

Almost neo balancing numbers

Ahmet Tekcan*

ABSTRACT

In this work, we defined almost neo balancing numbers and determined the general terms of them in terms of balancing and Lucas-balancing numbers. We also deduced some results on relationship with triangular, square triangular, Pell, Pell-Lucas numbers and these numbers. Further we formulate the sum of first n -terms of these numbers.

Keywords: balancing numbers, cobalancing numbers, Pell equations, triangular and square triangular numbers, set of representations, Pell numbers

2020 Mathematics Subject Classification: 11B37, 11B39, 11D09, 11D79.

1. Introduction

A positive integer n is called a balancing number [2] if the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r), \quad (1)$$

holds for some positive integer r which is called balancer. If n is a balancing number with balancer r , then from (1)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 1}}{2}. \quad (2)$$

Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (2), they noted that $8(0)^2 + 1 = 1$ and $8(1)^2 + 1 = 3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers.

Panda and Ray [18] defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r), \quad (3)$$

* Corresponding author.

holds for some positive integer r which is called cobalancer. If n is a cobalancing number with cobalancer r , then from (3)

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}. \quad (4)$$

From (4), they noted that $8(0)^2 + 8(0) + 1 = 1$ is a perfect square. So they accepted 0 to be a cobalancing number, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n denote the balancing number and let b_n denote the cobalancing number. Then from (2), B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and from (4), b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. Thus

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1},$$

are integers which are called Lucas-balancing number and Lucas-cobalancing number, respectively.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of the characteristic equation for Pell and Pell-Lucas numbers which are the numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$, respectively. Ray proved [20] that the Binet formulas for all balancing numbers are

$$B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2} \quad \text{and} \quad c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2},$$

for $n \geq 1$ (see also [7, 13, 19]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [10], Liptai proved that there is no Fibonacci balancing number except 1 and in [11] he proved that there is no Lucas-balancing number. In [23], Szalay considered the same problem and obtained some nice results by a different method. In [9], Kovács, Liptai, Olajos extended the concept of balancing numbers to the (a, b) -balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n - 1) + b) = (a(n + 1) + b) + \cdots + (a(n + r) + b),$$

for some positive integers n and r , then $an + b$ is an (a, b) -balancing number. The sequence of (a, b) -balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [12], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x with $x \leq y - 2$ is called a (k, l) -power numerical center for y if

$$1^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l) -power numerical centers. For positive integers k, x , let

$$\Pi_k(x) = x(x + 1) \cdots (x + k - 1).$$

Then it was proved in [9] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [34], Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [16], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [21], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [14], Özdemir introduced a new non-commutative number system called hybrid numbers. In [3], Bród, Szynal-Liana and Włoch defined balancing and Lucas-balancing hybrid numbers and in [22], Rubajczyk and Szynal-Liana defined cobalancing and Lucas-cobalancing hybrid numbers. In [5], Dash, Ota, Dash defined t -balancing numbers. In [27], Tekcan and Aydın determined the general terms of t -balancing and Lucas t -balancing numbers. In [28], Tekcan and Erdem determined the general terms of t -cobalancing and Lucas t -cobalancing numbers.

In [17], Panda and Panda defined almost balancing numbers. They said that a positive integer n is called an almost balancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - [1 + 2 + \cdots + (n-1)]| = 1, \quad (5)$$

holds for some positive integer r which is called the almost balancer. In [15], Panda defined almost cobalancing numbers. He said that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n+1) + (n+2) + \cdots + (n+r)] - (1 + 2 + \cdots + n)| = 1,$$

holds for some positive integer r which is called the almost cobalancer. In [29], Tekcan and Erdem determined the general terms of all almost balancing numbers and almost cobalancing numbers. In [25], Tekcan considered the sums and spectral norms of all almost balancing numbers and in [24], Tekcan derived some results on almost balancing numbers, triangular numbers and square triangular numbers. In [30, 32], Tekcan and Yıldız defined balcobalancing numbers and in [31, 33], they defined almost balcobalancing numbers. In [26], Tekcan and Akg üç defined almost neo cobalancing numbers.

2. Almost neo balancing numbers

In [4], Chailangka and Pakapongpun defined neo balancing numbers. They said that a positive integer n is called a neo balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r), \quad (6)$$

holds for some integer r which is called neo balancer corresponding to n . For example 2, 7, 36, 205 are neo balancing numbers with neo balancers $-1, 1, 13, 83$, respectively.

