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A normal distribution of Raab direction in binomial coefficients triangle

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ABSTRACT

In the present paper, we are interested in the distribution of the elements lying along the Raab direction in the binomial coefficients triangle. More precisely, we prove that the sequence $\{\binom{n-rk}{k}\}_{0\le k\le \lfloor n/(r+1)\rfloor}$ is asymptotically distributed according to a Gaussian law. We also provide some experimental evidences.

Keywords: binomial coefficients triangle; Gaussian distribution; Central Limit Theorem; Generating function

2020 Mathematics Subject Classification: 05A10, 60F05, 05A15.

1. Introduction and main result

Let n and r be two non-negative integers. The sequence of binomial coefficients $\{\binom{n-rk}{k}\}_k$, where $0 \le k \le \lfloor n/(r+1)\rfloor$ [9], appears in many combinatorial contexts. It represents the elements lying along all the parallel diagonals of direction $(1, r)$ in the binomial coefficients triangle [4]. Figures 1 and 2 illustrate two different directions $(1, 1)$ and $(1, 2)$ in the binomial coefficients triangle (BCT).

Raab in [9] established the recurrence relation for the sum of elements lying along the cited parallel diagonals. Let u_n be the general term of the sum of elements lying on the corresponding diagonal ray, for $n \ge 0$:

$$u_{n+1} = \sum_{k=0}^{\lfloor n/(1+r)\rfloor} \binom{n-rk}{k} x^{n-(r+1)k} y^k,$$

with x and y are variables and $u_0 = 0$. He proved that $\{u_n\}_n$ satisfies the following

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recurrence relation

$$u_{n+1} = xu_n + yu_{n-r},$$

with $u_0 = 0, u_k = x^{k-1}, k = 1, \ldots, r-1$. For more details see [2, 1].

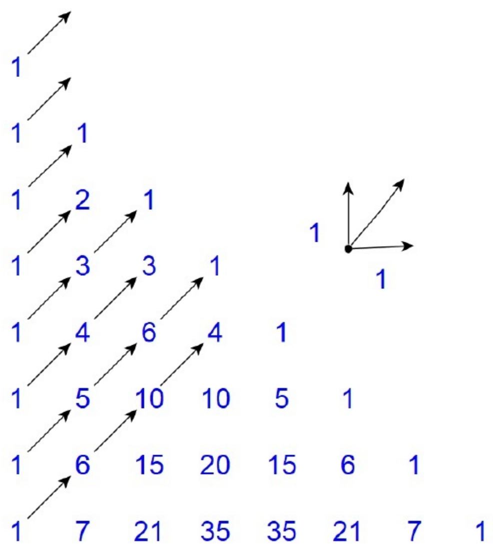


Fig. 1. Direction $(1, 1)$ in the BCT

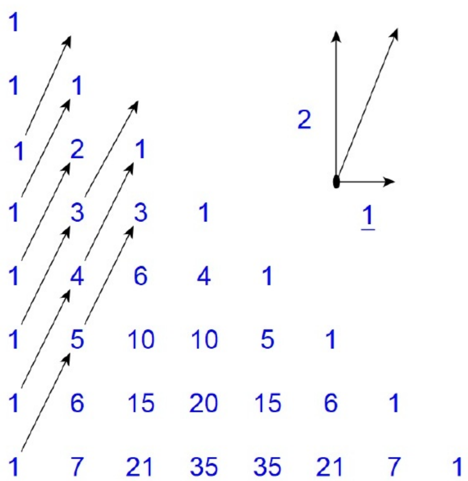


Fig. 2. Direction $(1, 2)$ in the BCT

Belbachir et al. [5, 4] generalized Raab's work to any finite direction, or a transversal ray in the BCT, denoted by (θ, α, r), where θ is the starting position of the transversal ray in the BCT, with $0 \leq \theta < \alpha$, $\alpha \in \mathbb{N}$. Here, α represent the number of horizontal steps, and r is the number of vertical steps, where $r \in \mathbb{Z}$ with $\alpha + r > 0$. This condition ensures that the direction is finite; otherwise, it will be infinite. They established the recurrence relation associated with the sum of elements lying along any finite direction.

