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On signless and normalized Laplacian spectra of a subgraph of the total graph of $\mathbb{Z}_n$

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ABSTRACT

In this paper, we study the signless Laplacian eigenvalues of the subgraph $Z^*(\Gamma(\mathbb{Z}_n))$ of the total graph of the integers modulo n, $\mathbb{Z}_n$, for certain values of n. We also identify specific values of n for which the graph is Q-integral. Finally, we discuss the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$.

Keywords: signless Laplacian eigenvalue, signless Laplacian spectrum, normalized Laplacian eigenvalue, normalized Laplacian spectrum

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1. Introduction

A simple graph G is defined with the vertex set $V(G) = \{v_1, v_2, \ldots, v_k\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_k\}$ with $v_i \sim v_j$ if and only if v_i is adjacent to v_j. Let $\mathbb{Z}_n$ be the ring of integers modulo n. The set of all zero divisors of $\mathbb{Z}_n$ denoted by $Z(\mathbb{Z}_n)$ and $Z^*(\mathbb{Z}_n)$ is the set of all non- zero zero divisors of $\mathbb{Z}_n$ that is, $Z^*(\mathbb{Z}_n) = \{x(\neq 0) \in \mathbb{Z}_n : (x, n) \neq 1\}$. Therefore, the number of non- zero zero divisor of n is, $n - \phi(n) - 1$. In this paper, we consider only simple and undirected graph. A graph G is said to be connected if and only if there exists a path between every pair of vertices of it. The number of edges of G incident to the vertex v is the degree of the vertex. There are different types of matrices for a graph G, incidence matrix, adjacency matrix, degree matrix, Laplacian matrix, signless Laplacian matrix, normalize Laplacian matrix etc. The adjacency matrix, $A(G)$ of a graph G is defined as $A(G) = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ or 0 according as $v_i \sim v_j$ or not. The Laplacian matrix, $L(G)$ and signless Laplacian matrix, $Q(G)$ of a graph G is defined

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as $D(G)-A(G)$ and $D(G)+A(G)$, respectively, where $D(G)$ is the degree matrix with all diagonal entries are the degree of the corresponding vertices and 0 otherwise. The normalized Laplacian matrix $\mathcal{L}(G)$ associated with the graph G is defined as,

$$\mathcal{L}(u,v)=\begin{cases}1 & \text{if } u=v \text{ and } \deg v\neq 0,\\ -\frac{1}{\sqrt{\deg_u \deg_v}} & \text{if u and v are adjacent,}\\ 0 & \text{otherwise.}\end{cases}$$

In fact, we have

$$\begin{aligned}\mathcal{L}(G)=&D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}\\ =&D(G)^{-\frac{1}{2}}(D(G)-A(G))D(G)^{-\frac{1}{2}}\\ =&I-D(G)^{-\frac{1}{2}}A(G)D(G)^{-\frac{1}{2}},\end{aligned}$$

with the convention $D^{-1}(v,v)=0$ for $\deg v=0$.

Both the matrices $\mathcal{L}(G)$ and $Q(G)$ are real positive semidefinite. 0 is the least eigenvalue in $\mathcal{L}(G)$ with multiplicity equal to the number of connected components of G. But 0 is the least signless Laplacian eigenvalue of $Q(G)$ if and only if G is bipartite and the multiplicity of 0 is equal to the number of bipartite components, which can be seen in [6].

The spectrum of $Q(G)$ and $\mathcal{L}(G)$ are said to be signless Laplacian spectrum, $\sigma_Q(G)$ and normalized Laplacian spectrum, $\sigma_{\mathcal{L}}(G)$, respectively. Let $\lambda_1,\lambda_2,\ldots,\lambda_k$ be the distinct eigenvalues of a matrix A with multiplicity $m_1,m_2,\ldots,m_k$, respectively, then we will denote the spectrum of A by

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_k\\ m_1 & m_2 & \ldots & m_k\end{pmatrix}.$$

A graph G is said to be complete if there exists an edge between every two distinct vertices of G. A complete graph G with n vertices is denoted by K_n and it is an $(n-1)$-regular graph. It is very easy to calculate the signless Laplacian spectrum and normalized Laplacian spectrum of K_n and these are

$$\sigma_Q(K_n)=\begin{pmatrix}2n-2 & n-2\\ 1 & n-1\end{pmatrix},$$

and

$$\sigma_{\mathcal{L}}(K_n)=\begin{pmatrix}0 & \frac{n}{n-1}\\ 1 & n-1\end{pmatrix},$$

respectively.

Here, $\phi(n)$ denotes the number of positive integers that are co-prime to n. For every positive integer n with the prime factorization $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$,

$$\phi(n)=(p_1^{e_1}-p_1^{e_1-1})(p_2^{e_2}-p_2^{e_2-1})\ldots(p_k^{e_k}-p_k^{e_k-1}).$$

For every prime number p,

$$\phi(p)+\phi(p^2)+\ldots+\phi(p^\alpha)=p^\alpha-1.$$

Any undefined terminology can be seen in [3] and [4].

In 2008, Anderson and Badawi [2] introduced the total graph, $T(\Gamma(R))$ of a commutative ring R. The vertices of $T(\Gamma(R))$ being the elements of R and two vertices x and y are adjacent if and only if $x + y \in Z(R)$. They introduced two induced subgraphs $Z(\Gamma(R))$ with vertices being the elements of $Z(R)$ and $Reg(\Gamma(R))$ with vertices being the regular or unit elements of R, of $T(\Gamma(R))$.

The Laplacian spectrum of the zero divisor graph of $\mathbb{Z}_n$ for different values of n, was first introduced by Sriparna Chattopadhya et al. in [5]. In [1], Mojgan Afkhami et al. established a way to find the signless Laplacian spectrum and normalized Laplacian spectrum of the zero divisor graph of $\mathbb{Z}_n$ for different values of n. They also proved some results related to the smallest and largest signless Laplacian eigenvalues of the same. Later, S. Pirzada et al. found the signless Laplacian spectrum of the zero divisor graph of $\mathbb{Z}_n$ for different values of n in [8] and [7]. They simplified the process by defining the zero divisor graph in G- generalized join graph and using the method to find the signless Laplacian spectrum of G-generalized join graph, described by B.-F. Wu et al. in [10].