By considering (5) and (6), we say that a positive integer n is called an almost neo balancing number if the Diophantine equation

$$\left| \begin{array}{c} (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r) \\ -[1 + 2 + \cdots + (n-1)] \end{array} \right| = 1, \quad (7)$$

holds for some positive integer r which is called almost neo balancer. From (7), we have two cases:

(1) If $(n-1) + (n-0) + (n+1) + (n+2) + \dots + (n+r) - [1+2+\dots+(n-1)] = 1$, then n is called an almost neo balancing number of first type, r is called an almost neo balancer of first type and in this case

$$r = \frac{-2n-1 + \sqrt{8n^2 - 16n + 17}}{2}. \quad (8)$$

For example 19, 106, 613 are almost neo balancing numbers of first type with almost neo balancers of first type are 6, 42, 252, respectively.

From (8), we note that for $n=1$ and $n=4$, $8(1)^2 - 16(1) + 17 = 3^2$ and $8(4)^2 - 16(4) + 17 = 9^2$ are perfect squares, but in this case $r=0$ and $r=0$ again. Nevertheless we accept 1 and 4 to be almost neo balancing numbers of first type, just like Behera and Panda accepted 0 and 1 to be balancing numbers.

Let B_n^{neo*} denote the almost neo balancing number of first type. Then from (8), B_n^{neo*} is an almost neo balancing number of first type if and only if $8(B_n^{neo*})^2 - 16B_n^{neo*} + 17$ is a perfect square. Thus

$$C_n^{neo*} = \sqrt{8(B_n^{neo*})^2 - 16B_n^{neo*} + 17}, \quad (9)$$

is an integer which is called an almost Lucas-neo balancing number of first type (We denote the almost neo balancer of first type by R_n^{neo*}).

(2) If $(n-1) + (n-0) + (n+1) + (n+2) + \dots + (n+r) - [1+2+\dots+(n-1)] = -1$, then n is called an almost neo balancing number of second type, r is called an almost neo balancer of second type and in this case

$$r = \frac{-2n-1 + \sqrt{8n^2 - 16n + 1}}{2}. \quad (10)$$

For example 12, 24, 65 are almost neo balancing numbers of second type with almost neo balancers of second type are 3, 8, 25, respectively.

From (10), we note that for $n=2, 3$ and $n=5$, $8(2)^2 - 16(2) + 1 = 1^2$, $8(3)^2 - 16(3) + 1 = 5^2$ and $8(5)^2 - 16(5) + 1 = 11^2$ are perfect squares, but in this case $r=-2, -1$ and $r=0$, respectively. Nevertheless we accept 2, 3 and 5 to be almost neo balancing numbers of second type.

Let B_n^{neo**} denote the almost neo balancing number of second type. Then from (10), B_n^{neo**} is an almost neo balancing number of second type if and only if $8(B_n^{neo**})^2 - 16B_n^{neo**} + 1$ is a perfect square. Thus

$$C_n^{neo**} = \sqrt{8(B_n^{neo**})^2 - 16B_n^{neo**} + 1}, \quad (11)$$

is an integer which is called an almost Lucas-neo balancing number of second type. (We denote the almost neo balancer of second type by R_n^{neo**}).

In this paper, we try to determine the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of balancing and Lucas-balancing numbers.

2.1. *Almost neo balancing numbers of first type*

We see that B_n^{neo*} is an almost neo balancing number of first type if and only if $8(B_n^{neo*})^2 - 16B_n^{neo*} + 17$ is a perfect square. So we set

$$8(B_n^{neo*})^2 - 16B_n^{neo*} + 17 = y^2,$$

for some positive integer y . Then

$$2[4(B_n^{neo*})^2 - 8B_n^{neo*}] + 17 = y^2,$$

and hence

$$2(2B_n^{neo*} - 2)^2 + 9 = y^2.$$

Taking $x = 2B_n^{neo*} - 2$, we get the Pell equation [1, 8]

$$2x^2 - y^2 = -9. \quad (12)$$

Let Ω^* denotes the set of all integer solutions of (12), that is,

$$\Omega^* = \{(x, y) : 2x^2 - y^2 = -9\}.$$

Then we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of (12) is $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$.*

Proof. For the Pell equation in (12), the indefinite form is $F = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$. Thus the set of representatives (see [6, p. 121]) is $\text{Rep} = \{[0 \ 3]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here we notice that $[0 \ 3]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$. It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix},$$

for $n \geq 1$. So

$$[x_n \ y_n] = [0 \ 3]M^n = [6B_n \ 3C_n].$$

Thus the set of all integer solutions is $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$. \square

