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The Gutman-Milovanović index and some Hamiltonian properties of a graph

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ABSTRACT

Let $G = (V, E)$ be a graph. The Gutman-Milovanović index of a graph G is defined as $\sum_{uv \in E} (d(u)d(v))^{\alpha}(d(u) + d(v))^{\beta}$, where α and β are any real numbers and $d(u)$ and $d(v)$ are the degrees of vertices u and v in G, respectively. In this note, we present sufficient conditions based on the Gutman-Milovanović index with $\alpha > 0$ and $\beta > 0$ for some Hamiltonian properties of a graph. We also present upper bounds for the Gutman-Milovanović index of a graph for different ranges of α and β.

Keywords: The Gutman-Milovanović index, Hamiltonian graph, traceable graph

2020 Mathematics Subject Classification: 05C45, 05C09.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notations and terminologies not defined here follow those in [1]. Let $G = (V, E)$ be a graph with n vertices and e edges, the degree of a vertex v in G is denoted by $d_G(v)$. We use δ and Δ to denote the minimum degree and maximum degree of G, respectively. The complement of a graph G is denoted by G^c. We use $G[S]$ to denote a subgraph induced by a subset S of $V(G)$. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\gamma(G)$, of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of V, we use $E(X, Y)$ to denote the set of all the edges in E such that one end vertex of each edge is in X and another end vertex of the edge is in Y. Namely,

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$E(X,Y) := \{ f : f = xy \in E, x \in X, y \in Y \}$. We use $G_1 \vee G_2$ to denote the the join of two disjoint graphs G_1 and G_2. The complete graph of order p is denoted by K_p. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

A variety of topological indexes for a graph have been introduced. Some notable ones are the Wiener index [11]. the Randić index [10], and the first Zagreb index and the second Zagreb index [5] of a graph. In 2020, Gutman, Milovanović, and Milovanović [4] introduced the concept of the Gutman-Milovanović index. For a graph G, its Gutman-Milovanović index, denoted $M_{\alpha,\beta}(G)$, is defined as $\sum_{uv\in E} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta}$, where α and β are any real numbers. The Gutman-Milovanović index is a generalization of both the first Zagreb index and second Zagreb index of a graph G since $M_{0,1}(G)$ and $M_{1,0}(G)$ are respectively the same as the first Zagreb index and second Zagreb index of a graph G. Some recent results on the Gutman-Milovanović index of a graph can be found in [3] and [9]. Motivated by the topological index conditions for some Hamiltonian properties of a graph in [6], [7], and [8], we in this note present sufficient conditions based on the Gutman-Milovanović index with $\alpha > 0$ and $\beta > 0$ for some Hamiltonian properties of a graph. We also present upper bounds for the Gutman-Milovanović index of a graph for different ranges of α and β. The main results are as follows.

Theorem 1.1. *Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices. If $\alpha > 0$, $\beta > 0$, and*

$$\begin{aligned} M_{\alpha,\beta}(G) \geq &(k+1)(n-k-1)^{\alpha+1}\Delta^{\alpha}(n-k-1+\Delta)^{\beta} \\ &+ (n-k-1)(n-k-2)2^{\beta-1}\Delta^{2\alpha+\beta}, \end{aligned}$$

then G is Hamiltonian or G is $K^c_{k+1} \vee K_k$.

Theorem 1.2. *Let G be a k-connected ($k \geq 1$) graph with n vertices. If $\alpha > 0$, $\beta > 0$, and*

$$\begin{aligned} M_{\alpha,\beta}(G) \geq &(k+2)(n-k-2)^{\alpha+1}\Delta^{\alpha}(n-k-2+\Delta)^{\beta} \\ &+ (n-k-2)(n-k-3)2^{\beta-1}\Delta^{2\alpha+\beta}, \end{aligned}$$

then G is traceable or G is $K^c_{k+2} \vee K_k$.