In this paper, we establish a simple method to find the signless Laplacian spectrum and the normalized Laplacian spectrum of the subgraph $Z^*(\Gamma(\mathbb{Z}_n))$ of the total graph $T(\Gamma(\mathbb{Z}_n))$ for $n = pq, p^t$ and $n = p^\alpha q^\beta$ with p and q are distinct primes and t, α and β are positive integers. In Section 2, we give some results about the structure of $Z^*(\Gamma(\mathbb{Z}_n))$ for some values of n. We also determine the signless Laplacian spectrum and normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$, for different values of n and discuss some related results.

2. Structure of $Z^*(\Gamma(\mathbb{Z}_n))$

In $Z^*(\Gamma(\mathbb{Z}_n))$, the vertices are all the non-zero zero divisors of $\mathbb{Z}_n$. Young [11] defined a new set $A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\}$, where d_i is a proper divisor of n. Young defined vertices of $Z^*(\Gamma(\mathbb{Z}_n))$ as the union of pairwise disjoint sets A_{d_i} and expressed as,

$$V^*(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \ldots \cup A_{d_k},$$

where $d_1, d_2 \ldots d_k$ are the proper divisors of n.

Now we state Lemma 2.1 from [11].

Lemma 2.1. [11] $|A_{d_i}| = \phi(\frac{n}{d_i})$.

For $i, j \in \{1, 2, \ldots, k\}$, the adjacency relationship between the set A_{d_i} and A_{d_j} plays a vital role in $Z^*(\Gamma(\mathbb{Z}_n))$. We state Lemma 2.2 and Lemma 2.3 from [4].

Lemma 2.2. [4] *In $Z^*(\Gamma(\mathbb{Z}_n))$, two vertices x and y are adjacent if and only if atleast one prime factor of $x + y$ is also a prime factor of n.*

Lemma 2.3. [4] *For $i, j \in \{1, 2, \ldots, k\}$ and for $n = pq, p^t$ and $n = p^r q^s$, where p and q are two distinct primes and r, s, t are positive integers, every vertex of A_{d_i} is adjacent to all vertices of A_{d_j} in $Z^*(\Gamma(\mathbb{Z}_n))$ if and only if $(d_i, d_j) \neq 1$.*

The structure of the graph $A_{d_i}(\Gamma(\mathbb{Z}_n))$ for all $i \in \{1, 2, \dots k\}$ was defined in [4]. Lemma 2.4 [4] states the structure of $A_{d_i}(\Gamma(\mathbb{Z}_n))$.

Lemma 2.4. [4] *For all $i \in \{1, 2, \dots k\}$, the induced subgraph $A_{d_i}(\Gamma(\mathbb{Z}_n))$ of $Z^*(\Gamma(\mathbb{Z}_n))$ with vertices being the elements of A_{d_i} is the complete graph $K_{\phi(\frac{n}{d_i})}$.*

In [9], Schwenk defined the G-generalised join graph or generalised composition graph, which plays a significant role to find the signless Laplacian spectrum and normalize Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for different values of n. For a graph G with the vertex set $V(G) = \{v_1, v_2 \dots v_k\}$, the G- generalized join graph $G[H_1, H_2 \dots H_k]$ of k pairwise disjoint graphs $H_1, H_2 \dots H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_j whenever $v_i \sim v_j$ in G. We state Lemma 2.5 from [4], which establishes the graph $Z^*(\Gamma(\mathbb{Z}_n))$ as γ_n-generalised join graph for $n = pq, p^r$ and $n = p^r q^s$, where p and q are two distinct primes and r, s are positive integers with at least one is greater than 1.

Lemma 2.5. [4] *For $n = pq, p^r$ and $n = p^r q^s$, where p and q are two distinct primes and r, s are positive integers with at least one is greater than 1,*

$$Z^*(\Gamma(\mathbb{Z}_n)) = \gamma_n[K_{\phi(\frac{n}{d_1})}, K_{\phi(\frac{n}{d_2})}, \dots K_{\phi(\frac{n}{d_k})}],$$

where γ_n is a graph with vertices being the elements of proper divisors of n and two vertices x and y are adjacent if and only if $(x, y) \neq 1$.

Moreover, for connectedness, we state Lemma 2.6 from [4].

Lemma 2.6. *$Z^*(\Gamma(\mathbb{Z}_n))$ is connected, provided $n \neq pq$ where p and q are primes.*

3. Signless Laplacian spectrum of G- generalised join graph

The signless Laplacian spectrum of G- generalised join graph was introduced by B.-F.Wu et al. in [10]. The following result, Theorem 3.1 [10], is used to find the signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for some values of n.

Theorem 3.1. [10] *Let G be a graph with vertex set $\{v_1, v_2, \dots, v_k\}$ and let $H_1, \dots H_k$ be k-pairwise disjoint r_1- regular,..., r_k- regular graphs with $m_1, \dots m_k$ vertices, respectively. Then the signless Laplacian spectrum of $G[H_1, H_2, \dots H_k]$ is given by*

$$\sigma_Q(G[H_1, H_2, \dots H_k]) = \left(\bigcup_{j=1}^{k} (M_j + (\sigma_Q(H_j) \setminus \{2r_j\})) \right) \bigcup \sigma(\mathcal{Q}(G)), \qquad (1)$$

where $\mathcal{Q}(G) = (q_{ij})_{k \times k}$ with

$$q_{ij}=\begin{cases}2r_i+M_i, & i=j,\\ \sqrt{m_im_j}, & v_i\sim v_j,\\ 0, & otherwise,\end{cases} \tag{2}$$

and

$$M_j=\begin{cases}\sum\limits_{v_i\sim v_j} m_i & if\ N_G(v_j)\neq\phi,\\ 0 & otherwise.\end{cases}$$

In (1), $\sigma_Q(H_j)\setminus\{2r_j\}$ means that one copy of eigenvalue $2r_j$ is removed from the multiset $\sigma_Q(H_j)$ and $M_j+(\sigma_Q(H_j)\setminus\{2r_j\}$ means that M_j is added to each element of $\sigma_Q(H_j)\setminus\{2r_j\}$.

Now, by using Theorem 3.1, we find the signless Laplacian spectrum $Z^*(\Gamma(\mathbb{Z}_n))$ where $n=pq, p^t$ and $n=p^\alpha q^\beta$ with p and q are distinct primes and t, α and β are positive integers.

