\{x_{i,j} \mid 1 \le i \le 2k\}$ to form a new vertex x_j of degree $4k$, we get the graph $(2k)P_2 \vee O_{2n}$. We still use f to denote the labeling for the graph $(2k)P_2 \vee O_{2n}$. From Observations (A) to (D) above, we get that $(2k)P_2 \vee O_{2n}$ admits a bijective edge labeling f with

(a) $f^+(x_j) = k(16kn + 4k + 2)$,
(b) $f^+(u_i) = 8kn^2 + 8kn + 5k + n$,
(c) $f^+(v_i) = 8kn^2 + 4kn - 3k + n + 1$,
(d) $f^+(u_i) > f^+(v_i)$.

Now,

$$\begin{aligned}(a)-(b) &= 16k^2n - 8kn^2 + 4k^2 - 6kn - 3k - n\\ &= (8kn+2k)(2k-n) - 6kn - 3k - n\\ &< 0 \quad \text{if } 2k \le n.\end{aligned}$$

Otherwise, $2k \ge n+1$ so that $(a)-(b) \ge 2kn - k - n > 0$ since $(n,k) \ne (1,1)$. Thus, $(a) \ne (b)$. Similarly,

$$\begin{aligned}(a)-(c) &= 16k^2n - 4kn^2 + 4k^2 - 8kn + 5k - n - 1\\ &= (8kn+2k)(2k-n) - k(6n+5) - n - 1\\ &< 0 \quad \text{if } 2k \le n.\end{aligned}$$

If $2k = n+1$, then $(a)-(c) = 2kn - 3k - n - 1 = 2k(2k-1) - 3k - (2k-1) - 1 = 4k^2 - 7k \ne 0$ Otherwise, $2k \ge n+2$ so that $a) - (c) \ge 10kn - k - n - 1 > 0$. Thus, $(a) \ne (c)$.

Therefore, f is a local antimagic 3-coloring and $\chi_{la}((2k)P_2 \vee O_{2n}) \le 3$. This completes the proof. □

Example 2.2. For $n = 2, k = 4$ so that $j = 1$, we have the following 9×8 matrix and $8(P_2 \vee O_4)$ as shown. For each $j \in [1,4]$, merge the vertices in $\{x_{i,j} \mid 1 \le i \le 8\}$, we get the $8P_2 \vee O_4$ and the labeling as defined above.

i	1	2	3	4	5	6	7	8
$f(u_ix_{i,1})$	65	66	67	68	69	70	71	72
$f(u_ix_{i,2})$	16	15	14	13	12	11	10	9
$f(u_ix_{i,3})$	49	53	54	55	50	51	52	56
$f(u_ix_{i,4})$	40	41	42	43	45	46	47	48
$f(u_iv_i)$	44	39	37	35	38	36	34	29
$f(v_ix_{i,1})$	1	2	3	4	5	6	7	8
$f(v_ix_{i,2})$	64	63	62	61	60	59	58	57
$f(v_ix_{i,3})$	17	21	22	23	18	19	20	24
$f(v_ix_{i,4})$	25	26	27	28	30	31	32	33

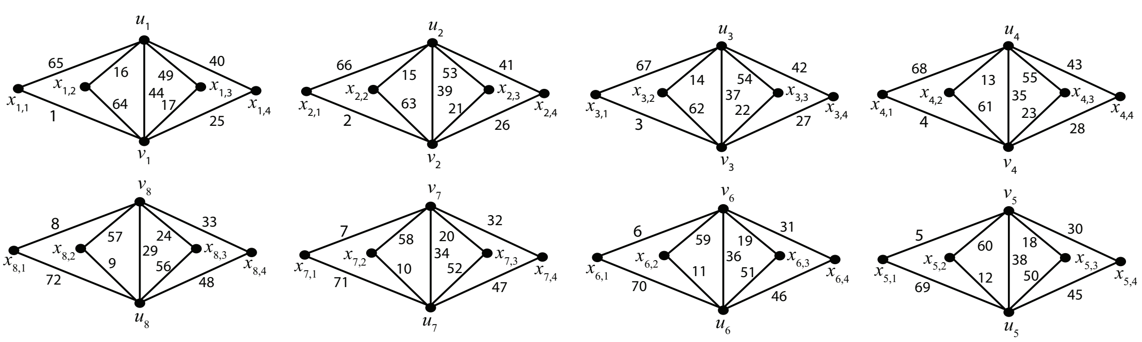

Fig. 2. Graph $8(P_2 \vee O_4)$.

Theorem 2.3. *For $n \geq 1, r \geq 2, s \geq 1$, $\chi_{la}(r((2s)P_2 \vee O_{2n})) = 3$.*

Proof. Let $k = rs \geq 2$. Begin with $2k(P_2 \vee O_{2n})$ and the local antimagic 3-coloring f as defined in the proof of Theorem 2.1. By Observations (C) and (D), we can now partition the $2k$ components into $2r$ blocks of s component(s) with the b-th block $(1 \leq b \leq 2r)$ containing vertices $u_i, v_i, x_{i,j}$ for $(b-1)s+1 \leq i \leq bs$, $r \geq 2, s \geq 1$. Now, for each $j \in [1, 2n]$, merge the vertices in $\{x_{i,j}, x_{2k+1-i,j} \mid (b-1)s+1 \leq i \leq bs\}$ to form a new vertex of degree $4s$, still denoted $x_{i,j}$, $1 \leq i \leq r$. We now get an r-component graph $r((2s)P_2 \vee O_{2n})$ with a bijective edge labeling, still denoted f, such that the degree $2n+1$ vertices, u_i and v_i, have induced vertex labels $(a) = f^+(u_i) = 8kn^2 + 8kn + 5k + n$ and $(b) = f^+(v_i) = 8kn^2 + 4kn - 3k + n + 1$, respectively. Moreover, the vertices $x_{i,j}$ have induced vertex label $(c) = f^+(x_{i,j}) = s(16kn + 4k + 2)$. Since $(a) > (b)$, we shall show that $(a), (b) \neq (c)$. If $n = 1$, the three induced vertex labels are $21k+1, 9k+2, s(20k+2)$ which are distinct. We now assume $n \geq 2$.