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$B_n^{neo*} = 3B_{n-1} + 1, \quad C_n^{neo*} = 3C_{n-1} \quad \text{and} \quad R_n^{neo*} = \frac{-6B_{n-1} + 3C_{n-1} - 3}{2},$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of (12) is $\Omega^* = \{(6B_n, 3C_n) : n \geq 1\}$. Since $x = 2B_n^{neo*} - 2$, we get

$$B_n^{neo*} = \frac{6B_{n-1} + 2}{2} = 3B_{n-1} + 1,$$

for $n \geq 1$. Thus from (9), we obtain

$$\begin{aligned} C_n^{neo*} &= \sqrt{8(B_n^{neo*})^2 - 16B_n^{neo*} + 17} \\ &= \sqrt{8(3B_{n-1} + 1)^2 - 16(3B_{n-1} + 1) + 17} \\ &= \sqrt{9(8B_{n-1}^2 + 1)} \\ &= 3C_{n-1}, \end{aligned}$$

for $n \geq 1$. From (8), we deduce that

$$R_n^{neo*} = \frac{-2(3B_{n-1} + 1) - 1 + 3C_{n-1}}{2} = \frac{-6B_{n-1} + 3C_{n-1} - 3}{2},$$

for $n \geq 1$. □

2.2. Almost neo balancing numbers of second type

We see that B_n^{neo**} is an almost neo balancing number of second type if and only if $8(B_n^{neo**})^2 - 16B_n^{neo**} + 1$ is a perfect square. So we set

$$8(B_n^{neo**})^2 - 16B_n^{neo**} + 1 = y^2,$$

for some positive integer y . Then

$$2[4(B_n^{neo**})^2 - 8B_n^{neo**}] + 1 = y^2,$$

and hence

$$2(2B_n^{neo**} - 2)^2 - 7 = y^2.$$

Taking $x = 2B_n^{neo**} - 2$, we get the Pell equation

$$2x^2 - y^2 = 7. \tag{13}$$

Let Ω^{**} denotes the set of all integer solutions of (13), that is,

$$\Omega^{**} = \{(x, y) : 2x^2 - y^2 = 7\}.$$

Then we can give the following theorem.

Theorem 2.3. *The set of all integer solutions of (13) is $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\} \cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$.*

Proof. For the Pell equation in (13), the indefinite form is again $F = (2, 0, -1)$. The set of representatives is $\text{Rep} = \{[\pm 2 \ 1]\}$ and $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Here we notice that $[2 \ 1]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) and $[2 \ -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. So

$$\begin{aligned} [x_{2n-1} \ y_{2n-1}] &= [2 \ 1]M^{n-1} = [2B_{n-1} + 2C_{n-1} \ 8B_{n-1} + C_{n-1}] \\ [x_{2n} \ y_{2n}] &= [2 \ -1]M^n = [-2B_n + 2C_n \ 8B_n - C_n]. \end{aligned}$$

Thus the set of all integer solutions is $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\} \cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$. \square

From Theorem 2.3, we can give the following result.

Theorem 2.4. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of second type are*

$$\begin{aligned} B_{2n-1}^{neo**} &= B_{n-1} + C_{n-1} + 1 \\ B_{2n}^{neo**} &= -B_n + C_n + 1 \\ C_{2n-1}^{neo**} &= 8B_{n-1} + C_{n-1} \\ C_{2n}^{neo**} &= 8B_n - C_n \\ R_{2n-1}^{neo**} &= \frac{6B_{n-1} - C_{n-1} - 3}{2} \\ R_{2n}^{neo**} &= \frac{10B_n - 3C_n - 3}{2}, \end{aligned}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.3 that the set of all integer solutions of (13) is $\Omega^{**} = \{(2B_{n-1} + 2C_{n-1}, 8B_{n-1} + C_{n-1}) : n \geq 1\} \cup \{(-2B_n + 2C_n, 8B_n - C_n) : n \geq 1\}$. Since $x = 2B_n^{neo**} - 2$, we get

$$B_{2n-1}^{neo**} = \frac{2B_{n-1} + 2C_{n-1} + 2}{2} = B_{n-1} + C_{n-1} + 1,$$

for $n \geq 1$. Thus from (11), we obtain

$$\begin{aligned} C_{2n-1}^{neo**} &= \sqrt{8(B_{2n-1}^{neo**})^2 - 16B_{2n-1}^{neo**} + 1} \\ &= \sqrt{8(B_{n-1} + C_{n-1} + 1)^2 - 16(B_{n-1} + C_{n-1} + 1) + 1} \\ &= \sqrt{8B_{n-1}^2 + 16B_{n-1}C_{n-1} + 8C_{n-1}^2 - 7} \\ &= \sqrt{C_{n-1}^2 - 1 + 16B_{n-1}C_{n-1} + 8(8B_{n-1}^2 + 1) - 7} \\ &= \sqrt{64B_{n-1}^2 + 16B_{n-1}C_{n-1} + C_{n-1}^2} \\ &= \sqrt{(8B_{n-1} + C_{n-1})^2} \end{aligned}$$