Theorem 3.2. *The signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{pq}))$ is given by*

$$\begin{pmatrix}2(\phi(p)-1) & 2(\phi(q)-1) & \phi(p)-2 & \phi(q)-2\\ 1 & 1 & \phi(p)-1 & \phi(q)-1\end{pmatrix}.$$

Proof. From Lemma 2.5 and Lemma 2.6, $Z^*(\Gamma(\mathbb{Z}_{pq}))$ is a disconnected graph and $Z^*(\Gamma(\mathbb{Z}_{pq}))$ $=K_{\phi(p)}\cup K_{\phi(p)}$. Therefore, the signless Laplacian matrix of it is a block diagonal matrix with two blocks $\phi(p)I-J$ and $\phi(q)I-J$, respectively, where I is the identity matrix and J is the matrix with all entries are equal to 1. Thus the characteristic polynomial of $Z^*(\Gamma(\mathbb{Z}_{pq}))$ is

$$(x-2(\phi(p)-1))(x-(\phi(p)-2))^{\phi(p)-1}(x-2(\phi(q)-1))(x-(\phi(q)-2))^{\phi(q)-1},$$

and the roots of this polynomial are the required signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{pq}))$. □

Theorem 3.3. *For $n=p^t$, the signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{p^t}))$ is given by*

$$\begin{pmatrix}2p^{t-1}-4 & p^{t-1}-3\\ 1 & p^{t-1}-2\end{pmatrix}.$$

Proof. It can be easily seen that

$$Z^*(\Gamma(\mathbb{Z}_{p^t}))=K_{p^{t-1}-1}.$$

Thus the signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{p^t}))$ is the signless Laplacian spectrum of $K_{p^{t-1}-1}$. Thus the result. □

Let $n = p^\alpha q^\beta$ with $\alpha > 1$ and $\beta \geq 1$. Therefore by Lemma 2.5 we have

$$Z^*(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})) = \gamma_{p^\alpha q^\beta}[K_{\phi(p^{\alpha-1}q^\beta)}, K_{\phi(p^{\alpha-2}q^\beta)}, \ldots, K_{\phi(q)}].$$

We provide the following calculation related to the signless Laplacian spectrum. Throughout the observation of $Z^*(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$, we compute M_i for each vertex $i \in V(\gamma_{p^\alpha q^\beta})$ as follows:

$$M_P = \sum_{i=2}^{\alpha} \left(\phi(p^{\alpha-i}q^\beta)\right) + \sum_{i=1}^{\alpha} \left(\sum_{j=1}^{\beta-1} \left(\phi(p^{\alpha-i}q^{\beta-j})\right)\right) + \sum_{i=1}^{\alpha-1} \left(\phi(p^{\alpha-i})\right),$$

and we calculate

$$\sum_{i=1}^{\alpha} \left(\sum_{j=1}^{\beta-1} \left(\phi(p^{\alpha-i}q^{\beta-j})\right)\right) + \sum_{i=1}^{\alpha-1} \left(\phi(p^{\alpha-i})\right) = p^{\alpha-1}q^{\beta-1} - 1,$$

and

$$\sum_{i=2}^{\alpha} \left(\phi(p^{\alpha-i}q^\beta)\right) = \phi(q^\beta)p^{\alpha-2}.$$

Thus

$$M_P = \phi(q^\beta)p^{\alpha-2} + p^{\alpha-1}q^{\beta-1} - 1. \tag{3}$$

In the same way, we get for $i \in \{2, 3, \ldots, \alpha\}$ and $j \in \{2, 3, \ldots, \alpha\}$

$$M_{p^i} = \phi(q^\beta)(p^{\alpha-1} - p^{\alpha-i} + p^{\alpha-(i+1)}) + p^{\alpha-1}q^{\beta-1} - 1, \tag{4}$$
$$M_q = \phi(p^\alpha)q^{\beta-2} + p^{\alpha-1}q^{\beta-1} - 1, \tag{5}$$
$$M_{q^j} = \phi(p^\alpha)(q^{\beta-1} - q^{\beta-j} + q^{\beta-(j+1)}) + p^{\alpha-1}q^{\beta-1} - 1, \tag{6}$$
$$M_{pq} = \phi(p^\alpha)q^{\beta-1} + \phi(q^\beta)p^{\alpha-1} + \phi(q^{\beta-1})p^{\alpha-2} + p^{\alpha-1}q^{\beta-2} - 1, \tag{7}$$
$$M_{pq^j} = \phi(p^\alpha)q^{\beta-1} + \phi(q^\beta)p^{\alpha-1} + \phi(q^{\beta-j})p^{\alpha-2} + p^{\alpha-1}(q^{\beta-1} - q^{\beta-j} + q^{\beta-(j+1)}) - 1, \tag{8}$$
$$M_{p^iq} = \phi(p^\alpha)q^{\beta-1} + \phi(q^\beta)p^{\alpha-1} + \phi(q^{\beta-1})(p^{\alpha-1} - p^{\alpha-i} + p^{\alpha-(i+1)}) + p^{\alpha-1}q^{\beta-2} - 1. \tag{9}$$

Hence

$$\begin{aligned} M_{p^iq^j} =& \sum_{i=1}^{\alpha} \left(\phi(p^{\alpha-i}q^\beta)\right) + \sum_{j=1}^{\beta} \left(\phi(p^\alpha q^{\beta-j})\right) + \sum_{i=1}^{\alpha} \left(\sum_{j=1}^{\beta-1} \left(\phi(p^{\alpha-i}q^{\beta-j})\right)\right) \\ & - \phi(p^{\alpha-i}q^{\beta-j}) + \sum_{i=1}^{\alpha-1} \left(\phi(p^{\alpha-i})\right) \\ =& \phi(p^\alpha)q^{\beta-1} + \phi(q^\beta)p^{\alpha-1} + \phi(q^{\beta-j})(p^{\alpha-1} - p^{\alpha-i} + p^{\alpha-(i+1)}) \\ & + p^{\alpha-1}(q^{\beta-1} - q^{\beta-j} + q^{\beta-(j+1)}) - 1. \end{aligned} \tag{10}$$

Now we have the signless Laplacian spectrum of $K_{\phi(p^{\alpha-1}q^\beta)}$. Also

$$\sum_{i=2}^{\alpha} \left(\phi(p^{\alpha-i}q^\beta)\right) + \phi(p^{\alpha-1}q^\beta) - 2 = \phi(q^\beta)(p^{\alpha-1}) - 2.$$

Thus,

$$M_p + (\sigma_Q(K_{\phi(p^{\alpha-1}q^{\beta})}) \setminus \{2(\phi(p^{\alpha-1}q^{\beta}) - 1)\}) = \begin{pmatrix} \phi(q^{\beta})(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3 \\ \phi(p^{\alpha-1}q^{\beta}) - 1 \end{pmatrix}.$$