Theorem 1.3. *Let G be a graph with n vertices and $\delta \geq 1$. If $\alpha > 0$ and $\beta > 0$, then*

$$M_{\alpha,\beta}(G) \leq \gamma(n-\gamma)^{\alpha+1}\Delta^{\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta},$$

with equality if and only if G is $K^c_{\gamma} \vee K_{n-\gamma}$.

Theorem 1.4. *Let G be a graph with n vertices and $\delta \geq 1$. If $\alpha > 0$ and $\beta < 0$, then*

$$M_{\alpha,\beta}(G) \leq \gamma(n-\gamma)^{\alpha+1}(2\delta)^{\beta}\Delta^{\alpha} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{\beta}\Delta^{2\alpha},$$

with equality if and only if G is K_n.

Theorem 1.5. *Let G be a graph with n vertices and $\delta \geq 1$. If $\alpha < 0$ and $\beta > 0$, then*

$$M_{\alpha,\beta}(G) \leq \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta},$$

with equality if and only if G is K_n.

Theorem 1.6. *Let G be a graph with n vertices and $\delta \geq 1$. If $\alpha < 0$ and $\beta < 0$, then*

$$M_{\alpha,\beta}(G) \leq (n-\gamma)(n+\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta},$$

with equality if and only if G is K_n.

2. Lemmas

We will use the following results as our lemmas.

Lemma 2.1. [2] *Let G be a k-connected graph of order $n \geq 3$. If $\gamma \leq k$, then G is Hamiltonian.*

Lemma 2.2. [2] *Let G be a k-connected graph of order n. If $\gamma \leq k+1$, then G is traceable.*

3. Proofs

Proof of Theorem 1.1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices satisfying the conditions in Theorem 1.1. Suppose G is not Hamiltonian. Then Lemma 2.1 implies that $\gamma \geq k+1$. Also, we have that $n \geq 2\delta+1 \geq 2k+1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. Let $I_1 := \{ u_1, u_2, ..., u_\gamma \}$ be a maximum independent set in G. Then $I := \{ u_1, u_2, ..., u_{k+1} \}$ is an independent set in G. Clearly, $\delta \leq d(x) \leq n-(k+1)$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V - I$. Thus we have that

$$\begin{aligned}
S_1 :=& \sum_{uv \in E, u \in I, v \in V-I} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\
\leq& \sum_{u \in I, v \in V-I} \left((n-k-1)^{\alpha}\Delta^{\alpha}(n-k-1+\Delta)^{\beta}\right) \\
\leq& (n-k-1)^{\alpha}\Delta^{\alpha}(n-k-1+\Delta)^{\beta}(n-k-1)(k+1) \\
=& (k+1)(n-k-1)^{\alpha+1}\Delta^{\alpha}(n-k-1+\Delta)^{\beta}.
\end{aligned}$$

We further have that

$$\begin{aligned}
S_2 :=& \sum_{uv \in E, u \in V-I, v \in V-I, u \neq v} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\
\leq& \sum_{u \in V-I, v \in V-I, u \neq v} \left(\Delta^{\alpha}\Delta^{\alpha}(\Delta+\Delta)^{\beta})\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}(n-k-1)(n-k-2)\, 2^{\beta}\Delta^{2\alpha+\beta} \\
&=(n-k-1)(n-k-2)\, 2^{\beta-1}\Delta^{2\alpha+\beta}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
M_{\alpha,\beta}(G) =& S_1 + S_2 \\
&\leq (k+1)(n-k-1)^{\alpha+1}\Delta^{\alpha}(n-k-1+\Delta)^{\beta} + (n-k-1)(n-k-2)2^{\beta-1}\Delta^{2\alpha+\beta}.
\end{aligned}
$$