Now, $(a) - (c)$ is:

$$8kn^2 - 16kns + 8kn - 4ks + 5k + n - 2s = (8kn+1)(n-2s+1) - k(4s-5) - 1 \quad (1)$$

If $n - 2s + 1 \leq 0$, then $2 \leq n < 2s$. Thus $s \geq 2$. We get $(1) < 0$.
If $n - 2s + 1 > 0$, then $n \geq 2s$. Thus,

$$\begin{aligned}(8kn+1)(n-2s+1) - k(4s-5) - 1 &\geq 8kn + 1 - 4ks + 5k - 1 \\ &= 4k(2n-s) + 5k > 0.\end{aligned}$$

Next, $(b) - (c)$ is:

$$\begin{aligned}&8kn^2 - 16kns + 4kn - 4ks - 3k + n - 2s + 1 \\ &= \left(4kn + \frac{1}{2}\right)(2n - 4s + 1) - k(4s+3) + \frac{1}{2}. \quad (2)\end{aligned}$$

If $2n - 4s + 1 \leq 0$, then clearly $(2) < 0$.
If $2n - 4s + 1 > 0$, then $n \geq 2s$. Thus,

$$\left(4kn + \frac{1}{2}\right)(2n - 4s + 1) - k(4s+3) + \frac{1}{2} \geq 4k(n-s) - 3k + 1 \geq 4ks - 3k + 1 > 0.$$

Therefore, f is a local antimagic 3-coloring and $\chi_{la}(r((2s)P_2 \vee O_{2n})) \leq 3$. This completes the proof. □

Example 2.4. Take $n = k = 2$ so that $r = 2, s = 1$, we can get the following 9×4 matrix and $2(2P_2 \vee O_4)$ with the defined labeling as follow. The induced vertex labels are $108, 77, 74$ respectively.

i	1	2	3	4
$f(u_ix_{i,1})$	33	34	35	36
$f(u_ix_{i,2})$	8	7	6	5
$f(u_ix_{i,3})$	25	27	26	28
$f(u_ix_{i,4})$	20	21	23	24
$f(u_iv_i)$	22	19	18	15
$f(v_ix_{i,1})$	1	2	3	4
$f(v_ix_{i,2})$	32	31	30	29
$f(v_ix_{i,3})$	9	11	10	12
$f(v_ix_{i,4})$	13	14	16	17

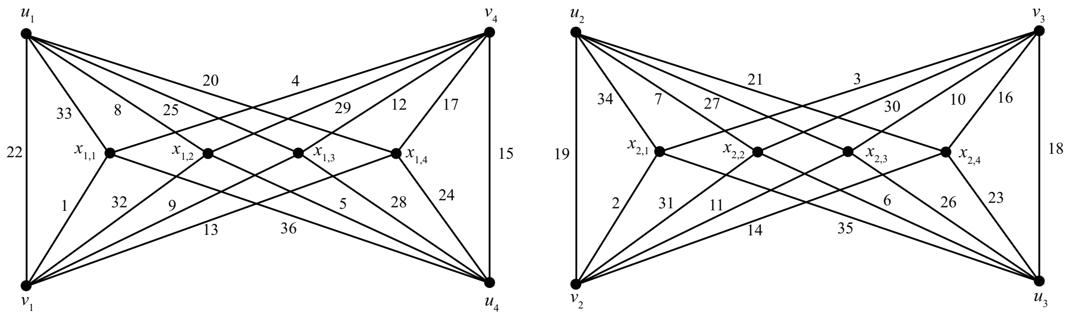

Fig. 3. Graph $2(2P_2 \vee O_4)$

In the graph $2(2P_2 \vee O_4)$ above, we can now split each degree $4s = 4$ vertex (with induced vertex label 74) into two degree $2s = 2$ vertices with induced vertex label 37. This new graph, say $G_{2,2}(4)$, is bipartite. Each component has a bipartition of equal size $2n + 2s = 6$. By Lemma 1.1, $\chi_{la}(G_{2,2}(4)) \geq 3$. Since the corresponding edge labeling induces a local antimagic 3-coloring, we have $\chi_{la}(G_{2,2}(4)) = 3$.

Begin with $r((2s)P_2 \vee O_{2n})$ and the local antimagic 3-coloring f defined in the proof of Theorem 2.3. In general, for $k = rs$, $n \geq 1$, $r \geq 2$, $s \geq 1$, let $G_{r,2s}(2n)$ be obtained from $r((2s)P_2 \vee O_{2n})$ by splitting each $x_{i,j}$ of degree $4s$ into two vertices, say $y_{i,j}$ and $z_{i,j}$, of degree $2s$ such that both have equal induced vertex labels under f. By Observation (F), we know that $G_{r,2s}(2n)$ admits a bijective edge labeling, say g, such that $g^+(u_i) = f^+(u_i) = 8kn^2 + 8kn + 5k + n$, $g^+(v_i) = f^+(v_i) = 8kn^2 + 4kn - 3k + n + 1$, and $g^+(y_{i,j}) = g^+(z_{i,j}) = \frac{1}{2}f^+(x_{i,j}) = s(8kn + 2k + 1)$.

Theorem 2.5. *For $n \geq 1$, $r \geq 2$, $s \geq 1$, $\chi_{la}(G_{r,2s}(2n)) = 3$.*

Proof. By definition, we know that $k = rs \geq 2$ and $G_{r,2s}(2n)$ is an r-component bipartite graph with each component of equal partite sets size $2n + 2s$. By Lemma 1.1, $\chi_{la}(G_{r,2s}(2n)) \geq 3$. Moreover, $G_{r,2s}(2n)$ admits a bijective edge labeling g with induced vertex labels $(a) = g^+(u_i) = 8kn^2 + 8kn + 5k + n$, $(b) = g^+(v_i) = 8kn^2 + 4kn - 3k + n + 1$, and $(c) = g^+(y_{i,j}) = g^+(z_{i,j}) = s(8kn + 2k + 1)$. Since $(a) > (b)$, we shall show that $(a), (b) \neq (c)$. Suppose $n = 1$. Since $k = rs \geq 2$, we have $(a) = 21k + 1$, $(b) = 9k + 2$ and $(c) = s(10k + 1)$ which are distinct. We now assume $n \geq 2$.