Similarly for $i \in \{1, 2, \ldots, \alpha\}$ and $j \in \{1, 2, \ldots, \beta\}$,

$$M_{p^i} + (\sigma_Q(K_{\phi(p^{\alpha-i}q^{\beta})}) \setminus \{2(\phi(p^{\alpha-i}q^{\beta}) - 1)\}) = \begin{pmatrix} \phi(q^{\beta})(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3 \\ \phi(p^{\alpha-i}q^{\beta}) - 1 \end{pmatrix},$$

$$M_{q^j} + (\sigma_Q(K_{\phi(p^{\alpha}q^{\beta-j})}) \setminus \{2(\phi(p^{\alpha}q^{\beta-j}) - 1)\}) = \begin{pmatrix} \phi(p^{\alpha})(q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 \\ \phi(p^{\alpha}q^{\beta-j}) - 1 \end{pmatrix},$$

$$\begin{aligned} &M_{p^iq^j} + (\sigma_Q(K_{\phi(p^{\alpha-i}q^{\beta-j})}) \setminus \{2(\phi(p^{\alpha-i}q^{\beta-j}) - 1)\}) \\ &= \begin{pmatrix} \phi(q^{\beta})(p^{\alpha-1}) + \phi(p^{\alpha})q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 \\ \phi(p^{\alpha-i}q^{\beta-j}) - 1 \end{pmatrix}. \end{aligned}$$

The multiplicity of each term can be determined. The multiplicity of $\phi(q^{\beta})(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3$ is $\sum_{i=1}^{\alpha} \big(\phi(p^{\alpha-i}q^{\beta}) - 1\big)$, that is, $\phi(q^{\beta})p^{\alpha-1} - \alpha$. Similarly, the multiplicity of $\phi(p^{\alpha})(q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3$ is $\phi(p^{\alpha})q^{\beta-1} - \beta$ and for $\phi(q^{\beta})(p^{\alpha-1}) + \phi(p^{\alpha})q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3$ is $p^{\alpha-1}q^{\beta-1} - \alpha\beta$. From Theorem 3.1, we have

$$\begin{aligned} \sigma_Q(Z^*(\Gamma(\mathbb{Z}_{p^{\alpha}q^{\beta}}))) = &\bigcup_{i=1}^{\alpha} \big(M_{p^i} + (\sigma_Q(K_{\phi(p^{\alpha-i}q^{\beta})}) \setminus \{2(\phi(p^{\alpha-i}q^{\beta}) - 1)\})\big) \\ &\bigcup_{j=1}^{\beta} \big(M_{q^j} + (\sigma_Q(K_{\phi(p^{\alpha}q^{\beta-j})}) \setminus \{2(\phi(p^{\alpha}q^{\beta-j}) - 1)\})\big) \\ &\bigcup_{i=1}^{\alpha} \left(\bigcup_{j=1}^{\beta-1} \big(M_{p^iq^j} + (\sigma_Q(K_{\phi(p^{\alpha-i}q^{\beta-j})}) \setminus \{2(\phi(p^{\alpha-i}q^{\beta-j}) - 1)\})\big) \right) \\ &\bigcup_{i=1}^{\alpha-1} \big(M_{p^iq^{\beta}} + (\sigma_Q(K_{\phi(p^{\alpha-i})}) \setminus \{2(\phi(p^{\alpha-i}) - 1)\})\big) \bigcup (\sigma_Q(\mathcal{Q}(G))). \end{aligned}$$

Hence a part of signless Laplacian spectrum of the above eigenvalues of $Z^*(\Gamma(\mathbb{Z}_{p^{\alpha}q^{\beta}}))$ is given by

$$\begin{pmatrix} \phi(q^{\beta})(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \phi(p^{\alpha})(q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \begin{matrix} \phi(q^{\beta})(p^{\alpha-1}) \\ +\phi(p^{\alpha})q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 \end{matrix} \\ \phi(q^{\beta})p^{\alpha-1} - \alpha & \phi(p^{\alpha})q^{\beta-1} - \beta & p^{\alpha-1}q^{\beta-1} - \alpha\beta \end{pmatrix}$$

and all other remaining signless Laplacian eigenvalues are calculated from the matrix $\mathcal{Q}(\mathcal{G})$, which is given in Eq. (2). By using Eqs (3)-(10), we calculate the matrix $\mathcal{Q}(\mathcal{G})$ in the following way:

$$q_{ij} = \begin{cases} \phi\big(q^{\beta}p^{\alpha-s}\big) + \phi\big(q^{\beta}\big)\,p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 3, & \text{if } i=j=p^s,\ s\in\{1,\dots,\alpha\},\\ \phi\big(p^{\alpha}q^{\beta-t}\big) + \phi\big(p^{\alpha}\big)\,q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 3, & \text{if } i=j=q^t,\ t\in\{1,\dots,\beta\},\\ \phi\big(p^{\alpha-s}q^{\beta-t}\big) + \phi\big(q^{\beta}\big)\,p^{\alpha-1} + \phi\big(p^{\alpha}\big)\,q^{\beta-1} & \\ \qquad + p^{\alpha-1}q^{\beta-1} - 3, & \text{if } i=j=p^sq^t,\ s\in\{1,\dots,\alpha\},\\ & \quad t\in\{1,\dots,\beta\},\\ \sqrt{\phi(p^{\alpha-s}q^{\beta})\ \phi(p^{\alpha-t}q^{\beta})}, & \text{if } i=p^s,\ j=p^t,\ i\neq j,\ s,t\in\{1,\dots,\alpha\},\\ \sqrt{\phi(p^{\alpha-s}q^{\beta})\ \phi(p^{\alpha-t}q^{\beta-u})}, & \text{if } i=p^s,\ j=p^tq^u,\ s,t\in\{1,\dots,\alpha\},\\ & \quad u\in\{1,\dots,\beta\},\\ \sqrt{\phi(p^{\alpha}q^{\beta-s})\ \phi(p^{\alpha}q^{\beta-t})}, & \text{if } i=q^s,\ j=q^t,\ i\neq j,\ s,t\in\{1,\dots,\beta\},\\ \sqrt{\phi(p^{\alpha}q^{\beta-s})\ \phi(p^{\alpha-t}q^{\beta-u})}, & \text{if } i=q^s,\ j=p^tq^u,\ s,u\in\{1,\dots,\beta\},\\ & \quad t\in\{1,\dots,\alpha\},\\ \sqrt{\phi(p^{\alpha-s}q^{\beta-t})\ \phi(p^{\alpha-u}q^{\beta-v})}, & \text{if } i=p^sq^t,\ j=p^uq^v,\ i\neq j,\\ & \quad s,u\in\{1,\dots,\alpha\},\ t,v\in\{1,\dots,\beta\},\\ \sqrt{\phi(p^{\alpha-s}q^{\beta})\phi(p^{\alpha-t}q^{\beta-u})}, & \text{if } i=p^tq^u,\ j=p^s,\ \text{and } s\in\{1,2,\dots,\alpha\},\\ & \quad t\in\{1,2,\dots,\alpha\}, u\in\{1,2,\dots,\beta\},\\ \sqrt{\phi(p^{\alpha}q^{\beta-s})\phi(p^{\alpha-t}q^{\beta-u})}, & \text{if } i=p^tq^u,\ j=q^s,\ \text{and } s\in\{1,2,\dots,\beta\},\\ & \quad t\in\{1,2,\dots,\alpha\}, u\in\{1,2,\dots,\beta\},\\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

