UTILITAS MATHEMATICA 125 (2025) 103–108

Utilitas Mathematica
www.combinatorialpress.com/um

Combinatorial Press

Unique paired vs edge-vertex minimum dominating sets in trees

Mateusz Miotk, Michał Zakrzewski* and Paweł Żyliński

ABSTRACT

We prove that the class of trees with unique minimum edge-vertex dominating sets is equivalent to the class of trees with unique minimum paired dominating sets.

Keywords: paired domination, edge-vertex domination, unique dominating set, tree

2020 Mathematics Subject Classification: 05C05, 05C59.

1. Introduction

A subset D of V_G is said to be a *dominating set* of a graph $G = (V_G, E_G)$ if each vertex belonging to the set $V_G \setminus D$ has a neighbour in D [1, 20]. A set $D \subseteq V_G$ is a *paired-dominating set* of G if D is a dominating set of G and the induced subgraph $G[D]$ has a perfect matching. The *paired-domination number* of G, denoted by $\gamma_{\rm pr}(G)$, is defined to be the minimum cardinality of a paired-dominating set D of G, and any minimum paired dominating set of G is referred to as a $\gamma_{\rm pr}$-set [12, 13]. Next, an edge $e \in E_G$ is said to *ev-dominate* a vertex $v \in V_G$ if e is incident to v or e is incident to a vertex adjacent to v. A set $M \subseteq E_G$ is an *edge-vertex dominating set* (or simply, an *ev-dominating set*) of G if each vertex of G is *ev*-dominated by some edge in M. Finally, the *edge-vertex domination number of* G, denoted by $\gamma_{ev}(G)$, is the minimum cardinality of an *ev*-dominating set of G, and any minimum edge-vertex dominating set of G is referred to as a $\gamma_{\rm ev}$-set [17].

Clearly, any matching in a minimum paired-dominating set of a graph G constitutes an edge-vertex dominating set of G, but not vice versa, that is, all the vertices of edges forming a minimum edge-vertex dominating set of G do not always constitute a paired-dominating set of G. However, a fundamental, but not common, relation between edge-vertex domination and paired domination was proved by Hedetniemi et al. [15] who es-

* Corresponding author.

Received 19 Nov 2025; Revised 01 Dec 2025; Accepted 02 Dec 2025; Published Online 25 Dec 2025.
DOI: 10.61091/um125-07

tablished that for any graph G with no isolated vertex we have $2\gamma_{\mathrm{ev}}(G) = \gamma_{\mathrm{pr}}(G)$.

Theorem 1.1. [15] *If G is a graph with no isolated vertex, then $2\gamma_{\mathrm{ev}}(G) = \gamma_{\mathrm{pr}}(G)$.*

In this short note, we present another one, when restricted to unique dominating set. Namely, a commonly used approach for constructive characterizations of trees with unique minimum dominating sets, for different models of domination [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 16, 19, 23, 21, 22, 24, 25] is to define a set of basic operations such that starting from the initial pre-defined (finite) set of small trees, playing a role of seeds, any tree with the unique minimum dominating set (in the relevant model) can be constructed by applying, successively, a finite sequence of these operations. In particular, Chellali and Haynes [4] characterized the class of trees with unique minimum paired dominating sets by providing the relevant set of four basic graph operations, whereas Senthilkumar et al. [22] – the relevant set of five basic graph operations in the case of edge-vertex domination. However, we establish the following result.

Theorem 1.2. *The class of trees having unique minimum edge-vertex dominating sets is equivalent to the class of trees having unique minimum paired dominating sets.*

Observe that if a graph G has a cycle, then uniqueness of a γ_{pr}-set of G does not imply uniqueness of a γ_{ev}-set of G, see Figure 1 for an example. However, as a side result of the proof of Theorem 1.2, we obtain that such a γ_{pr}-uniqueness forces all γ_{ev}-sets of G to be spanned on the same subset of vertices (Corollary 2.3 and 2.4).

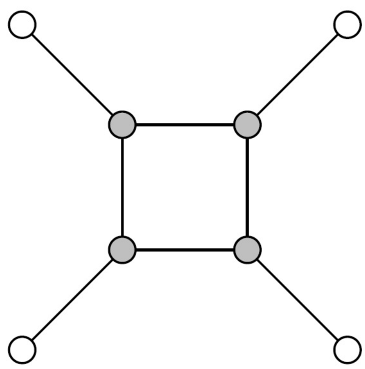

Fig. 1. The graph G has a unique γ_{pr}-set (gray vertices), whereas its minimum γ_{ev}-set is not unique: there are two such sets, but both spanned on the same set of (gray) vertices

2. Main results

We assume that all graphs are simple, i.e., they contain neither loops nor parallel edges. Recall that the *open neighborhood* of a vertex v in a connected graph $G = (V_G, E_G)$ is the set $N_G(v) = \{u \in V_G : uv \in E_G\}$, whereas for an edge subset $M \subseteq E_G$, $V_G(M)$ denotes the set of vertices of all edges in M. We start with the following crucial lemma (its proof, in general, follows the argument in the proof of Theorem 73 in [18] and Theorem 2.2 in [15], but proceeds with a more detailed analysis).