From the conditions in Theorem 1.1, we have that

$$
\begin{aligned}
M_{\alpha,\beta}(G) =&(k+1)(n-k-1)^{\alpha+1}\Delta^{\alpha}(n-k-1+\Delta)^{\beta} \\
&+ (n-k-1)(n-k-2)2^{\beta-1}\Delta^{2\alpha+\beta}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
S_1 =&(k+1)(n-k-1)^{\alpha+1}\Delta^{\alpha}(n-k-1+\Delta)^{\beta}, \\
S_2 =&(n-k-1)(n-k-2)\, 2^{\beta-1}\Delta^{2\alpha+\beta}.
\end{aligned}
$$

Therefore $d(x) = n-k-1$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-k-1)$, and $d(y) = \Delta = n-1$ for each $y \in V-I$, Therefore G is $K_{k+1}^c \vee K_{n-(k+1)}$. Since $n \geq 2k+1$, we have that $|V-I| \geq k$. If $|V-I| \geq (k+1)$, then G is Hamiltonian, a contradiction. Thus $|V-I| = k$ and G is $K_{k+1}^c \vee K_k$. □

Proof of Theorem 1.2. If $n=1$ or $n=2$, then it is trivial that G is traceable. Let G be a k-connected ($k \geq 1$) graph with $n \geq 3$ vertices satisfying the conditions in Theorem 1.2. Suppose G is not traceable. Then Lemma 2.2 implies that $\gamma \geq k+2$. Also, we have that $n \geq 2\delta + 2 \geq 2k+2$ otherwise $\delta \geq k \geq (n-1)/2$ and G is traceable. Let $I_1 := \{ u_1, u_2, ..., u_\gamma \}$ be a maximum independent set in G. Then $I := \{ u_1, u_2, ..., u_{k+2} \}$ is an independent set in G. Clearly, $\delta \leq d(x) \leq n-(k+2)$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V-I$. Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$
\begin{aligned}
S_1 :=& \sum_{uv \in E, u \in I, v \in V-I} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\
&\leq (k+2)(n-k-2)^{\alpha+1}\Delta^{\alpha}(n-k-2+\Delta)^{\beta}, \\
S_2 :=& \sum_{uv \in E, u \in V-I, v \in V-I, u \neq v} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\
&\leq (n-k-2)(n-k-3)\, 2^{\beta-1}\Delta^{2\alpha+\beta}.
\end{aligned}
$$

Therefore

$$
M_{\alpha,\beta}(G) = S_1 + S_2
$$

$$\leq (k+2)(n-k-2)^{\alpha+1}\Delta^{\alpha}(n-k-2+\Delta)^{\beta} + (n-k-2)(n-k-3)2^{\beta-1}\Delta^{2\alpha+\beta}.$$

From the conditions in Theorem 1.2, we have that

$$\begin{aligned} M_{\alpha,\beta}(G) =& (k+2)(n-k-2)^{\alpha+1}\Delta^{\alpha}(n-k-2+\Delta)^{\beta} \\ &+ (n-k-2)(n-k-3)2^{\beta-1}\Delta^{2\alpha+\beta}. \end{aligned}$$

Thus

$$\begin{aligned} S_1 =& (k+2)(n-k-2)^{\alpha+1}\Delta^{\alpha}(n-k-2+\Delta)^{\beta}, \\ S_2 =& (n-k-2)(n-k-2)2^{\beta-1}\Delta^{2\alpha+\beta}. \end{aligned}$$

Therefore $d(x) = n-k-2$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-k-2)$, and $d(y) = \Delta = n-1$ for each $y \in V-I$. Since $n \geq 2k+2$, we have that $|V-I| \geq k$. If $|V-I| \geq (k+1)$, then G is traceable, a contradiction. Thus $|V-I| = k$ and G is $K_{k+2}^{c} \vee K_k$. □