It may be difficult to find the eigenvalues of the matrix $\mathcal{Q}(\mathcal{G})$ of higher order. To find eigenvalues of such kind of matrices we can take help of Matlab.

From the above discussion, we state Theorem 3.4.

Theorem 3.4. *The signless Laplacian spectrum of* $Z^*(\Gamma(\mathbb{Z}_n))$ *is given by*

$$\begin{pmatrix} \phi(q^{\beta})(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \phi(p^{\alpha})(q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \phi(p^{\alpha})(q^{\beta-1}) + \phi(q^{\beta})(p^{\alpha-1}) \\ & & + p^{\alpha-1}q^{\beta-1} - 3 \\ \phi(q^{\beta})p^{\alpha-1} - \alpha & \phi(p^{\alpha})q^{\beta-1} - \beta & p^{\alpha-1}q^{\beta-1} - \alpha\beta \end{pmatrix},$$

and all the other eigenvalues can be calculated from the matrix $\mathcal{Q}(\mathcal{G})$ *with (11).*

Example 3.5. For $Z*(\Gamma(\mathbb{Z}_{36}))$, the signless Laplacian spectrum of $Z*(\Gamma(\mathbb{Z}_{36}))$ is given by

$$\begin{pmatrix} 15 & 9 & 21 \\ 10 & 4 & 2 \end{pmatrix},$$

and the remaining eigenvalues are calculated from the matrix $\mathcal{Q}(G)$, where

$$\mathcal{Q}(G) = \begin{pmatrix} 21 & 6 & \sqrt{12} & \sqrt{12} & \sqrt{6} & 0 & 0 \\ 6 & 21 & \sqrt{12} & \sqrt{12} & \sqrt{6} & 0 & 0 \\ \sqrt{12} & \sqrt{12} & 23 & 2 & \sqrt{2} & \sqrt{8} & 2 \\ \sqrt{12} & \sqrt{12} & 2 & 23 & \sqrt{2} & \sqrt{8} & 2 \\ \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{2} & 22 & 2 & \sqrt{2} \\ 0 & 0 & \sqrt{8} & \sqrt{8} & 2 & 13 & 4 \\ 0 & 0 & 2 & 2 & \sqrt{2} & 4 & 11 \end{pmatrix}.$$

Note that if all the signless Laplacian eigenvalues of a graph G are integers, then we say that G is Q-integral. In Lemma 3.6, Wu et al. [10] established a result on the Q-integral eigenvalues of the G-generalized join graph.

Lemma 3.6. [10] *Let H be a graph with $V(H) = \{1, 2, \dots, k\}$, let G_i be a r_i- regular graph of order n_i $(i = 1, 2, \dots, k)$ and $G = H[G_1, G_2, \dots, G_k]$ and $Q(H)$ is defined as in Theorem 3.1, then G is Q- integral if and only if $G_1, G_2, \dots, G_k$ are Q- integral and the eigenvalues of $Q(H)$ are integers. Moreover, if $G_1, G_2, \dots G_k$ are all r-regular graphs of order n, then G is Q-integral if and only if $G_1, G_2, \dots, G_k$ and H are Q- integral.*

Lemma 3.7. *For $n = p^\alpha q^\beta$ with p and q are distinct prime numbers and α and β are two positive integers, then $Z^*(\Gamma(\mathbb{Z}_n))$ is Q- integral if and only if all the eigenvalues of $\mathcal{Q}(G)$ are integers.*

Proof. From Theorem 3.4 a part of signless Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ is given by

$$\begin{pmatrix} \phi(q^\beta)(p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \phi(p^\alpha)(q^{\beta-1}) + p^{\alpha-1}q^{\beta-1} - 3 & \begin{matrix}\phi(p^\alpha)(q^{\beta-1}) + \phi(q^\beta)(p^{\alpha-1}) \\ +p^{\alpha-1}q^{\beta-1} - 3\end{matrix} \\ \phi(q^\beta)p^{\alpha-1} - \alpha & \phi(p^\alpha)q^{\beta-1} - \beta & p^{\alpha-1}q^{\beta-1} - \alpha\beta \end{pmatrix}$$

and all other eigenvalues can be calculated from the symmetric matrix $\mathcal{Q}(G)$ defined in Theorem 3.4.

Since $\phi(n)$ is always a positive integer for every positive integer n, by using Lemma 3.6 we can establish the result. □

Using Theorem 3.2 and Theorem 3.3, it is verified that for $n = pq$ and $n = p^t$, $Z^*(\Gamma(\mathbb{Z}_n))$ is Q-integral.

4. Normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$

In this section, we discuss the normalized Lapalcian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for $n = pq$ and $n = p^t$ and $n = p^\alpha q^\beta$, where p and q are two distinct primes and t, α and β are positive integers. It can be easy to find the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for different values of n, by using Theorem 4.1, given by Wu et al. [10] in 2014.