Bender in 1973 established conditions for the nature of the bivariate generating function associated with a double indexed sequence of non-negative numbers $a_{n,l}$ $(0 \leq l \leq n)$ which implies asymptotic normality for the finite sequence $\{a_{n,l}\}_l$ for fixed n.

In the present paper, we use Bender's approach [8] to prove that the sequence $\{\binom{n-rk}{k}\}_k$ follows asymptotically a normal Gaussian distribution.

Set $P_n(k) = \dfrac{\binom{n-rk}{k}}{\sum\limits_j \binom{n-rj}{j}}$. The main result of this paper is the following.

Theorem 1.1. *The sequence $\{\binom{n-rk}{k}\}_k$ is asymptotically normal with mean μ_n and variance σ_n^2 and we have*

$$\lim_{n\to\infty} \sup_x \left| \sum_{k \leq \sigma_n x + \mu_n} P_n(k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \right| = 0,$$

with

$$\mu_n = \frac{n(1-\beta)}{1+r-r\beta} \qquad \text{and} \qquad \sigma_n^2 = \frac{n\beta(1-\beta)}{(1+r-r\beta)^3},$$

where β is a simple root of the equation $1 - z - z^{1+r} = 0$, with $|\beta|$ is the smallest one.

As specified by Bender in [8], there is only one root inside the circle of convergence (β is the specified simple root).

In order to prove Theorem 1.1, we define the bivariate generating function associated with $\{\binom{n-rk}{k}\}_{n,k}$ and we prove that it satisfies Bender's conditions hence satisfy the Central Limit Theorem (CLT).

2. Proof of the main result

We start by giving Bender's result. For this, consider a double indexed sequence of non-negative numbers $\{a_{n,l}\}_{0 \leq l \leq n}$ and the corresponding normalized sequence $P_n(l) = \dfrac{a_{n,l}}{\sum\limits_l a_{n,l}}$.

Theorem 2.1. [8] *Let $f(z,w)$ have a power series expansion*

$$f(z,w) = \sum_{n,l \geq 0} a_{n,l} z^n w^l,$$

where $a_{n,l} \geq 0$, for all n and l. Suppose there exists

- *an $A(s)$ continuous and non-zero near 0,*
- *an $\beta(s)$ with bounded third derivative near 0,*
- *a non-negative integer m, and*
- *$\epsilon, \delta > 0$ such that*

$$(1 - \frac{z}{\beta(s)})^m f(z, e^s) - \frac{A(s)}{1 - z/\beta(s)}, \tag{1}$$

is analytic and bounded for

$$|s| < \epsilon \quad \text{and} \quad |z| < |\beta(0)| + \delta.$$

Define

$$\mu = -\frac{\beta^{'}(0)}{\beta(0)} \quad \textit{and} \quad \sigma^2 = \mu^2 - \frac{\beta^{''}(0)}{\beta(0)}. \tag{2}$$

If $\sigma \neq 0$, *then*

$$\lim_{n\to\infty} \sup_x \left| \sum_{l \le \sigma_n x + \mu_n} P_n(l) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \right| = 0,$$

holds with $\mu_n = n\mu$ *and* $\sigma_n^2 = n\sigma^2$.

In [8], Bender discussed the case where the bivariate generating function associated with $\{a_{n,l}\}_{n,l}$ is of the form $\dfrac{Q(z,w)}{H(z,w)^{m+1}}$, where Q is an analytic function and H is a polynomial.

He established new conditions in order to ensure the asymptotic normality of the coefficients. Herein we quote the conditions.

Claim 1. [8] Suppose that

$$f(z,w) = \frac{Q(z,w)}{H(z,w)^{m+1}},$$

if $H(z,w)$ is a polynomial in z with coefficients continuous in w, $H(z,1)$ has a simple root at β and all other roots have larger absolute value, $Q(z,w)$ is analytic for w near 1 and $z < \beta + \epsilon$, $Q(\beta,1) \neq 0$.