Now, $(a) - (c)$ is:

$$8kn^2 - 8kns + 8kn - 2ks + 5k + n - s = (8kn + 1)(n - s + 1) - k(2s - 5) - 1. \quad (3)$$

If $n - s + 1 \leq 0$, then $s \geq n + 1 \geq 3$. We get (3) < 0.
If $n - s + 1 > 0$, then $n \geq s$. Thus,

$$\begin{aligned}(8kn + 1)(n - s + 1) - k(2s - 5) - 1 &\geq 8kn + 1 - 2ks + 5k - 1 \\ &= 2k(4n - s) + 5k > 0.\end{aligned}$$

Next, $(b) - (c)$ is:

$$8kn^2 - 8kns + 4kn - 2ks - 3k + n - s + 1 = \left(4kn + \frac{1}{2}\right)(2n - 2s + 1) - k(2s + 3) + \frac{1}{2}. \quad (4)$$

If $2n - 2s + 1 \leq 0$, then clearly (4) < 0.
If $2n - 2s + 1 = 1$, then $n = s$. Thus, $(4kn + \frac{1}{2})(2n - 2s + 1) - k(2s + 3) + \frac{1}{2} = k(4n - 2s - 3) + 1 = k(2s - 3) + 1 \neq 0$ for all k, s.
If $2n - 2s + 1 \geq 2$, then $2n \geq 2s + 1$. Thus,

$$\left(4kn + \frac{1}{2}\right)(2n - 2s + 1) - k(2s + 3) + \frac{1}{2} \geq k(8n - 2s - 3) + \frac{3}{2} > 0.$$

Therefore, g is a local antimagic 3-coloring and $\chi_{la}(G_{r,2s}(2n)) \leq 3$. This completes the proof. □

Example 2.6. Take $n = 2$, $k = 3$ so that $r = 3$, $s = 1$, the graph $G_{3,2}(4)$ with the local antimagic 3-coloring defined under Theorem 2.5 is given in Figure 4.

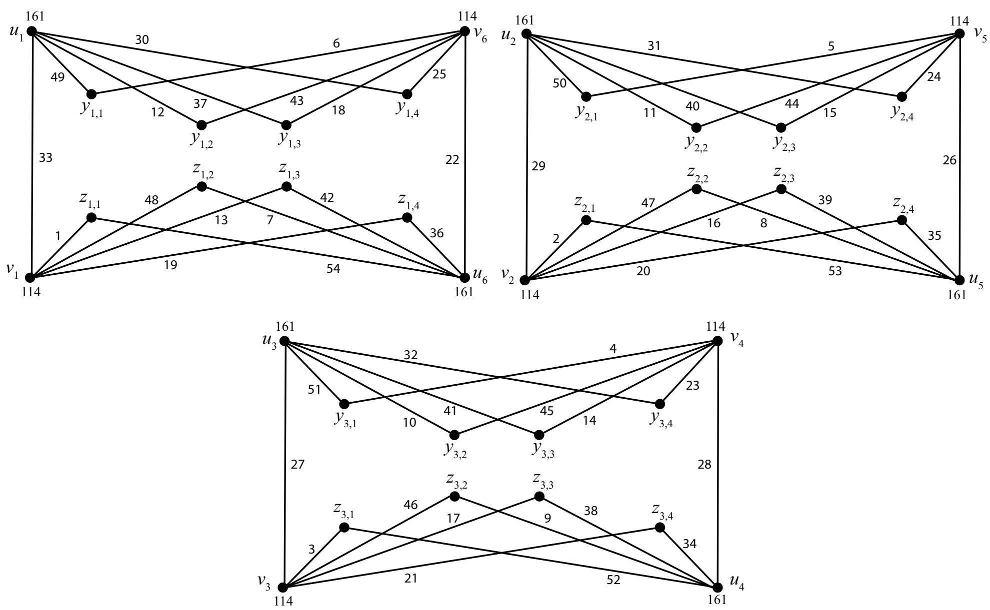

Fig. 4. Graph $G_{3,2}(4)$

Another way to get a new graph is the delete-add process which the authors used in [9, 8]. In the above example, we can delete edges $v_4x_{1,1}, v_1x_{1,1}$ and $v_3x_{2,1}, v_2x_{2,1}$, and then add edges $v_4x_{2,1}, v_1x_{2,1}$ with labels $4, 1$ respectively, also add edges $v_3x_{1,1}, v_2x_{1,1}$ with labels $3, 2$ respectively. The new graph is connected but not a graph we have obtained earlier. In general, let $\mathcal{G}_{r,2s}(2n)$ be the family of all non-isomorphic graphs that can be obtained by applying the delete-add process to $r((2s)P_2 \vee O_{2n})$ and $G_{r,2s}(2n)$. Note that we must have $s \geq 2$ to apply this process to $G_{r,2s}(2n)$. We immediately have the following corollary.

Corollary 2.7. *For $n \geq 1, r \geq 2, s \geq 1$, every graph $G \in \mathcal{G}_{r,2s}(2n)$ has $\chi_{la}(G) = 3$.*

In [8, Theorem 2.2], the authors defined a family of $(r+1)$-componenet tripartite graphs $G_{2n}(2r+1, 2s+1)$ obtained from $(2k+1)(P_2 \vee O_{2n})$. Let $2k = (2r+1)(2s)$ for $r, s \geq 1$, we can also define a similar family of $(r+1)$-component tripartite graphs $G_{2n}(2r+1, 2s)$ obtained from $2k(P_2 \vee O_{2n})$ by keeping the local antimagic 3-coloring f defined under Theorem 2.1.

Theorem 2.8. *Suppose $n \geq 1$. If $2k = (2r+1)(2s)$ for $r, s \geq 1$, then $\chi_{la}(G_{2n}(2r+1, 2s)) = 3$.*

Proof. By definition, $\chi_{la}(G_{2n}(2r+1, 2s)) \geq \chi(G_{2n}(2r+1, 2s)) = 3$. Moreover, we can conclude that $G_{2n}(2r+1, 2s)$ admits a bijective edge labeling, say g, with induced vertex labels $(a) = 8kn^2 + 8kn + 5k + n$, $(b) = 8kn^2 + 4kn - 3k + n + 1$ and $(c) = s(16kn + 4k + 2)$ such that adjacent vertices must have distinct induced vertex labels. From the proof of Theorem 2.3, we can conclude that g is a local antimagic 3-coloring so that $\chi_{la}(G_{2n}(2r+1, 2s)) \leq 3$. This completes the proof. □

Example 2.9. We now use $n = 2, k = 3$ so that $r = s = 1$. We get the 9×6 matrix and the graph $G_4(3, 2)$ in Figure 5 as shown below.

i	1	2	3	4	5	6
$f(u_ix_{i,1})$	49	50	51	52	53	54
$f(u_ix_{i,2})$	12	11	10	9	8	7
$f(u_ix_{i,3})$	37	40	41	38	39	42
$f(u_ix_{i,4})$	30	31	32	34	35	36
$f(u_iv_i)$	33	29	27	28	26	22
$f(v_ix_{i,1})$	1	2	3	4	5	6
$f(v_ix_{i,2})$	48	47	46	45	44	43
$f(v_ix_{i,3})$	13	16	17	14	15	18
$f(v_ix_{i,4})$	19	20	21	23	24	25