$$= 8B_{n-1} + C_{n-1},$$

for $n \geq 1$. From (10), we deduce that

$$R_{2n-1}^{neo**} = \frac{-2B_{2n-1}^{neo**} - 1 + C_{2n-1}^{neo**}}{2} = \frac{6B_{n-1} - C_{n-1} - 3}{2},$$

for $n \geq 1$. The others can be proved similarly. \square

3. Balancing numbers and almost neo balancing numbers

In the previous section, we determined the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of balancing and Lucas-balancing numbers. Conversely we can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of almost neo balancing numbers and almost Lucas-neo balancing numbers of first and second type as follows.

Theorem 3.1. *The general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are*

$$\begin{aligned} B_n &= \frac{B_{n+1}^{neo*} - 1}{3} \\ b_n &= \frac{-2B_{n+1}^{neo*} + C_{n+1}^{neo*} - 1}{6} \\ C_n &= \frac{C_{n+1}^{neo*}}{3} \\ c_n &= \frac{4B_{n+1}^{neo*} - C_{n+1}^{neo*} - 4}{3}, \end{aligned}$$

for $n \geq 1$, or

$$\begin{aligned} B_n &= \frac{B_{2n}^{neo**} + C_{2n}^{neo**} - 1}{7} \\ b_n &= \frac{6B_{2n}^{neo**} - C_{2n}^{neo**} - 13}{14} \\ C_n &= \frac{8B_{2n}^{neo**} + C_{2n}^{neo**} - 8}{7} \\ c_n &= \frac{-4B_{2n}^{neo**} + 3C_{2n}^{neo**} + 4}{7}, \end{aligned}$$

for $n \geq 1$.

Proof. It can be easily derived from Theorems 2.2 and 2.4. \square

Thus we construct one-to-one correspondence between all balancing numbers and all almost neo balancing numbers.

4. Relationship with Pell and Pell-Lucas numbers

Recall that general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers can be given in terms of Pell numbers, namely,

$$B_n = \frac{P_{2n}}{2}, b_n = \frac{P_{2n-1} - 1}{2}, C_n = P_{2n} + P_{2n-1} \text{ and } c_n = P_{2n-1} + P_{2n-2}. \quad (14)$$

Similarly we can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type in terms of Pell numbers as follows.

Theorem 4.1. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned} B_n^{neo*} &= \frac{3P_{2n-2} + 2}{2}, n \geq 1, \\ C_n^{neo*} &= 3P_{2n-2} + 3P_{2n-3}, n \geq 2, \\ R_n^{neo*} &= \frac{3P_{2n-3} - 3}{2}, n \geq 2, \end{aligned}$$

and of second type are

$$\begin{aligned} B_{2n-1}^{neo**} &= \frac{3P_{2n-2} + 2P_{2n-3} + 2}{2}, n \geq 2 \\ B_{2n}^{neo**} &= \frac{P_{2n} + 2P_{2n-1} + 2}{2}, n \geq 1 \\ C_{2n-1}^{neo**} &= 5P_{2n-2} + P_{2n-3}, n \geq 2 \\ C_{2n}^{neo**} &= 3P_{2n} - P_{2n-1}, n \geq 1 \\ R_{2n-1}^{neo**} &= \frac{2P_{2n-2} - P_{2n-3} - 3}{2}, n \geq 2 \\ R_{2n}^{neo**} &= \frac{2P_{2n} - 3P_{2n-1} - 3}{2}, n \geq 1. \end{aligned}$$

Proof. Recall that $B_n = \frac{P_{2n}}{2}$ and $b_n = \frac{P_{2n-1}-1}{2}$ by (14). Thus from Theorems 2.2 and 2.4, we get the desired result. \square

Conversely, we can give the general terms of even and odd ordered Pell numbers in terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type as follows.

Theorem 4.2. *The general terms of even and odd ordered Pell numbers are*

$$\begin{aligned} P_{2n} &= \frac{2B_{n+1}^{neo*} - 2}{3} \\ P_{2n-1} &= \frac{-2B_{n+1}^{neo*} + C_{n+1}^{neo*} + 2}{3}, \end{aligned}$$

for $n \geq 1$, or

$$P_{2n} = \frac{2B_{2n}^{neo**} + 2C_{2n}^{neo**} - 2}{7}$$

$$P_{2n-1} = \frac{6B_{2n}^{neo**} - C_{2n}^{neo**} - 6}{7},$$

for $n \geq 1$.