Proof of Theorem 1.3. Suppose G is a graph with n vertices and $\delta \geq 1$. Let I be an independent set such that $|I| = \gamma$. Then $1 \leq \gamma \leq n-1$ and $1 \leq n-\gamma \leq n-1$. Clearly, $\delta \leq d(x) \leq n-\gamma$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V-I$. Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$\begin{aligned} S_1 :=& \sum_{uv \in E, u \in I, v \in V-I} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ \leq& \gamma(n-\gamma)^{\alpha+1}\Delta^{\alpha}(n-\gamma+\Delta)^{\beta}, \\ S_2 :=& \sum_{uv \in E, u \in V-I, v \in V-I, u \neq v} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ \leq& (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} M_{\alpha,\beta}(G) =& S_1 + S_2 \\ \leq& \gamma(n-\gamma)^{\alpha+1}\Delta^{\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta}. \end{aligned}$$

If

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)^{\alpha+1}\Delta^{\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta},$$

then

$$\begin{aligned} S_1 &= \gamma(n-\gamma)^{\alpha+1}\Delta^{\alpha}(n-\gamma+\Delta)^{\beta}, \\ S_2 &= (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta}. \end{aligned}$$

In reviewing all the above arguments in this proof, we have that $d(x) = n-\gamma$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-\gamma)$, and $d(y) = \Delta = n-1$ for each

$y \in V-I$. Therefore G is $K_\gamma^c \vee K_{n-\gamma}$. If G is $K_\gamma^c \vee K_{n-\gamma}$, a simple computation verifies that

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)^{\alpha+1}\Delta^\alpha(n-\gamma+\Delta)^\beta + (n-\gamma)(n-\gamma-1)2^{\beta-1}\Delta^{2\alpha+\beta}.$$

□

Proof of Theorem 1.4. Suppose G is a graph with n vertices and $\delta \geq 1$. Let I be an independent set such that $|I| = \gamma$. Then $1 \leq \gamma \leq n-1$ and $1 \leq n-\gamma \leq n-1$. Clearly, $\delta \leq d(x) \leq n-\gamma$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V-I$. Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$\begin{aligned}
S_1 &:= \sum_{uv\in E, u\in I, v\in V-I} (d(u)d(v))^\alpha (d(u)+d(v))^\beta \\
&\leq \sum_{u\in I, v\in V-I} \left((n-\gamma)^\alpha \Delta^\alpha (\delta+\delta)^\beta\right) \\
&= \gamma(n-\gamma)^{\alpha+1}(2\delta)^\beta\Delta^\alpha, \\
S_2 &:= \sum_{uv\in E, u\in V-I, v\in V-I, u\neq v} (d(u)d(v))^\alpha (d(u)+d(v))^\beta \\
&\leq \sum_{u\in V-I, v\in V-I, u\neq v} \left(\Delta^\alpha\Delta^\alpha(\delta+\delta)^\beta\right) \\
&= (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^\beta\Delta^{2\alpha}.
\end{aligned}$$

Therefore

$$\begin{aligned}
M_{\alpha,\beta}(G) &= S_1 + S_2 \\
&\leq \gamma(n-\gamma)^{\alpha+1}(2\delta)^\beta\Delta^\alpha + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^\beta\Delta^{2\alpha}.
\end{aligned}$$

If

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)^{\alpha+1}(2\delta)^\beta\Delta^\alpha + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^\beta\Delta^{2\alpha},$$

then

$$\begin{aligned}
S_1 &= \gamma(n-\gamma)^{\alpha+1}(2\delta)^\beta\Delta^\alpha, \\
S_2 &= (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^\beta\Delta^{2\alpha}.
\end{aligned}$$

In reviewing all the above arguments in this proof, we have that $d(x) = n-\gamma$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-\gamma)$, and $d(y) = \delta = \Delta = n-1$ for each $y \in V-I$. Therefore G is K_n. If G is K_n, a simple computation verifies that

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)^{\alpha+1}(2\delta)^\beta\Delta^\alpha + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^\beta\Delta^{2\alpha}.$$