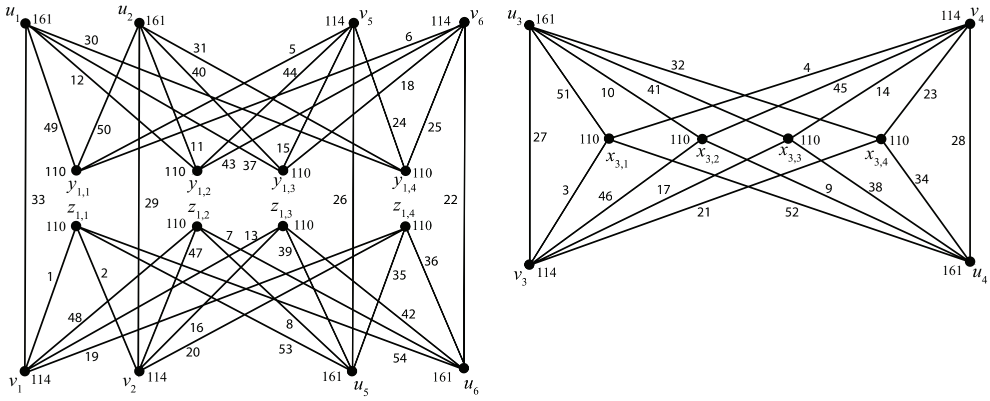

Fig. 5. Graph $G_4(3,2)$ with a local antimagic 3-coloring

We can also apply the delete-add process to $G_{2n}(2r+1,2s)$ to obtain various connected but non-isomorphic graphs of the same order and size, all of which have local antimagic chromatic number 3. In the example above, we may choose to delete edges $u_1y_{1,4}, v_6y_{1,4}, u_3x_{3,1}, v_4x_{3,1}$ and then add edges $u_1x_{3,1}, v_6x_{3,1}, u_3y_{1,4}, v_4y_{1,4}$ with labels $30, 25, 51, 4$ respectively. Let $\mathcal{G}_{2n}(2r+1,2s)$ be the families of all such graphs. We immediately have the following corollary.

Corollary 2.10. *For $n, r, s \geq 1$, every graph $G \in \mathcal{G}_{2n}(2r+1,2s)$ has $\chi_{la}(G) = 3$.*

Note that $k(2P_2 \vee O_{2n})$ in Theorem 2.3 (respectively $G_{k,2}(2n)$ in Theorem 2.5) has $2k$ vertices $v_1, \ldots, v_{2k}$, with induced vertex label $8kn^2 + 4kn - 3k + n + 1$. Suppose we can partition $\{v_i \mid 1 \leq i \leq 2k\}$ into blocks of equal size $s \geq 2$ such that all the vertices in the same block do not have common neighbors.

(a) Let $\mathcal{J}_1(k, 2n, s)$ be the family of graphs obtained from $k(2P_2 \vee O_{2n})$ by merging all the vertices in the same block.

(b) Let $\mathcal{J}_2(k, 2n, s)$ be the family of graphs obtained from $G_{k,2}(2n)$ by merging all the vertices in the same block.

Theorem 2.11. *For $k, s \geq 2$, $n \geq 1$, each graph $G \in \mathcal{J}_1(k, 2n, s)$ has $\chi_{la}(G) = 3$.*

Proof. By definition, $\chi_{la}(G) \geq \chi(G) = 3$. Keeping the bijective edge labeling defined for $k(2P_2 \vee O_{2n})$, we can immediately conclude that G admits a bijective labeling with induced vertex labels $(a) = 8kn^2 + 8kn + 5k + n$, $(b) = s(8kn^2 + 4kn - 3k + n + 1)$ and $(c) = 16kn + 4k + 2$. It is easy to verify that all the induced vertex labels are distinct and the labeling is local antimagic. Thus, $\chi_{la}(G) \leq 3$. Consequently, $\chi_{la}(G) = 3$. □

Similarly, we also have the following theorem with the proof omitted.

Theorem 2.12. *Suppose $k, s \geq 2$, $n \geq 1$ and $G \in \mathcal{J}_2(k, 2n, s)$. If G is bipartite with every component of equal partite set size or G is tripartite, then $\chi_{la}(G) = 3$.*

The i-th component of the graph $k(2P_2 \vee O_{2n})$ or graph $G_{k,2}(2n)$ has vertices u_i, v_i,

u_{2k+1-i}, v_{2k+1-i}, $1 \le i \le k$. We now define two new families of connected graphs.

(a) Let $J_1(k, 2n)$ be a connected graph obtained from $k(2P_2 \vee O_{2n})$ by merging vertices in $\{v_i, v_{2k-i} \mid 1 \le i \le k-1\} \cup \{v_k, v_{2k}\}$. Here $J_1(k, 2n) \in \mathcal{J}_1(k, 2n, 2)$.

(b) Let $J_2(k, 2n)$ be a connected graph obtained from $G_{k,2}(2n)$ by merging vertices in $\{v_i, v_{2k-i} \mid 1 \le i \le k-1\} \cup \{v_k, v_{2k}\}$. Here $J_2(k, 2n) \in \mathcal{J}_2(k, 2n, 2)$.

Example 2.13. Using $6(2P_2 \vee O_2)$, we get partition $\{v_i \mid 1 \le i \le 12\}$ into 4 blocks of size 3 like $\{v_{3a-2}, v_{3a-1}, v_{3a} \mid 1 \le a \le 4\}$ to get a 2-component graph, or else $\{v_1, v_2, v_3\}$, $\{v_8, v_9, v_{10}\}$, $\{v_5, v_6, v_7\}$ and $\{v_4, v_{11}, v_{12}\}$ to get a connected graph in $\mathcal{J}_1(6, 2, 3)$. Keeping the edge labeling, the induced vertex labels are $127, 168, 122$.

Suppose we group the k (isomorphic) components of $k(2P_2 \vee O_{2n})$ into t graphs, $t \ge 2$. Let the a-th graph be J_a which contains $k_a \ge 2$ components. By a similar merging process as for $J_1(k, 2n)$, we obtain a connected graph, denoted by $H_1(k_a, 2n)$. So, the new graph $H_1(k_1, 2n) + H_1(k_2, 2n) + \cdots + H_1(k_t, 2n)$ is a t-component tripartite graph with $k = k_1 + \cdots + k_t$. Thus, by keeping the edge labeling of $k(2P_2 \vee O_{2n})$, we have the following corollary immediately.