Proof. Since $P_{2n} = 2B_n$ and $P_{2n-1} = 2b_n + 1$, we get the desired result from Theorem 3.1. \square

Thus we construct one-to-one correspondence between Pell numbers and almost neo balancing numbers.

As in (14), we note that the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers can be given in terms of Pell-Lucas numbers, namely,

$$B_n = \frac{Q_{2n} + Q_{2n-1}}{8}, \quad b_n = \frac{Q_{2n} - Q_{2n-1} - 4}{8}, \quad C_n = \frac{Q_{2n}}{2} \quad \text{and} \quad c_n = \frac{Q_{2n-1}}{2}. \quad (15)$$

Similarly we can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type in terms of Pell-Lucas numbers as follows.

Theorem 4.3. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned} B_n^{neo*} &= \frac{3Q_{2n-2} + 3Q_{2n-3} + 8}{8}, \quad n \geq 2 \\ C_n^{neo*} &= \frac{3Q_{2n-2}}{2}, \quad n \geq 1 \\ R_n^{neo*} &= \frac{3Q_{2n-2} - 3Q_{2n-3} - 12}{8}, \quad n \geq 2, \end{aligned}$$

and of second type are

$$\begin{aligned} B_{2n-1}^{neo**} &= \frac{5Q_{2n-2} + Q_{2n-3} + 8}{8}, \quad n \geq 2 \\ B_{2n}^{neo**} &= \frac{3Q_{2n} - Q_{2n-1} + 8}{8}, \quad n \geq 1 \\ C_{2n-1}^{neo**} &= \frac{3Q_{2n-2} + 2Q_{2n-3}}{2}, \quad n \geq 2 \\ C_{2n}^{neo**} &= \frac{Q_{2n} + 2Q_{2n-1}}{2}, \quad n \geq 1 \\ R_{2n-1}^{neo**} &= \frac{Q_{2n-2} + 3Q_{2n-3} - 12}{8}, \quad n \geq 2 \\ R_{2n}^{neo**} &= \frac{-Q_{2n} + 5Q_{2n-1} - 12}{8}, \quad n \geq 1. \end{aligned}$$

Proof. Applying Theorems 2.2 and 2.4, we get the desired result from (15). \square

Conversely, we can give the general terms of even and odd ordered Pell-Lucas numbers in terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and of second type as follows.

Theorem 4.4. *The general terms of even and odd ordered Pell-Lucas numbers are*

$$Q_{2n} = \frac{2C_{n+1}^{neo*}}{3}$$

$$Q_{2n-1} = \frac{8B_{n+1}^{neo*} - 2C_{n+1}^{neo*} - 8}{3},$$

for $n \geq 1$, or

$$Q_{2n} = \frac{16B_{2n}^{neo**} + 2C_{2n}^{neo**} - 16}{7}$$

$$Q_{2n-1} = \frac{-8B_{2n}^{neo**} + 6C_{2n}^{neo**} + 8}{7},$$

for $n \geq 1$.

Proof. Recall that $B_n = \frac{Q_{2n} + Q_{2n-1}}{8}$ and $C_n = \frac{Q_{2n}}{2}$ by (15). Thus from Theorem 3.1, we get the desired result. \square

Thus we construct one-to-one correspondence between Pell-Lucas numbers and all almost neo balancing numbers.

5. Relationship with triangular and square triangular numbers

Recall that triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n+1)}{2}.$$

It is known that there is a correspondence between balancing (and also cobalancing) numbers and triangular numbers. Indeed from (1), we note that n is a balancing number if and only if n^2 is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2. \quad (16)$$

Similarly from (3), n is a cobalancing number if and only if n^2+n is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2 + n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n.$$

As in (16), we can give the following theorem.

Theorem 5.1. *B_n^{neo*} is an almost neo balancing number of first type if and only if $(B_n^{neo*})^2 - 2B_n^{neo*} + 2$ is a triangular number, that is,*

$$T_{B_n^{neo*}+R_n^{neo*}} = (B_n^{neo*})^2 - 2B_n^{neo*} + 2,$$

and B_n^{neo**} is an almost neo balancing number of second type if and only if $(B_n^{neo**})^2 - 2B_n^{neo**}$ is a triangular number, that is,

$$T_{B_n^{neo**}+R_n^{neo**}} = (B_n^{neo**})^2 - 2B_n^{neo**}.$$

Proof. Let n be an almost neo balancing number of first type. Then from (7), we get

$$2n - 1 + nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = 1,$$

and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 - 2n + 2.$$

Thus

$$T_{B_n^{neo*}+R_n^{neo*}} = (B_n^{neo*})^2 - 2B_n^{neo*} + 2.$$

Similarly, let n be an almost neo balancing number of second type. Then from (7), we get

$$2n - 1 + nr + \frac{r(r+1)}{2} - \frac{(n-1)n}{2} = -1,$$

and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 - 2n.$$

Thus

$$T_{B_n^{neo**}+R_n^{neo**}} = (B_n^{neo**})^2 - 2B_n^{neo**},$$

as we wanted. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n . Notice that