Then the Theorem 2.1 is applicable.

Let $\beta(\theta)$ be a root of $H(z, e^{\theta}) = 0$. Differentiating we obtain

$$\beta^{'} \frac{\partial H}{\partial z} + e^{\theta} \frac{\partial H}{\partial w} = 0,$$

$$\beta^{''} \frac{\partial H}{\partial z} + (\beta^{'})^2 \frac{\partial^2 H}{\partial z^2} + 2e^{\theta} r^{'} \frac{\partial H}{\partial z \partial w} + e^{\theta} \frac{\partial H}{\partial w} + e^{2\theta} \frac{\partial H}{\partial w^2} = 0.$$

When $\theta = 0$ we obtain

$$\beta^{'} = -\frac{\partial H}{\partial w} \Big/ \frac{\partial H}{\partial z},$$

$$\beta^{''} = \frac{-1}{\frac{\partial H}{\partial z}} \left(\beta^{'2} \frac{\partial^2 H}{\partial z^2} + 2\beta^{'} \frac{\partial H}{\partial z \partial w} + \frac{\partial H}{\partial w} + \frac{\partial^2 H}{\partial w^2} \right).$$

In order to prove Theorem 1.1, we need to define the bivariate generating function associated with $\{\binom{n-rk}{k}\}_{n,k}$ given by Tanny and Zuker (see Lemma 2.2 below). We then prove that these sequences satisfy Bender's special case conditions Claim 1.

Lemma 2.2. [10] *The bivariate generating function associated with* $\{\binom{n-rk}{k}\}_{n,k}$ *is given by*

$$\sum_{n,k} \binom{n-rk}{k} z^n w^k = \frac{1}{1 - z - wz^{r+1}}.$$

We need also the following lemma given by Wolfram in [11].

Lemma 2.3. [11] *The characteristic polynomial $p(x) = x^p - x^{p-1} - \cdots - 1$ associated with the multibonacci sequence $v_n = v_{n-1} + \cdots + v_{n-p}$ admits only simple roots.*

For more properties concerning this sequence see [3].

Proof of Theorem 1.1. Let consider the bivariate generating function associated with $\{\binom{n-rk}{k}\}_{n,k}$,

$$\sum_{n,k} \binom{n-rk}{k} z^n w^k = \frac{1}{1 - z - wz^{r+1}} = \frac{Q(z,w)}{H(z,w)},$$

then, we have

- $H(z,w) = 1 - z - wz^{r+1}$ is a polynomial in z where the coefficients are continuous function of w.
- By Lemma 2.3, the roots of $H(z,1)$ are simple.
- Let β be the root of $H(z,1)$, with the smallest absolute value. The absolutely smallest root is less than 1 for w near 1, because $\frac{\partial H}{\partial z} < 0$ for $z, w > 0$ (see[8]).
- $Q(z,w) = 1$ is trivially analytic and $Q(\beta, 1) \neq 0$.

These conditions being checked, we can apply Bender's result and conclude that the sequence $\{\binom{n-rk}{k}\}_{n,k}$ is asymptotically normal with mean μ_n and variance σ_n^2.

We compute μ and σ^2 given by (2). Let $\beta(\theta)$ be the root of $H(z, e^\theta)$. We have

$$\beta'(\theta) = -\frac{\partial H}{\partial w} / \frac{\partial H}{\partial z},$$
$$\beta''(\theta) = \frac{-1}{\frac{\partial H}{\partial z}} \left(\beta'^2 \frac{\partial^2 H}{\partial z^2} + 2\beta' \frac{\partial H}{\partial z \partial w} + \frac{\partial H}{\partial w} + \frac{\partial^2 H}{\partial w^2} \right),$$

thus

$$\mu = -\frac{\beta'(0)}{\beta(0)} \qquad \text{and} \qquad \sigma^2 = \mu^2 - \frac{\beta''(0)}{\beta(0)}.$$