Corollary 2.14. *For $n \ge 1$ and $t, k_a \ge 2$, the graph $G = H_1(k_1, 2n) + H_1(k_2, 2n) + \cdots + H_1(k_t, 2n)$ has $\chi_{la}(G) = 3$.*

Using $G_{k,2}(2n)$, we can also construct the graph $G = H_2(k_1, 2n) + H_2(k_2, 2n) + \cdots + H_2(k_t, 2n)$ which is tripartite if some of k_i is odd. Otherwise, each $H_2(k_i, 2n)$ is a bipartite graph with equal partite set size $2k_i n + \frac{3k_i}{2}$. Thus, G is a bipartite graph with equal partite set size $\sum\limits_{i=1}^{t}(2k_i n + \frac{3k_i}{2})$. We also have the following corollary immediately.

Corollary 2.15. *For $n \ge 1$ and $t, k_a \ge 2$, the graph $G = H_2(k_1, 2n) + H_2(k_2, 2n) + \cdots + H_2(k_t, 2n)$ has $\chi_{la}(G) = 3$.*

Example 2.16. For example, take $6(2P_2 \vee O_2)$, we can get either $H_1(2, 2) + H_1(4, 2)$ if we merge the vertices in $\{v_1, v_2\}$, $\{v_{11}, v_{12}\}$, $\{v_3, v_9\}$, $\{v_4, v_8\}$, $\{v_5, v_7\}$ and $\{v_6, v_{10}\}$, or else we get $2H_1(3, 2n)$ if we merge the vertices in $\{v_1, v_{11}\}$, $\{v_2, v_{10}\}$, $\{v_3, v_{12}\}$, $\{v_4, v_8\}$, $\{v_5, v_7\}$ and $\{v_6, v_9\}$. The induced vertex labels are $127, 112, 122$.

3. Graphs of size $(4n + 3) \times 2k$

Consider $(2k)(P_2 \vee O_{2n+1})$ of order $2k(2n + 3)$ and size $2k(4n + 3)$ for $k, n \ge 1$. We shall construct the following Table 5, which shows the label of each edge under a labeling f.

We now have the following observations.

(1) Each integer in $[1, 2k(4n + 3)]$ serves as an edge label once.

(2) For each column, the sum of the first $2n + 2$ entries is

$$f^+(u_i) = 4kn + 6k + 1 + \sum_{j=1}^{n}[8k(2n - j) + 16k + 1] = (n + 1)(12nk + 4k + 1) + 2k.$$

(3) For each column, the sum of the last $2n+2$ entries is
$f^+(v_i) = 4k+1+\sum_{j=1}^{n}[4k+1+8kj] = (n+1)(4nk+4k+1)$. Clearly $f^+(u_i) > f^+(v_i)$.

(4) Consider $1 \le i \le 2k$. For odd $j \in [1, 2n+1]$, the terms $f(u_ix_{i,j}) + f(v_ix_{i,j})$ form an arithmetic sequence with first term $8kn+10k$, last term $8kn+6k+2$ and common difference -2. For even $j \in [1, 2n+1]$, the terms $f(u_ix_{i,j}) + f(v_ix_{i,j})$ form an arithmetic sequence with first term $8kn+6k+2$, last term $8kn+10k$ and common difference 2. Thus, for each $j \in [1, 2n+1]$, sum of rows $f(u_ix_{i,j})$ and $f(v_ix_{i,j})$ entries is $2k(8kn+8k+1)$.

Note that, if $\{A_i\}_{i=1}^m$ is an increasing (resp. decreasing) sequence, then $\{A_{m+1-i}\}_{i=1}^m$ is a decreasing (resp. increasing) sequence.

Thus, for $1 \le i \le k$, $f(u_ix_{i,j})+f(v_ix_{i,j})+f(u_{2k+1-i}x_{2k+1-i,j})+f(v_{2k+1-i}x_{2k+1-i,j}) = K$, where K is independent of i. By the discussion above, $K = 2(8kn+8k+1)$.

(5) Suppose $k = rs$ for $r \ge 2, s \ge 1$. We can divide the $2k$ columns into $2r$ blocks of s column(s) with the b-th block containing $(b-1)s+1, (b-1)s+2, \ldots, bs$ columns. From Observation (3), we can conclude that sum of rows $f(u_ix_{i,j})$ and $f(v_ix_{i,j})$ entries in block b-th and $(2r+1-b)$-th collectively is a constant $s(16kn+16k+2)$.

(6) For each $1 \le i \le k$ and $1 \le j \le 2n+1$, $f(u_ix_{i,j}) + f(v_{2k+1-i}x_{2k+1-i,j}) = f(v_ix_{i,j}) + f(u_{2k+1-i}x_{2k+1-i,j}) = 8kn+8k+1$.

Table 5. $(4n+3) \times 2k$ matrix for $(2k)(P_2 \vee O_{2n+1})$

i	1	2	$\cdots$	2k-1	2k
$f(u_ix_{i,1})$	4n(2n-1)+10k	4n(2n-1)+10k-1	$\cdots$	4n(2n-1)+8k+2	4n(2n-1)+8k+1
$f(u_ix_{i,2})$	4n(2n-1)+6k+1	4n(2n-1)+6k+2	$\cdots$	4n(2n-1)+8k-1	4n(2n-1)+8k
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$f(u_ix_{i,2j-1})$	4k(2n-j)+10k	4k(2n-j) +10k-1	$\cdots$	4k(2n-j)+8k+2	4k(2n-j) + 8k+1
$f(u_ix_{i,2j})$	4k(2n-j) +6k+1	4k(2n-j) + 6k+2	$\cdots$	4k(2n-j) +8k-1	4k(2n-j) + 8k
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$f(u_ix_{i,2n-1})$	4kn+10k	4kn+10k-1	$\cdots$	4kn+8k+2	4kn+8k+1
$f(u_ix_{i,2n})$	4kn+6k+1	4kn+6k+2	$\cdots$	4kn+8k-1	4kn+8k
$f(u_ix_{i,2n+1})$	4kn+6k	4kn+ 6k-1	$\cdots$	4kn+4k+2	4kn+4k+1
$f(u_iv_i)$	1	2	$\cdots$	2k-1	2k
$f(v_ix_{i,1})$	4k	4k-1	$\cdots$	2k+2	2k+1
$f(v_ix_{i,2})$	4k+1	4k+2	$\cdots$	6k-1	6k
$f(v_ix_{i,3})$	8k	8k-1	$\cdots$	6k+2	6k+1
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$f(v_ix_{i,2j})$	4kj+1	4kj+2	$\cdots$	4kj+2k-1	4kj + 2k
$f(v_ix_{i,2j+1})$	4kj+4k	4kj+4k-1	$\cdots$	4kj+2k+2	4kj+2k+1
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$f(v_ix_{i,2n+1})$	4kn+4k	4kn+4k-1	$\cdots$	4kn+2k+2	4kn+2k+1

Theorem 3.1. *For $n, k \ge 1$, $\chi_{la}((2k)P_2 \vee O_{2n+1}) = 3$.*

Proof. Note that $\chi_{la}((2k)P_2 \vee O_{2n+1}) \ge \chi((2k)P_2 \vee O_{2n+1}) = 3$. Consider $(2k)(P_2 \vee O_{2n+1})$.