Theorem 4.1. *Let G be a graph with vertex set $\{v_1, v_2, \dots, v_k\}$ and let $H_1, H_2, \dots, H_k$ be k-pairwise disjoint r_i- regular graphs with $m_1, m_2, \dots, m_k$ vertices, respectively. Then the normalized Laplacian spectrum of $G[H_1, H_2, \dots, H_k]$ is given by*

$$\sigma_{\mathcal{L}}(G[H_1, H_2, \dots, H_k]) = \left(\bigcup_{j=1}^{k} \left(\frac{M_j}{r_j + M_j} + \frac{r_j}{r_j + M_j}(\sigma_{\mathcal{L}}(H_j) \setminus \{0\}) \right) \right) \bigcup \sigma(\mathcal{L}(G)),$$

where $\mathcal{L}(G) = (I_{ij})_{k \times k}$ with

$$I_{ij} = \begin{cases} \frac{M_j}{r_j+M_j}, & i = j \text{ and } d_G(v_i) \neq 0, \\ -\sqrt{\frac{m_i m_j}{(r_i+M_i)(r_j+M_j)}}, & v_i \sim v_j, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

and

$$M_j = \begin{cases} \sum\limits_{v_i \sim v_j} m_i, & \text{if } N_G(v_j) \neq \phi, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\sigma_{\mathcal{L}}(H_j) \setminus \{0\}$ means that one copy of the eigenvalue 0 is removed from the multiset $\sigma_{\mathcal{L}}(H_j)$ and $\frac{r_j}{r_j+M_j}(\sigma_{\mathcal{L}}(H_j) \setminus \{0\})$ means $\frac{r_j}{r_j+M_j}$ is multiplied to the remaining values of $\sigma_{\mathcal{L}}(H_j) \setminus \{0\}$. Also by $\frac{M_j}{r_j+M_j} + \frac{r_j}{r_j+M_j}(\sigma_{\mathcal{L}}(H_j) \setminus \{0\})$, $\frac{M_j}{r_j+M_j}$ is added to each term of the multiset $\frac{r_j}{r_j+M_j}(\sigma_{\mathcal{L}}(H_j) \setminus \{0\})$. Moreover, it is clear that $\mathcal{L}(G)$ is a symmetric matrix.

The normalized Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is established in [1], which is stated in Theorem 4.2.

Theorem 4.2. *Let $d_1, d_2, \dots, d_k$ be the proper divisors of n. Then the normalized Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is given by*

$$\sigma_{\mathcal{L}}(\Gamma(\mathbb{Z}_n)) = \left(\bigcup_{j=1}^{k} \left(\frac{M_{d_j}}{r_j + M_{d_j}} + \frac{r_j}{r_j + M_{d_j}}(\sigma_{\mathcal{L}}(\Gamma(A_{d_j})) \setminus \{0\}) \right) \right) \bigcup \sigma(\mathcal{L}(G_n)),$$

where r_j is equal to $\phi(\frac{n}{d_j}) - 1$ or 0.

We determine the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for some values of n by using Theorem 4.1 and Theorem 4.2. In Theorem 4.5, we discuss the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ for $n = p^\alpha q^\beta$, where p and q are distinct primes and α and β are two non-negative integers.

Theorem 4.3. *The normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{pq}))$ is given by*

$$\begin{pmatrix} \frac{q-1}{q-2} & \frac{p-1}{p-2} & 0 \\ p-2 & q-2 & 2 \end{pmatrix}.$$

Proof. By Lemma 2.5, $Z^*(\Gamma(\mathbb{Z}_{pq})) = \gamma_{pq}[K_{\phi(q)}, K_{\phi(p)}]$. Therefore, the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{pq}))$ is given by

$$\begin{aligned}\sigma_{\mathcal{L}}(Z^*(\Gamma(\mathbb{Z}_{pq}))) = &\left(\frac{M_p}{\phi(q)-1+M_p} + \frac{\phi(q)-1}{\phi(q)-1+M_p}(\sigma_{\mathcal{L}}(K_{\phi(q)} \setminus \{0\}))\right) \\ &\bigcup \left(\frac{M_q}{\phi(p)-1+M_q} + \frac{\phi(p)-1}{\phi(p)-1+M_q}(\sigma_{\mathcal{L}}(K_{\phi(p)} \setminus \{0\}))\right) \\ &\bigcup \sigma(\mathcal{L}(G_{pq})).\end{aligned}$$

Here $M_p = 0$ and $M_q = 0$.

Putting the values of M_p and M_q, we obtain

$$\sigma_{\mathcal{L}}(Z^*(\Gamma(\mathbb{Z}_{pq}))) = \begin{pmatrix} \frac{q-1}{q-2} & \frac{p-1}{p-2} \\ p-2 & q-2 \end{pmatrix} \bigcup \sigma(\mathcal{L}(G_{pq})).$$

We see that

$$\mathcal{L}(G_{pq}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the result. □

We notice that

$$Z^*(\Gamma(\mathbb{Z}_{p^t})) = K_{p^{t-1}-1}.$$

Thus the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{p^t}))$ is given in Theorem 4.4

Theorem 4.4. *The normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{p^t}))$ is given by*

$$\begin{pmatrix} 0 & \frac{p^{t-1}-1}{p^{t-1}-2} \\ 1 & p^{t-1}-2 \end{pmatrix}.$$

Theorem 4.5. *For $n = p^\alpha q^\beta$, the normalized Laplacain spectrum of $Z^*(\Gamma(\mathbb{Z}_n))$ is given by*

$$\begin{pmatrix} A & B & C \\ \phi(q^\beta)p^{\alpha-1}-\alpha & \phi(p^\alpha)q^{\beta-1}-\beta & p^{\alpha-1}q^{\beta-1}-\alpha\beta+1 \end{pmatrix} \tag{13}$$

where

$$\begin{aligned} A &= \frac{\phi(q^\beta)p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^\beta)p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}, \\ B &= \frac{\phi(p^\alpha)q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(p^\alpha)q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2}, \\ C &= \frac{\phi(q^\beta)p^{\alpha-1}+\phi(p^\alpha)q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^\beta)p^{\alpha-1}+\phi(p^\alpha)q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2}, \end{aligned}$$

and the remaining eigenvalues are determine from the symmetric matrix $\mathcal{L}(G) = (I_{ij})_{k\times k}$, given by

$$I_{ij} = \frac{\phi(q^\beta)p^{\alpha-2}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^\beta)p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1},$$

if $i = j = p$, *and* $d_G(p) \neq 0$.

$$I_{ij} = \frac{\phi(q^\beta)(p^{\alpha-1} - p^{\alpha-t} + p^{\alpha-(t+1)}) + p^{\alpha-1}q^{\beta-1} - 1}{\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = p^t,\ t \in \{2, 3, \ldots, \alpha\}$ *and* $d_G(p^t) \neq 0$.

$$I_{ij} = \frac{\phi(p^\alpha)q^{\beta-2} + p^{\alpha-1}q^{\beta-1} - 1}{\phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = q$, *and* $d_G(q) \neq 0$.

$$I_{ij} = \frac{\phi(p^\alpha)(q^{\beta-1} - q^{\beta-t} + q^{\beta-(t+1)}) + p^{\alpha-1}q^{\beta-1} - 1}{\phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = q^t,\ t \in \{2, 3, \ldots, \beta\}$, *and* $d_G(q) \neq 0$.