Consequently,

$$\mu = \frac{1-\beta}{(1+r) - r\beta} \qquad \text{and} \qquad \sigma^2 = \frac{\beta(1-\beta)}{(1+r-r\beta)^3}.$$

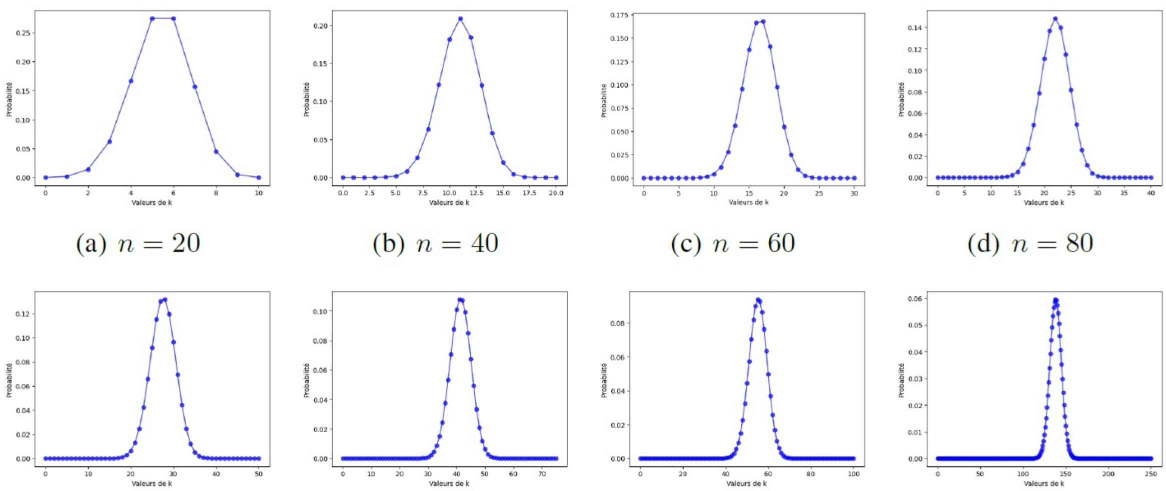
Then $\{\binom{n-rk}{k}\}_{n,k}$ is asymptotically normal with mean $\mu_n = n\mu$ and variance $\sigma_n^2 = n\sigma^2$. $\square$

In what follows, we give some examples of $\{\binom{n-rk}{k}\}_k$ with different fixed values of r.

Table 1. Some examples of $\{\binom{n-rk}{k}\}_k$ with different fixed values of r

$(1,r)$	$\binom{n-rk}{k}$	$H(z,1)$	β	μ	σ^2
$(1,0)$	$\binom{n}{k}$	$1-2z$	0.5	0.5	0.25
$(1,1)$	$\binom{n-k}{k}$	$1-z-z^2$	0, 62	0.28	0.09
$(1,2)$	$\binom{n-2k}{k}$	$1-z-z^3$	0.68	0.20	0.05
$(1,3)$	$\binom{n-3k}{k}$	$1-z-z^4$	0, 72	0, 15	0, 03
$(1,4)$	$\binom{n-4k}{k}$	$1-z-z^5$	0, 75	0, 125	0, 02
$(1,5)$	$\binom{n-5k}{k}$	$1-z-z^6$	0, 78	0, 10	0, 02
$(1,6)$	$\binom{n-6k}{k}$	$1-z-z^7$	0, 8	0, 09	0, 02
$(1,7)$	$\binom{n-7k}{k}$	$1-z-z^8$	0, 81	0, 08	0, 01
$(1,8)$	$\binom{n-8k}{k}$	$1-z-z^9$	0, 82	0, 07	0, 01
$(1,9)$	$\binom{n-9k}{k}$	$1-z-z^{10}$	0, 84	0, 07	$9 \cdot 10^{-3}$