Since G has size $2k(4n+3)$, we can now define a bijection $f : E(G) \to [1, 2k(4n+3)]$ according to the table above. Clearly, for $1 \le i \le 2k$, $f^+(u_i) = (n+1)(12nk+4k+1)+2k > f^+(v_i) = (n+1)(4nk+4k+1)$. Now, for each $j \in [1, 2n+1]$, merging the vertices in $\{x_{i,j} \mid 1 \le i \le 2k\}$, to form a new vertex x_j of degree $4k$, we get the graph $(2k)P_2 \vee O_{2n+1}$. From Observations (1) to (3) above, we get that $(2k)P_2 \vee O_{2n+1}$ admits a bijective edge labeling with

(a) $f^+(x_j) = 2k(8nk+8k+1) = 16k^2(n+1)+2k$,
(b) $f^+(u_i) = (n+1)(12nk+4k+1)+2k$, and
(c) $f^+(v_i) = (n+1)(4nk+4k+1)$.

Now, $(a)-(b) = -(n+1)[4k(3n-4k-1)+1] < 0$ if $3n-4k-1 \ge 0$. Otherwise, $(a)-(b) > 0$. Thus, $(a) \ne (c)$. Similarly, $(a)-(c) = (4kn+4k)(4k-n)-4kn-2k-n-1 < 0$ if $4k-n \le 1$. Otherwise, $(a)-(c) > 0$. Thus, $(a) \ne (c)$. Therefore, f is a local antimagic 3-coloring and $\chi_{la}((2k)P_2 \vee O_{2n+1}) \le 3$. This completes the proof. □

Example 3.2. Taking $n = 2, k = 3$, we have the following 11×6 matrix and $6(P_2 \vee O_5)$ as shown. For each $j \in [1, 5]$, merge the vertices in $\{x_{i,j} \mid 1 \le i \le 6\}$, we get a $6P_2 \vee O_5$ and the labeling as defined above.

i	1	2	3	4	5	6
$f(u_ix_{i,1})$	66	65	64	63	62	61
$f(u_ix_{i,2})$	55	56	57	58	59	60
$f(u_ix_{i,3})$	54	53	52	51	50	49
$f(u_ix_{i,4})$	43	44	45	46	47	48
$f(u_ix_{i,5})$	42	41	40	39	38	37
$f(u_iv_i)$	1	2	3	4	5	6
$f(v_ix_{i,1})$	12	11	10	9	8	7
$f(v_ix_{i,2})$	13	14	15	16	17	18
$f(v_ix_{i,3})$	24	23	22	21	20	19
$f(v_ix_{i,4})$	25	26	27	28	29	30
$f(v_ix_{i,5})$	36	35	34	33	32	31

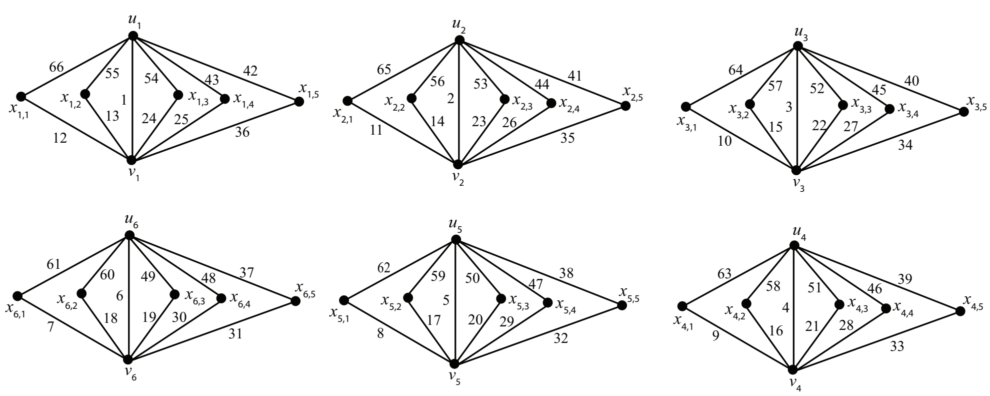

Fig. 6. $6(P_2 \vee O_5)$

Theorem 3.3. *For $n \geq 1, r \geq 2, s \geq 1$, $\chi_{la}(r((2s)P_2 \vee O_{2n+1})) = 3$.*

Proof. Let $k = rs \geq 2$. Begin with $2k(P_2 \vee O_{2n+1})$ and the local antimagic 3-coloring f as defined in the proof of Theorem 3.1. By Observation (4), we can now partition the $2k$ components into $2r$ blocks of s component(s) with the b-th block $(1 \leq b \leq 2r)$ containing vertices $u_i, v_i, x_{i,j}$ for $(b-1)s+1 \leq i \leq bs, r \geq 2, s \geq 1$. Now, for each $j \in [1, 2n+1]$, merge the vertices in $\{x_{i,j}, x_{2k+1-i,j} \mid (b-1)s+1 \leq i \leq bs\}$ to form a new vertex of degree $4s$, still denoted $x_{i,j}$, $1 \leq i \leq r$. We now get an r components graph $r((2s)P_2 \vee O_{2n+1})$ with a bijective edge labeling such that the degree $2n+2$ vertices, u_i and v_i, have induced vertex labels $(a) = f^+(u_i) = (n+1)(12kn+4k+1)+2k$ and $(b) = f^+(v_i) = (n+1)(4kn+4k+1)$ respectively. Moreover, the vertices $x_{i,j}$ have induced vertex label $(c) = f^+(x_{i,j}) = s(16kn+16k+2)$. Since $(a) > (b)$, we shall show that $(a), (b) \neq (c)$.