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. We can give the general terms of S_n , s_n and t_n in terms of balancing and cobalancing numbers, namely,

$$S_n = B_n^2, \quad s_n = B_n \quad \text{and} \quad t_n = B_n + b_n. \quad (17)$$

Their Binet formulas are

$$S_n = \frac{\alpha^{4n} + \beta^{4n} - 2}{32}, \quad s_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} \quad \text{and} \quad t_n = \frac{\alpha^{2n} + \beta^{2n} - 2}{4}, \quad (18)$$

for $n \geq 1$. We can give the general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first and second type in terms of s_n and t_n as follows.

Theorem 5.2. *The general terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\begin{aligned} B_n^{neo*} &= 3s_{n-1} + 1, \\ C_n^{neo*} &= 6t_{n-1} + 3, \\ R_n^{neo*} &= -3s_{n-1} + 3t_{n-1}, \end{aligned}$$

for $n \geq 1$, and of second type are

$$\begin{aligned} B_{2n-1}^{neo**} &= s_{n-1} + 2t_{n-1} + 2 \\ B_{2n}^{neo**} &= -s_n + 2t_n + 2 \\ C_{2n-1}^{neo**} &= 8s_{n-1} + 2t_{n-1} + 1 \\ C_{2n}^{neo**} &= 8s_n - 2t_n - 1 \\ R_{2n-1}^{neo**} &= 3s_{n-1} - t_{n-1} - 2 \\ R_{2n}^{neo**} &= 5s_n - 3t_n - 3, \end{aligned}$$

for $n \geq 1$.

Proof. Notice that $s_n = B_n$ and $t_n = B_n + b_n$ by (17). Thus

$$\begin{aligned} t_n &= B_n + b_n \\ &= \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}} + \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\alpha^{2n}(1 + \alpha^{-1}) + \beta^{2n}(-1 - \beta^{-1})}{4\sqrt{2}} - \frac{1}{2} \\ &= \frac{\frac{\alpha^{2n} + \beta^{2n}}{2}}{2} - \frac{1}{2} \\ &= \frac{C_n - 1}{2}. \end{aligned}$$

So $C_n = 2t_n + 1$. Applying Theorems 2.2 and 2.4, we get the desired result. \square

Conversely, we can give the following theorem.

Theorem 5.3. *The general terms of S_n , s_n and t_n are*

$$\begin{aligned} S_n &= \frac{(B_{n+1}^{neo*})^2 - 2B_{n+1}^{neo*} + 1}{9} \\ s_n &= \frac{B_{n+1}^{neo*} - 1}{3} \\ t_n &= \frac{C_{n+1}^{neo*} - 3}{6}, \end{aligned}$$

for $n \geq 1$, or

$$S_n = \frac{(B_{2n}^{neo**})^2 + (C_{2n}^{neo**})^2 + 2B_{2n}^{neo**}C_{2n}^{neo**} - 2B_{2n}^{neo**} - 2C_{2n}^{neo**} + 1}{49}$$

$$s_n = \frac{B_{2n}^{neo**} + C_{2n}^{neo**} - 1}{7}$$

$$t_n = \frac{8B_{2n}^{neo**} + C_{2n}^{neo**} - 15}{14},$$

for $n \geq 1$.

Proof. From (18), we get

$$\begin{aligned} S_n &= \frac{\alpha^{4n} + \beta^{4n} - 2}{32} \\ &= \left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}} \right)^2 \\ &= \frac{9\left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}}\right)^2 + 6\left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}}\right) + 1 - 6\left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}}\right) - 2 + 1}{9} \\ &= \frac{\left[3\left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}}\right) + 1\right]^2 - 2\left[3\left(\frac{\alpha^{4n} - \beta^{4n}}{4\sqrt{2}}\right) + 1\right] + 1}{9} \\ &= \frac{(B_{n+1}^{neo*})^2 - 2B_{n+1}^{neo*} + 1}{9}, \end{aligned}$$

by Theorem 2.2. The others can be proved similarly. \square

Thus we construct one-to-one correspondence between almost neo balancing numbers and square triangular numbers.