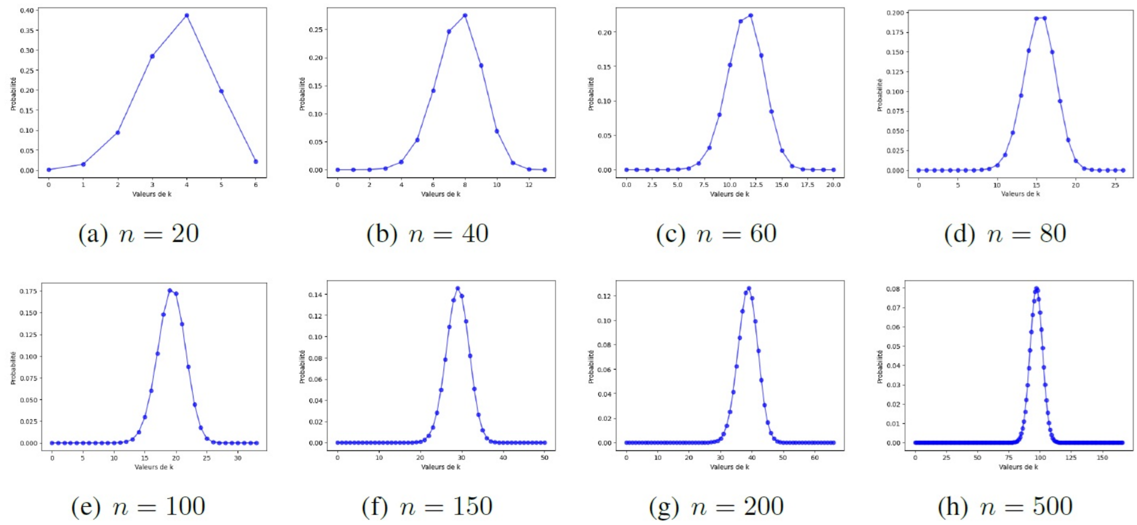
Figures 3, 4 and 5 represent the distribution of the binomial coefficients associated with different directions and for different values of n.



(a) $n = 20$ (b) $n = 40$ (c) $n = 60$ (d) $n = 80$

(e) $n = 100$ (f) $n = 150$ (g) $n = 200$ (h) $n = 500$

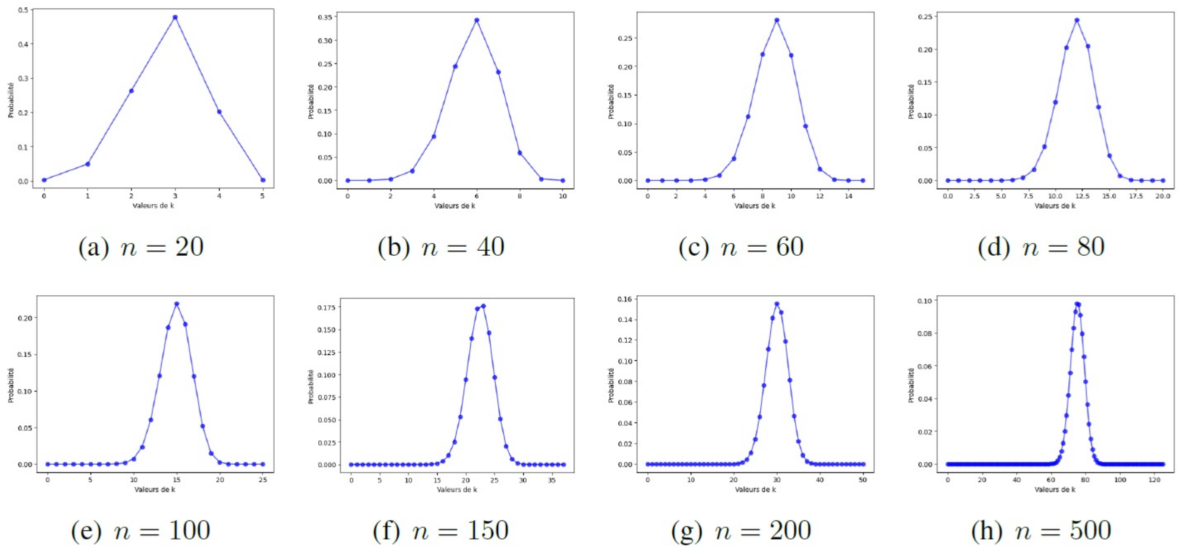
Fig. 3. Distribution of binomial coefficients associated with the direction $(1,1)$