$$I_{ij} = \frac{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + \phi(q^{\beta-1})p^{\alpha-2} + p^{\alpha-1}q^{\beta-2} - 1}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = pq$, *and* $d_G(pq) \neq 0$.

$$I_{ij} = \frac{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + \phi(q^{\beta-1})(p^{\alpha-1} - p^{\alpha-t} + p^{\alpha-(t+1)}) + p^{\alpha-1}q^{\beta-2} - 1}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = p^t q$, $t \in \{2, 3, \ldots, \alpha\}$*and* $d_G(p^t q) \neq 0$.

$$I_{ij} = \frac{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + \phi(q^{\beta-t})p^{\alpha-2} + p^{\alpha-1}(q^{\beta-1} - q^{\beta-t} + q^{\beta-(t+1)}) - 1}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1},$$

if $i = j = pq^t$, $t \in \{2, 3, \ldots, \beta\}$ *and* $d_G(pq^t) \neq 0$.

$$\begin{aligned} I_{ij} =& \frac{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + \phi(q^{\beta-s})(p^{\alpha-1} - p^{\alpha-t} + p^{\alpha-(t+1)})}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1}, \\ &+ \frac{p^{\alpha-1}(q^{\beta-1} - q^{\beta-s} + q^{\beta-(s+1)}) - 1}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1}, \end{aligned}$$

if $i = j = p^t q^s$, $t \in \{2, 3, \ldots, \alpha\}$ $s \in \{2, 3, \ldots, \beta\}$ *and* $d_G(p^t q^s) \neq 0$.

$$I_{ij} = -\sqrt{\frac{\phi(p^{\alpha-s}q^\beta)\phi(p^{\alpha-t}q^\beta)}{(\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 2)^2}},$$

if $i = p^s,\ j = p^t$ *and* $s \neq t,\ s, t \in \{1, 2, \ldots, \alpha\}$.

$$I_{ij} = -\sqrt{\frac{\phi(p^{\alpha-s}q^\beta)\phi(p^{\alpha-t}q^{\beta-u})}{(\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 2)(\phi(p^\alpha)q^{\beta-1} + \phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 2)}},$$

if $i = p^s$ *and* $j = p^t q^u$ *with* $s, t \in \{1, 2, \ldots, \alpha\},\ u \in \{1, 2, \ldots, \beta\}$.

$$I_{ij} = -\sqrt{\frac{\phi(p^\alpha q^{\beta-s})\phi(p^\alpha q^{\beta-t})}{(\phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1)^2}},$$

if $i = q^s,\ j = q^t$ *and* $s \neq t,\ s,t \in \{1,2,\ldots,\alpha\}$.

$$I_{ij} = -\sqrt{\frac{\phi(p^\alpha q^{\beta-s})\phi(p^{\alpha-t}q^{\beta-u})}{(\phi(p^\alpha)q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2)(\phi(p^\alpha)q^{\beta-1}+\phi(q^\beta)p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2)}},$$

if $i = q^s$ *and* $j = p^t q^u$ *with* $s,u \in \{1,2,\ldots,\beta\},\ u \in \{1,2,\ldots,\beta\}$.

Proof. The structure of $Z^*(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$ was established in Lemma 2.5. From Theorem 4.1, we have

$$\begin{aligned}
&\sigma_{\mathcal{L}}(Z^*(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})))\\
&=\bigcup_{i=1}^{\alpha}\left(\frac{M_{p^i}}{\phi(p^{\alpha-i}q^\beta)-1+M_{p^i}}+\frac{\phi(p^{\alpha-i}q^\beta)-1}{\phi(p^{\alpha-i}q^\beta)-1+M_{p^i}}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i}q^\beta)})\setminus\{0\})\right)\\
&\bigcup_{j=1}^{\beta}\left(\frac{M_{q^j}}{\phi(p^\alpha q^{\beta-j})-1+M_{q^j}}+\frac{\phi(p^\alpha q^{\beta-j})-1}{\phi(p^\alpha q^{\beta-j})-1+M_{q^j}}(\sigma_{\mathcal{L}}(K_{\phi(p^\alpha q^{\beta-j})})\setminus\{0\})\right)\\
&\bigcup_{i=1}^{\alpha}\left(\bigcup_{j=1}^{\beta-1}\left(\frac{M_{p^iq^j}}{\phi(p^{\alpha-i}q^{\beta-j})-1+M_{p^iq^j}}+\frac{\phi(p^{\alpha-i}q^{\beta-j})-1}{\phi(p^{\alpha-i}q^{\beta-j})-1+M_{p^iq^j}}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i}q^{\beta-j})})\setminus\{0\})\right)\right)\\
&\bigcup_{i=1}^{\alpha-1}\left(\frac{M_{p^i}}{\phi(p^{\alpha-i})-1+M_{p^iq^\beta}}+\frac{\phi(p^{\alpha-i})}{\phi(p^{\alpha-i})-1+M_{p^iq^\beta}}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i})})\setminus\{0\})\right)\bigcup\sigma(\mathcal{L}(G_n)).
\end{aligned} \tag{14}$$

Here $M_p, M_{p^i}, M_q, M_{q^j}$ and $M_{p^iq^j}$ are calculated in (3)-(10). Now for all $i \in \{1,2,\ldots,\alpha\}$ and $j \in \{1,2,\ldots,\beta\}$,

$$\phi(p^{\alpha-i}q^\beta)-1+M_{p^i} = \phi(q^\beta)(p^{\alpha-1})+p^{\alpha-1}q^{\beta-1}-2,$$

$$\phi(p^\alpha q^{\beta-j})-1+M_{q^j} = \phi(p^\alpha)(q^{\beta-1})+p^{\alpha-1}q^{\beta-1}-2,$$

and

$$\phi(p^{\alpha-i}q^{\beta-j})-1+M_{p^iq^j} = \phi(p^\alpha)q^{\beta-1}+\phi(q^\beta)p^{\alpha-1}++p^{\alpha-1}q^{\beta-1}-2,\ j \in \{1,2,\ldots,\beta-1\},$$
$$\phi(p^{\alpha-i})-1+M_{p^iq^\beta} = \phi(p^\alpha)q^{\beta-1}+\phi(q^\beta)p^{\alpha-1}++p^{\alpha-1}q^{\beta-1}-2,\ i \in \{1,2,\ldots,\alpha-1\}.$$