6. Sums of almost neo balancing numbers

Theorem 6.1. *The sum of first n -terms of almost neo balancing numbers, almost Lucas-neo balancing numbers and almost neo balancers of first type are*

$$\sum_{i=1}^n B_i^{neo*} = \frac{15B_{n-1} - 3B_{n-2} + 4n - 3}{4}$$

$$\sum_{i=1}^n C_i^{neo*} = \frac{21B_{n-1} - 3B_{n-2} + 3}{2}$$

$$\sum_{i=1}^n R_i^{neo*} = \frac{3B_{n-1} - 3n + 3}{2},$$

for $n \geq 1$, and of second type are

$$\sum_{i=1}^n B_i^{neo**} = \begin{cases} 3B_{\frac{n}{2}} + n, & n \geq 2 \text{ even} \\ 7B_{\frac{n-1}{2}} - B_{\frac{n-3}{2}} + n, & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^n C_i^{neo**} = \begin{cases} 51B_{\frac{n-2}{2}} - 9B_{\frac{n-4}{2}} - 3, & n \geq 2 \text{ even} \\ 20B_{\frac{n-1}{2}} - 4B_{\frac{n-3}{2}} - 3, & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^n R_i^{neo**} = \frac{1}{2} \begin{cases} 15B_{\frac{n-2}{2}} - 3B_{\frac{n-4}{2}} - 3n - 3, & n \geq 2 \text{ even} \\ 6B_{\frac{n-1}{2}} - 2B_{\frac{n-3}{2}} - 3n - 3, & n \geq 1 \text{ odd.} \end{cases}$$

Proof. Recall that $B_1 + B_2 + \cdots + B_n = \frac{5B_n - B_{n-1} - 1}{4}$. Thus Theorem 2.2, we get

$$\begin{aligned} \sum_{i=1}^n B_i^{neo*} &= \sum_{i=1}^n (3B_{i-1} + 1) \\ &= 3\left(\sum_{i=1}^n B_{i-1}\right) + n \\ &= 3\left(\frac{5B_{n-1} - B_{n-2} - 1}{4}\right) + n \\ &= \frac{15B_{n-1} - 3B_{n-2} + 4n - 3}{4} \end{aligned}$$

for $n \geq 2$. The others can be proved similarly. \square

References

- [1] E. J. Barbeau. *Pell's Equation*. Springer-Verlag, New York, 2003.
- [2] A. Behera and G. K. Panda. On the square roots of triangular numbers. *The Fibonacci Quarterly*, 37(2):98–105, 1999. https://doi.org/10.1080/00150517.1999.12428864?urlappend=%3Futm_source%3Dresearchgate.net%26utm_medium%3Darticle.
- [3] D. Bród, A. Szynal-Liana, and I. Włoch. Balancing hybrid numbers, their properties and some identities. *Indian Journal of Mathematics*, 63(1):143–157, 2021.
- [4] N. Chailangka and A. Pakapongpun. Neo balancing numbers. *International Journal of Mathematics and Computer Science*, 16(4):1653–1664, 2021.
- [5] K. K. Dash, R. S. Ota, and S. Dash. t -balancing numbers. *International Journal of Contemporary Mathematical Sciences*, 7(41):1999–2012, 2012.
- [6] D. E. Flath. *Introduction to Number Theory*. Wiley, 1989.
- [7] G. K. Gözeri, A. Özkoç, and A. Tekcan. Some algebraic relations on balancing numbers. *Utilitas Mathematica*, 103:217–236, 2017.
- [8] M. Jacobson and H. Williams. *Solving the Pell Equation*. CMS Books in Mathematics. Springer Science and Business Media, LLC, 2009.
- [9] T. Kovacs, K. Liptai, and P. Olajos. On (a, b) -balancing numbers. *Publicationes Mathematicae Debrecen*, 77(3–4):485–498, 2010. <https://doi.org/10.5486/PMD.2010.4857>.
- [10] K. Liptai. Fibonacci balancing numbers. *The Fibonacci Quarterly*, 42(4):330–340, 2004. <https://doi.org/10.1080/00150517.2004.12428404>.
- [11] K. Liptai. Lucas balancing numbers. *Acta Mathematica Universitatis Ostraviensis*, 14:43–47, 2006.