(a) $n = 20$ (b) $n = 40$ (c) $n = 60$ (d) $n = 80$

(e) $n = 100$ (f) $n = 150$ (g) $n = 200$ (h) $n = 500$

Fig. 4. Distribution of binomial coefficients associated with the direction $(1, 2)$



(a) $n = 20$ (b) $n = 40$ (c) $n = 60$ (d) $n = 80$

(e) $n = 100$ (f) $n = 150$ (g) $n = 200$ (h) $n = 500$

Fig. 5. Distribution of binomial coefficients associated with the direction $(1, 3)$

3. Unimodality and peaks

A positive real sequence $\{a_l\}_{0\leq l\leq n}$ is said to be unimodal if there exists integers l_0 and l_1, $0 \leq l_0 \leq l_1 \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{l_0-1} < a_{l_0} = \cdots = a_{l_1} > a_{l_1+1} \geq \cdots a_n.$$

The integers m, with $l_0 \leq m \leq l_1$, are called the modes of $\{a_l\}_{0\leq l\leq n}$. If $l_0 = l_1$, we talk about a peak, otherwise, the set of modes is called a plateau. It is well known that if n is even, then the n^{th} row of the BCT has a peak at $n/2$, otherwise (when n is odd) there is a plateau with two elements $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$. Also some extensions, for the direction

$(1,1)$ were studied in [9, 3, 7, 6]. It was proved by Tanny and Zuker[10] that the sequence $\{\binom{n-rk}{k}\}_k$ is unimodal more over it is log-concave, also they derived a general formula for $k_{n,r}$ the least integer at which the maximum of $\{\binom{n-rk}{k}\}_k$ occurs. Here we can consider that the peaks of $\{\binom{n-rk}{k}\}_k$ is the nearest non-negative integer close to the mean μ_n($\lfloor \mu_n \rfloor$ or $\lceil \mu_n \rceil$), for a larger n, where $\mu_n = n(1-\beta)/(1+r-r\beta)$. Hence we obtain the same result given by Tanny and Zuker in [10].

The following Tables 2 and 3 give the comparative values between the peaks of $\{\binom{n-k}{k}\}_k$ and $\{\binom{n-2k}{k}\}_k$ and the corresponding mean μ_n.

Table 2. Comparative table between the peaks k_0 of $\{\binom{n-k}{k}\}_k$ and the corresponding means μ_n

n	μ_n	k_0	$\lvert\mu_n - k_0\rvert$	$\binom{n-k_0}{k_0}$
1	0.28	0	0.28	1
2	0.56	0	0.56	1
3	0.84	1	0.16	2
4	1.12	1	0.12	3
5	1.4	1	0.4	4
6	1.68	2	0.32	6
7	1.96	2	0.04	10
8	2.24	2	0.24	15
9	2.52	2	0.52	21
10	2.80	3	0.2	35
11	3.08	3	0.08	56
12	3.36	3	0.36	84
13	3.64	4	0.36	126
14	3.92	4	0.08	210
15	4.2	4	0.2	330
16	4.48	4	0.48	495
17	4.76	5	0.24	792
18	5.04	5	0.04	1287
19	5.32	5	0.32	2002
20	5.60	5	0.6	3003
21	5.88	6	0.12	5005
22	6.16	6	0.16	8008
23	6.44	6	0.44	12376
24	6.72	7	0.28	19448
25	7.00	7	0	31824