Now, $(a)-(c)$ is:

$$\begin{aligned} &12kn^2 - 16kns + 16kn - 16ks + 6k + n - 2s + 1 \\ &= 4kn(3n - 4s + 4) - 16ks + 6k + n - 2s + 1. \end{aligned} \tag{5}$$

If $3n - 4s + 4 \leq 2$ (i.e., $-4s \leq -3n - 2$), then

$$\begin{aligned} (5) &\leq 8kn - 16ks + 6k + n - 2s + 1 \\ &\leq 8kn - 12kn - 8k + 6k + n - 2s + 1 = -4kn - 2k + n - 2s + 1 < 0. \end{aligned}$$

If $3n - 4s + 4 \geq 3$ (i.e., $-4s \geq -3n - 1$), then

$$\begin{aligned} (5) &\geq 12kn - 16ks + 6k + n - 2s + 1 \\ &\geq 2k + n - 2s + 1 = 2rs + n - 2s + 1 > 0. \end{aligned}$$

Next, $(b)-(c)$ is :

$$\begin{aligned} &4kn^2 - 16kns + 8kn - 16ks + 4k + n - 2s + 1 \\ &= 4kn(n - 4s + 2) - 16ks + 4k + n - 2s + 1. \end{aligned} \tag{6}$$

If $n - 4s + 2 \leq 0$, then $(6) \leq -16ks + 4k + 2s - 1 = -4k(3s-1) - s(4k-2) - 1 < 0$. If $n - 4s + 2 \geq 1$, then $(6) \geq 4kn - 16ks + 4k + 2s \geq 4k(4s-1) - 16ks + 4k + 2s = 2s > 0$.

Therefore, f is a local antimagic 3-coloring and $\chi_{la}(r((2s)P_2 \vee O_{2n+1})) = 3$. This completes the proof. □

Begin with $r((2s(P_2 \vee O_{2n+1})$ and the local antimagic 3-coloring f defined in the proof of Theorem 3.3. In general, for $k = rs, n \geq 1, r \geq 2, s \geq 1$, we can also define $G_{r,2s}(2n+1)$ (similar to $G_{r,2s}(2n)$ as in Theorem 2.5) obtained from $r((2s)P_2 \vee O_{2n+1})$ by splitting each $x_{i,j}$ of degree $4s$ into two vertices, say $y_{i,j}$ and $z_{i,j}$, of degree $2s$ such that both have equal induced vertex labels under f. By Observation (6), we know that $G_{r,2s}(2n+1)$ admits a bijective edge labeling, say g, such that $g^+(u_i) = f^+(u_i) = (n+1)(12kn+4k+1)+2k$, $g^+(v_i) = f^+(v_i) = (n+1)(4kn+4k+1)$ and $g^+(y_{i,j}) = g^+(z_{i,j}) = \frac{1}{2}f^+(x_{i,j}) = s(8kn + 8k + 1)$.

Theorem 3.4. *For $n \geq 1$, $r \geq 2$, $s \geq 1$, $\chi_{la}(G_{r,2s}(2n+1)) = 3$.*

Proof. By definition, we know that $k = rs \geq 2$ and $G_{r,2s}(2n+1)$ is an r-component bipartite graph with each component of equal partite sets size $2n+2s+1$. By Lemma 1.1, $\chi_{la}(G_{r,2s}(2n+1)) \geq 3$. Moreover, $G_{r,2s}(2n+1)$ admits a bijective edge labeling g with induced vertex labels $(a) = g^+(u_i) = (n+1)(12kn+4k+1)+2k$, $(b) = g^+(v_i) = (n+1)(4kn+4k+1)$, and $g^+(y_{i,j}) = g^+(z_{i,j}) = s(8kn+8k+1)$. Since $(a) > (b)$, we shall show that $(a), (b) \neq (c)$.

Now, $(a) - (c)$ is:

$$\begin{aligned} &12kn^2 - 8kns + 16kn - 8ks + 6k + n - s + 1 \\ &\quad = 4kn(3n - 2s + 4) - 8ks + 6k + n - s + 1. \end{aligned} \tag{7}$$

If $3n - 2s + 4 \leq 2$ (i.e., $-2s \leq -3n - 2$), then

$$\begin{aligned} (7) &\leq 8kn - 8ks + 6k + n - s + 1 \\ &\leq 8kn - 12kn - 8k + 6k + n - s + 1 = -4kn - 2k + n - s + 1 < 0. \end{aligned}$$

If $3n - 2s + 4 \geq 3$ (i.e., $-2s \geq -3n - 1$), then

$$\begin{aligned} (7) &\geq 12kn - 8ks + 6k + n - s + 1 \\ &\geq 2k + n - s + 1 = 2rs + n - s + 1 > 0. \end{aligned}$$

Next, $(b) - (c)$ is :

$$\begin{aligned} &4kn^2 - 8kns + 8kn - 8ks + 4k + n - s + 1 \\ &\quad = 4kn(n - 2s + 2) - 8ks + 4k + n - s + 1. \end{aligned} \tag{8}$$

If $n - 2s + 2 \leq 0$, then $(8) \leq -8ks + 4k + s - 1 = -2k(3s-2) - s(2k-1) - 1 < 0$. If $n - 2s + 2 \geq 1$, then $(8) \geq 4kn - 8ks + 4k + s \geq 4k(2s-1) - 8ks + 4k + s = s > 0$.

Therefore, g is a local antimagic 3-coloring and $\chi_{la}(G_{r,2s}(2n+1)) \leq 3$. This completes the proof. □

Example 3.5. Using the $6(P_2 \vee O_5)$ as in Figure 6, we get a $3(2P_2 \vee O_5)$ that admits a local antimagic 3-coloring with induced vertex labels $261, 111, 146$ respectively. Moreover, we can also get a $G_{3,2}(5)$ (see Figure 7) that admits a local antimagic 3-coloring with induced vertex labels $261, 111, 73$ respectively.

We can now define $\mathcal{G}_{r,2s}(2n+1)$ (similar to $\mathcal{G}_{r,2s}(2n)$ as in Corollary 2.7) being the family of all non-isomorphic graphs that can be obtained by applying the delete-add process to $r((2s)P_2 \vee O_{2n+1})$ and $G_{r,2s}(2n+1)$. The following corollary is also immediate.