□

Proof of Theorem 1.5. Suppose G is a graph with n vertices and $\delta \geq 1$. Let I be an independent set such that $|I| = \gamma$. Then $1 \leq \gamma \leq n-1$ and $1 \leq n-\gamma \leq n-1$. Clearly,

$\delta \leq d(x) \leq n-\gamma$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V-I$. Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$\begin{aligned} S_1 &:= \sum_{uv\in E, u\in I, v\in V-I} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ &\leq \sum_{u\in I, v\in V-I} \left(\delta^{\alpha}\delta^{\alpha}(n-\gamma+\Delta)^{\beta}\right) \\ &= \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta}, \\ S_2 &:= \sum_{uv\in E, u\in V-I, v\in V-I, u\neq v} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ &\leq \sum_{u\in V-I, v\in V-I, u\neq v} \left(\delta^{\alpha}\delta^{\alpha}(\Delta+\Delta)^{\beta})\right) \\ &= (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} M_{\alpha,\beta}(G) =& S_1+S_2 \\ \leq& \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta}. \end{aligned}$$

If

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta},$$

then

$$\begin{aligned} S_1 =& \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta}, \\ S_2 =& (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta}. \end{aligned}$$

In reviewing all the above arguments in this proof, we have that $d(x) = n-\gamma = \delta$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-\gamma)$, and $d(y) = \delta = \Delta = n-1$ for each $y \in V-I$. Therefore G is K_n. If G is K_n, a simple computation verifies that

$$M_{\alpha,\beta}(G) = \gamma(n-\gamma)\delta^{2\alpha}(n-\gamma+\Delta)^{\beta} + (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha}\Delta^{\beta}.$$

□

Proof of Theorem 1.6. Suppose G is a graph with n vertices and $\delta \geq 1$. Let I be an independent set such that $|I| = \gamma$. Then $1 \leq \gamma \leq n-1$ and $1 \leq n-\gamma \leq n-1$. Clearly, $\delta \leq d(x) \leq n-\gamma$ for each $x \in I$ and $\delta \leq d(y) \leq \Delta$ for each $y \in V-I$. Using the arguments similar to the ones in Proof of Theorem 1.1, we have that

$$\begin{aligned} S_1 &:= \sum_{uv\in E, u\in I, v\in V-I} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ &\leq \sum_{u\in I, v\in V-I} \left(\delta^{\alpha}\delta^{\alpha}(\delta+\delta)^{\beta}\right) \\ &= \gamma(n-\gamma)2^{\beta}\delta^{2\alpha+\beta}, \end{aligned}$$

$$\begin{aligned} S_2 := & \sum_{uv\in E, u\in V-I, v\in V-I, u\neq v} (d(u)d(v))^{\alpha}(d(u)+d(v))^{\beta} \\ \leq & \sum_{u\in V-I, v\in V-I, u\neq v} \left(\delta^{\alpha}\delta^{\alpha}(\delta+\delta)^{\beta})\right) \\ = & (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta}. \end{aligned}$$

Therefore

$$M_{\alpha,\beta}(G) = S_1 + S_2 \leq (n-\gamma)(n+\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta}.$$

If

$$M_{\alpha,\beta}(G) = (n-\gamma)(n+\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta},$$

then

$$\begin{aligned} S_1 = & \gamma(n-\gamma)2^{\beta}\delta^{2\alpha+\beta}, \\ S_2 = & (n-\gamma)(n-\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta}. \end{aligned}$$

In reviewing all the above arguments in this proof, we have that $d(x) = n-\gamma = \delta$ for each $x \in I$, $G[V-I]$ is a complete graph of order $(n-\gamma)$, and $d(y) = \delta = n-1$ for each $y \in V-I$. Therefore G is K_n. If G is K_n, a simple computation verifies that

$$M_{\alpha,\beta}(G) = (n-\gamma)(n+\gamma-1)2^{\beta-1}\delta^{2\alpha+\beta}.$$

□

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