n	μ_n	k_0	$\lvert\mu_n - k_0\rvert$	$\binom{n-k_0}{k_0}$
26	7.28	7	0.28	50388
27	7.56	7	0.56	77520
28	7.84	8	0.16	125970
29	8.12	8	0.12	203490
30	8.4	8	0.4	319770
31	8.68	9	0.32	497420
32	8.96	9	0.04	817190
33	9.24	9	0.24	1307504
34	9.52	9	0.52	2042975
35	9.8	10	0.2	3268760
36	10.08	10	0.08	5311735
37	10.36	10	0.36	8436285
38	10.64	10	0.64	13123110
39	10.92	11	0.08	21474180
40	11.20	11	0.2	34597290
41	11.48	11	0.48	54627300
42	11.76	12	0.24	86493225
43	12.04	12	0.04	141120525
44	12.32	12	0.32	225792840
45	12.60	12	0.6	354817320
46	12.88	13	0.12	573166440
47	13.16	13	0.16	927983760
48	13.44	13	0.44	1476337800
49	13.72	14	0.28	2319959400
50	14.00	14	0	3796297200

Table 3. Comparative table between the peaks k_0 of $\{\binom{n-2k}{k}\}_k$ and the corresponding means μ_n

n	μ_n	k_0	$\|\mu_n - k_0\|$	$\binom{n-2k_0}{k_0}$
1	0.2	0	0.2	1
2	0.4	0	0.4	1
3	0.6	0	0.6	1
4	0.8	1	0.2	2
5	1.0	1	0	3
6	1.2	1	0.2	4
7	1.4	1	0.4	5
8	1.6	1	0.6	6
9	1.8	2	0.2	10
10	2.0	2	0	15
11	2.2	2	0.2	21
12	2.4	2	0.4	28
13	2.6	2	0.6	36
14	2.8	3	0.2	56
15	3.0	3	0	84
16	3.2	3	0.2	120
17	3.4	3	0.4	165
18	3.6	3	0.6	220
19	3.8	4	0.2	330
20	4.0	4	0	495
21	4.2	4	0.2	715
22	4.4	4	0.4	1001
23	4.6	4	0.6	1365
24	4.8	5	0.2	2002
25	5.0	5	0	3003
26	5.2	5	0.2	4368
27	5.4	5	0.4	6188
28	5.6	5	0.6	8568
29	5.8	6	0.2	12376
30	6	6	0	18564
31	6.2	6	0.2	27132
32	6.4	6	0.4	38760
33	6.6	6	0.6	54264
34	6.8	7	0.2	77520
35	7	7	0	116280
36	7.2	7	0.2	170544
37	7.4	7	0.4	245157
38	7.6	7	0.6	346104
39	7.8	8	0.2	490314
40	8	8	0	735471
41	8.2	8	0.2	1081575
42	8.4	8	0.4	1562275
43	8.6	8	0.6	2220075
44	8.8	9	0.2	3124550
45	9	9	0	4686825
46	9.2	9	0.2	6906900
47	9.4	9	0.4	10015005
48	9.6	9	0.6	14307150
49	9.8	9	0.2	20160075
50	10	10	0	30045015

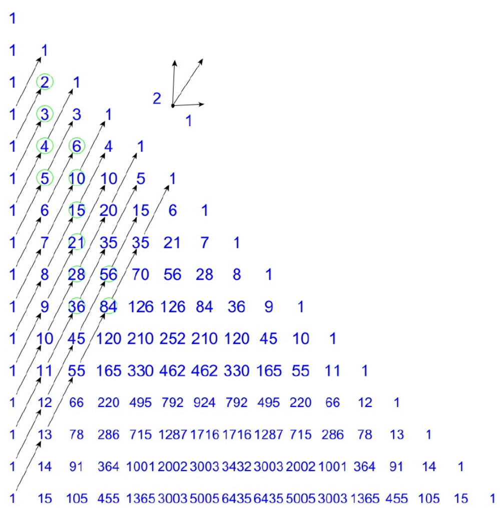


Fig. 6. The green circles describe the position of the peaks in the direction (1, 2) in the BCT

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