-
- [12] K. Liptai, F. Luca, A. Pinter, and L. Szalay. Generalized balancing numbers. *Indagationes Mathematicae (New Series)*, 20(1):87–100, 2009. [https://doi.org/10.1016/S0019-3577\(09\)80005-0](https://doi.org/10.1016/S0019-3577(09)80005-0).
- [13] P. Olajos. Properties of balancing, cobalancing and generalized balancing numbers. *Annales Mathematicae et Informaticae*, 37:125–138, 2010.
- [14] M. Özdemir. Introduction to hybrid numbers. *Advances in Applied Clifford Algebras*, 28(1):Paper No. 11, 32 pp. 2018. <https://doi.org/10.1007/s00006-018-0833-3>.
- [15] A. K. Panda. *Some variants of the balancing sequences*. PhD thesis, National Institute of Technology Rourkela, India, 2017.
- [16] G. K. Panda, T. Komatsu, and R. K. Davala. Reciprocal sums of sequences involving balancing and lucas-balancing numbers. *Mathematical Reports*, 20(70):201–214, 2018.
- [17] G. K. Panda and A. K. Panda. Almost balancing numbers. *Journal of the Indian Mathematical Society*, 82(3–4):147–156, 2015.
- [18] G. K. Panda and P. K. Ray. Cobalancing numbers and cobalancers. *International Journal of Mathematics and Mathematical Sciences*, 8:1189–1200, 2005. <https://doi.org/10.1155/IJMMS.2005.1189>.
- [19] G. K. Panda and P. K. Ray. Some links of balancing and cobalancing numbers with pell and associated pell numbers. *Bulletin of the Institute of Mathematics, Academia Sinica*, 6(1):41–72, 2011.
- [20] P. K. Ray. *Balancing and cobalancing numbers*. PhD thesis, Department of Mathematics, National Institute of Technology Rourkela, India, 2009.
- [21] P. K. Ray. Balancing and lucas-balancing sums by matrix methods. *Mathematical Reports*, 17(67):225–233, 2015.
- [22] M. Rubajczyk and A. Szynal-Liana. Cobalancing hybrid numbers. *Annales Universitatis Mariae Curie-Skłodowska, Sectio A Mathematica*, 78(1):87–95, 2024. <https://doi.org/10.17951/a.2024.78.1.87-95>.
- [23] L. Szalay. On the resolution of simultaneous pell equations. *Annales Mathematicae et Informaticae*, 34:77–87, 2007.
- [24] A. Tekcan. Almost balancing, triangular and square triangular numbers. *Notes on Number Theory and Discrete Mathematics*, 25(1):108–121, 2019. <https://doi.org/10.7546/nntdm.2019.25.1.108-121>.
- [25] A. Tekcan. Sums and spectral norms of all almost balancing numbers. *Creative Mathematics and Informatics*, 28(2):203–214, 2019. <https://doi.org/10.37193/CMI.2019.02.12>.
- [26] A. Tekcan and E. Akgüç. Almost neo cobalancing numbers. *Notes on Number Theory and Discrete Mathematics*, 31(1):113–126, 2025. <https://doi.org/10.7546/nntdm.2025.31.1.113-126>.
- [27] A. Tekcan and S. Aydın. On t -balancers, t -balancing numbers and lucas t -balancing numbers. *Libertas Mathematica*, 41(1):37–51, 2021. <https://doi.org/10.52547/ijmsi.17.2.147>.

-
- [28] A. Tekcan and A. Erdem. t -cobalancing numbers and t -cobalancers. *Notes on Number Theory and Discrete Mathematics*, 26(1):45–58, 2020. <https://doi.org/10.7546/nntdm.2020.26.1.45-58>.
- [29] A. Tekcan and A. Erdem. General terms of all almost balancing numbers of first and second type. *Communications in Mathematics*, 31(1):155–167, 2023.
- [30] A. Tekcan and M. Yıldız. Balcobalancing numbers and balcobalancers. *Creative Mathematics and Informatics*, 30(2):203–222, 2021. <https://doi.org/10.37193/CMI.2021.02.11>.
- [31] A. Tekcan and M. Yıldız. Almost balcobalancing numbers. *Annales Universitatis Scientiarum Budapestinensis, Sectio Computatorica*, 53:71–83, 2022.
- [32] A. Tekcan and M. Yıldız. Balcobalancing numbers and balcobalancers ii. *Creative Mathematics and Informatics*, 31(2):247–258, 2022. <https://doi.org/10.37193/CMI.2022.02.10>.
- [33] A. Tekcan and M. Yıldız. Almost balcobalancing numbers ii. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, 35(1):24–50, 2024.
- [34] A. Tengely. Balancing numbers which are products of consecutive integers. *Publicationes Mathematicae Debrecen*, 83(1–2):197–205, 2013. <https://doi.org/10.5486/PMD.2013.5654>.

Ahmet Tekcan

Bursa Uludag University, Faculty of Science, Department of Mathematics, Bursa, Turkiye

E-mail tekcan@uludag.edu.tr