Each term of the Eq. (14) is determined in the following manner:

$$\begin{aligned}
&\frac{M_p}{\phi(p^{\alpha-1}q^\beta)-1+M_p}+\frac{\phi(p^{\alpha-1}q^\beta)-1}{\phi(p^{\alpha-1}q^\beta)-1+M_p}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-1}q^\beta)}\setminus\{0\}))\\
&\quad=\frac{\phi(q^\beta)(p^{\alpha-2})+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^\beta)(p^{\alpha-1})+p^{\alpha-1}q^{\beta-1}-2}+\frac{\phi(p^{\alpha-1}q^\beta)-1}{\phi(q^\beta)(p^{\alpha-1})+p^{\alpha-1}q^{\beta-1}-2}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-1}q^\beta)}\setminus\{0\})).
\end{aligned}$$

We have the normalize Laplacian spectrum of $K_{\phi(p^{\alpha-1}q^\beta)}$. Thus second part with multiplicity is $\begin{pmatrix}\left(\frac{\phi(p^{\alpha-1}q^\beta)-1}{\phi(q^\beta)(p^{\alpha-1})+p^{\alpha-1}q^{\beta-1}-2}\right)\left(\frac{\phi(p^{\alpha-1}q^\beta)}{\phi(p^{\alpha-1}q^\beta)-1}\right)\\ \phi(p^{\alpha-1}q^\beta)-1\end{pmatrix}$.

Thus by calculation we get,

$$\frac{M_p}{\phi(p^{\alpha-1}q^{\beta})-1+M_p}+\frac{\phi(p^{\alpha-1}q^{\beta})-1}{\phi(p^{\alpha-1}q^{\beta})-1+M_p}\left(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-1}q^{\beta})}\setminus\{0\})\right)$$
$$=\begin{pmatrix}\frac{\phi(q^{\beta})(p^{\alpha-1})+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}\\ \phi(p^{\alpha-1}q^{\beta})-1\end{pmatrix}. \quad (15)$$

Using the same process, we get other terms as follows. For $i\geq 2$,

$$\frac{M_{p^i}}{\phi(p^{\alpha-i}q^{\beta})-1+M_{p^i}}+\frac{\phi(p^{\alpha-i}q^{\beta})-1}{\phi(p^{\alpha-i}q^{\beta})-1+M_{p^i}}\left(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i}q^{\beta})}\setminus\{0\})\right)$$
$$=\begin{pmatrix}\frac{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}\\ \phi(p^{\alpha-i}q^{\beta})-1\end{pmatrix}. \quad (16)$$

Continuing the same process we can calculate for $i\in\{1,2,\ldots,\alpha\}$

$$\frac{M_{q^j}}{\phi(p^{\alpha}q^{\beta-j})-1+M_{q^j}}+\frac{\phi(p^{\alpha}q^{\beta-j})-1}{\phi(p^{\alpha}q^{\beta-j})-1+M_{q^j}}\left(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha}q^{\beta-j})}\setminus\{0\})\right)$$
$$=\begin{pmatrix}\frac{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}\\ \phi(p^{\alpha}q^{\beta-j})-1\end{pmatrix}, \quad j\in\{1,2,\ldots\beta\}. \quad (17)$$

$$\frac{M_{p^iq^j}}{\phi(p^{\alpha-i}q^{\beta-j})-1+M_{p^iq^j}}+\frac{\phi(p^{\alpha-i}q^{\beta-j})-1}{\phi(p^{\alpha-i}q^{\beta-j})-1+M_{p^iq^j}}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i}q^{\beta-j})}\setminus\{0\}))$$
$$=\begin{pmatrix}\frac{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2}\\ \phi(p^{\alpha-i}q^{\beta-j})-1\end{pmatrix}, \quad j\in\{1,2,\ldots\beta-1\}. \quad (18)$$

Again for $i\in\{1,2,\ldots,\alpha-1\}$,

$$\frac{M_{p^iq^{\beta}}}{\phi(p^{\alpha-i})-1+M_{p^iq^{\beta}}}+\frac{\phi(p^{\alpha-i})-1}{\phi(p^{\alpha-i})-1+M_{p^iq^{\beta}}}(\sigma_{\mathcal{L}}(K_{\phi(p^{\alpha-i})}\setminus\{0\}))$$
$$=\begin{pmatrix}\frac{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2}\\ \phi(p^{\alpha-i})-1\end{pmatrix}. \quad (19)$$

The algebraic multiplicity of each of the eigenvalue can be determine very easily. The algebraic multiplicity of $\frac{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}$ is $\sum_{i=1}^{\alpha}(\phi(p^{\alpha-i}q^{\beta})-1)=\phi(q^{\beta})p^{\alpha-1}-\alpha$. Similarly, the algebraic multiplicity of $\frac{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+p^{\alpha-1}q^{\beta-1}-2}$ is $\phi(p^{\alpha})q^{\beta-1}-\beta$ and the algebraic multiplicity of $\frac{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-1}{\phi(q^{\beta})p^{\alpha-1}+\phi(p^{\alpha})q^{\beta-1}+p^{\alpha-1}q^{\beta-1}-2}$ is $p^{\alpha-1}q^{\beta-1}-\alpha\beta+1$.

Thus a part of the normalized Laplacian spectrum of $Z^*(\Gamma(\mathbb{Z}_{p^{\alpha}q^{\beta}}))$ is given by

$$\sigma_{\mathcal{L}(Z^*(\Gamma(\mathbb{Z}_{p^{\alpha}q^{\beta}})))}=\begin{pmatrix}D & E & F\\ \phi(q^{\beta})p^{\alpha-1}-\alpha & \phi(p^{\alpha})q^{\beta-1}-\beta & p^{\alpha-1}q^{\beta-1}-\alpha\beta+1\end{pmatrix}$$

where

$$D = \frac{\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 1}{\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 2},$$
$$E = \frac{\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 1}{\phi(q^\beta)p^{\alpha-1} + p^{\alpha-1}q^{\beta-1} - 2},$$
$$F = \frac{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 1}{\phi(q^\beta)p^{\alpha-1} + \phi(p^\alpha)q^{\beta-1} + p^{\alpha-1}q^{\beta-1} - 2}.$$

All other normalized Laplacian eigenvalues of $Z^*(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$ are obtained from $\mathcal{L}(G)$ defined in Theorem 4.1.

With the help of the Eqs (15)–(19) and the values of M_i's in matrix defined in Theorem 4.1, we can establish the matrix (13). Hence the theorem is proved. □

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