Corollary 3.6. *For $n \geq 1, r \geq 2, s \geq 1$, every graph $G \in \mathcal{G}_{r,2s}(2n+1)$ has $\chi_{la}(G) = 3$.*

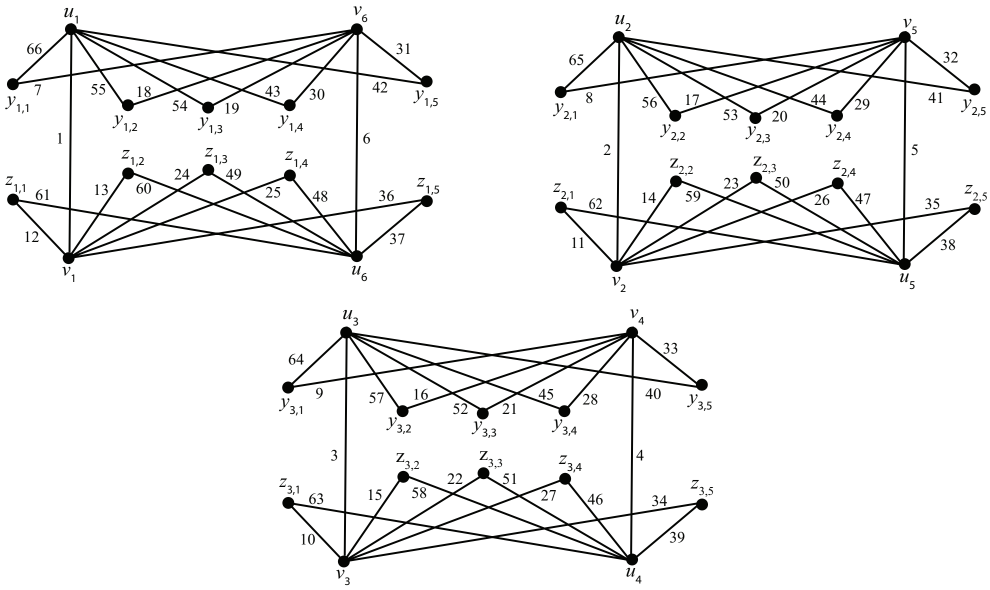

Fig. 7. $G_{3,2}(5)$

Similar to Theorem 2.8, we can also define $G_{2n+1}(2r+1, 2s)$ and the following theorem. The proof is thus omitted.

Theorem 3.7. *Suppose* $n \geq 1$*. If* $2k = (2r+1)(2s)$ *for* $r, s \geq 1$*, then* $\chi_{la}(G_{2n+1}(2r+1, 2s)) = 3$*.*

Similar to Corollary 2.10, we also have the following corollary immediately.

Corollary 3.8. *For* $n, r, s \geq 1$*, every graph* $G \in \mathcal{G}_{2n+1}(2r+1, 2s)$ *has* $\chi_{la}(G) = 3$*.*

Similar to Theorems 2.11 and 2.12, we also let $\mathcal{J}_1(k, 2n+1, s)$ (respectively, $\mathcal{J}_2(k, 2n+1, s)$) be the family of graphs obtained from $k(2P_2 \vee O_{2n+1})$ (respectively, $G_{k,2}(2n+1)$) after partitioning the $2k$ vertices $u_1, \ldots, u_{2k}$, with induced vertex label $(n+1)(12kn+4k+1)+2k$.

Theorem 3.9. *For* $k, s \geq 2$*,* $n \geq 1$*, each graph* $G \in \mathcal{J}_1(k, 2n+1, s)$ *has* $\chi_{la}(G) = 3$*.*

Proof. By definition, $\chi_{la}(G) \geq \chi(G) = 3$. Keeping the bijective edge labeling defined for $k(2P_2 \vee O_{2n+1})$, we can immediately conclude that G admits a bijective edge labeling with induced vertex labels $(a) = s[(n+1)(12kn+4k+1)+2k]$, $(b) = (n+1)(4kn+4k+1)$ and $(c) = 16kn+16k+2$. It is easy to show that $(a) > (b) > (c)$ and the labeling is local antimagic. Thus, $\chi_{la}(G) \leq 3$. Consequently, $\chi_{la}(G) = 3$. □

Similarly, we also have the following theorem with the proof omitted.

Theorem 3.10. *Suppose* $k, s \geq 2, n \geq 1$ *and* $G \in \mathcal{J}_2(k, 2n, s)$*. If* G *is bipartite with every component of equal partite set size or* G *is tripartite, then* $\chi_{la}(G) = 3$*.*

Similar to Corollaries 2.14 and 2.15, we also define $t \geq 2$ components graphs $H_1(k_1, 2n+$

$1)+H_1(k_2,2n+1)+\cdots+H_1(k_t,2n+1)$ and $H_2(k_1,2n+1)+H_2(k_2,2n+1)+\cdots+H_2(k_t,2n+1)$ using $k(P_2 \vee O_{2n+1})$ and $G_{k,2}(2n+1)$ respectively.

Corollary 3.11. *For $n \ge 1$ and $t, k_a \ge 2$, the graph $G_1 = H_1(k_1, 2n+1) + H_1(k_2, 2n+1) + \cdots + H_1(k_t, 2n+1)$ and $G_2 = H_2(k_1, 2n+1) + H_2(k_2, 2n+1) + \cdots + H_2(k_t, 2n+1)$ has $\chi_{la}(G_i) = 3$ for $i = 1, 2$.*

Further note that the following graphs are $(2n+2)$-regular:
(1) $(n+1)P_2 \vee O_{2n+1}$.
(2) $r((n+1)P_2 \vee O_{2n+1})$.
(3) $G_{r,2n+2}(2n+1)$.
(4) each graph in $\mathcal{G}_{r,2n+2}(2n+1)$.
Thus, together with Theorem 2.8 in [1] and Theorems 6 and 7 in [2], we have the following corollary.

Corollary 3.12. *For each even $n \ge 2$, there are (possibly connected) n-regular bipartite or tripartite graphs with local antimagic chromatic number 3.*

4. Conclusion and open problems

This paper is a natural extension to [9, 8]. We successfully proved that the join of a 1-regular graph and a null graph has local antimagic chromatic number 3. Moreover, $\chi_{la}(a(bP_2 \vee O_m)) = 3$ for all $a, b, m \ge 2$ except a is even and b is odd.

Conjecture 4.1. *For $r, s \ge 1$ and $m \ge 2$, $\chi_{la}(2r((2s+1)P_2 \vee O_m)) = 3$.*

Interested readers can refer to [12, 7] for more results on local antimagic chromatic number of the join of a path or a cycle and a null graph. More results on the join of regular graphs are available in [4]. We end this paper with the following two problems.

Problem 4.2. *Let G be a path or a cycle. Determine $\chi_{la}(mG \vee O_n)$ for $m, n \ge 2$.*

Problem 4.3. *Show that for each odd $n \ge 3$, there are (possibly connected) n-regular bipartite or tripartite graphs with local antimagic chromatic number 3.*

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