Lemma 2.1. *Let D_{ev} be a γ_{ev}-set of a graph $G = (V_G, E_G)$. If there exist two edges in D_{ev} sharing a vertex, then there exist two (distinct) γ_{ev}-sets D'_{ev} and D''_{ev} of G such that $V_G(D'_{\text{ev}}) \neq V_G(D''_{\text{ev}})$ and no two edges in D'_{ev}, resp. D''_{ev}, share a vertex.*

Proof. Let D_{ev} be a γ_{ev}-set of a graph $G = (V_G, E_G)$. We have the following claim (it follows immediately from minimality of D_{ev}).

Claim. There are no three (distinct) edges in D_{ev} such that they constitute either a 4-vertex path or a 3-vertex cycle in G.

Assume now that there exist two edges $e_1, e_2 \in D_{\text{ev}}$ sharing a vertex, say $e_1 = x_1x_2$ and $e_2 = x_2x_3$. Since D_{ev} is a γ_{ev}-set, there exists $x_0 \in N_G(x_1)$ such that x_0 is ev-dominated only by edge e_1, and analogously, there exists $x_4 \in N_G(x_3)$ such that x_4 is ev-dominated only by edge e_2. Consider now the sets $X^1_{\text{ev}} = (D_{\text{ev}} \setminus \{e_1\}) \cup \{x_0x_1\}$ and $Y^1_{\text{ev}} = (D_{\text{ev}} \setminus \{e_2\}) \cup \{x_3x_4\}$; we shall refer to such edge replacement operation on D_{ev} as D_{ev}-*twinning*. Clearly, X^1_{ev} and Y^1_{ev} are distinct γ_{ev}-sets of G and $X^1_{\text{ev}} \cap Y^1_{\text{ev}} = D_{\text{ev}} \setminus \{e_1, e_2\}$. Moreover, taking into account the above claim, each of them has less (but the same) number of pairs of edges sharing a vertex. Consequently, if they have no such pair of distinct edges having a vertex in common, we are done.

Otherwise, we focus only on the set X^1_{ev} and apply X^1_{ev}-twinning, which now results in two distinct γ_{ev}-sets X^2_{ev} and Y^2_{ev}. We continue (iteratively) X^i_{ev}-twinning procedure as long as X^i_{ev} (and so Y^i_{ev}) has at least one pair of edges sharing a vertex. Eventually, we arrive to the situation when X^i_{ev} (and so Y^i_{ev}) has no two edges having a vertex in common. Clearly, keeping in mind minimality of the initial set D_{ev}, it follows from the construction that the sets X^i_{ev} and Y^i_{ev} are the two sought γ_{ev}-sets of G. □

Next, let G be a graph of order at least three (the base case of a 2-vertex tree is obvious) and assume that all γ_{ev}-sets of G are spanned on the same (unique) vertex set D. Let D_{ev} be any γ_{ev}-set of G. Taking into account Lemma 2.1, uniqueness of D implies that no two elements of D_{ev} share a vertex. Consequently, the set D of size $2|D_{\text{ev}}|$ – perfectly matchable by D_{ev} – is a minimum paired-dominating set of G by Theorem 1.1. Furthermore, we claim that D is unique one. Indeed, suppose to the contrary that there exists another γ_{pr}-set D' of G. Then, a perfect matching M' in the induced subgraph $G[D']$ is an edge-vertex dominating set of G with $|D'|/2 = |D_{\text{ev}}| = \gamma_{\text{ev}}(G)$, which contradicts uniqueness of D (since $D' \neq D$).

Corollary 2.2. *If a graph G has a unique γ_{ev}-set, then G has a unique γ_{pr}-set.*

Assume now that a graph G has a unique γ_{pr}-set D_{pr}. We first claim that $V_G(D'_{\text{ev}}) = V_G(D''_{\text{ev}})$ for any two γ_{ev}-sets of G. Indeed, consider a γ_{ev}-set D_{ev} of G, with $2|D_{\text{ev}}| = |D_{\text{pr}}|$ (by Theorem 1.1), imposed by a perfect matching in $G[D_{\text{pr}}]$. It follows from Lemma 2.1 that no two edges in (any) γ_{ev}-set D_{ev} of G share a vertex (since otherwise, the relevant sets

$V_G(D'_{\mathrm{ev}})$ and $V_G(D''_{\mathrm{ev}})$ in Lemma 2.1 are two distinct γ_{pr}-sets of G, a contradiction with D_{pr} being unique). So suppose now that G has two distinct minimum edge-vertex dominating sets, say X'_{ev} and X''_{ev}, with no two edges sharing a vertex. If $V_G(X'_{\mathrm{ev}}) \neq V_G(X''_{\mathrm{ev}})$, then both $V_G(X'_{\mathrm{ev}})$ and $V_G(X''_{\mathrm{ev}})$ are paired-dominating sets of G of size $\gamma_{\mathrm{pr}}(G)$ – a contradiction with uniqueness of D_{pr}. Therefore, we must have $V_G(X'_{\mathrm{ev}}) = V_G(X''_{\mathrm{ev}})$, which makes us in a position to prove Theorem 1.2.

Observe that if $V_G(X'_{\mathrm{ev}}) = V_G(X''_{\mathrm{ev}})$, then the induced subgraph $G[V_G(X'_{\mathrm{ev}}) \cup V_G(X''_{\mathrm{ev}})]$ has a cycle (since both D'_{ev} and D''_{ev} are perfect matchings in $G[V_G(X'_{\mathrm{ev}}) \cup V_G(D''_{\mathrm{ev}})]$), and thus G is not a tree. Therefore, if G is a tree, then we must have $X'_{\mathrm{ev}} = X''_{\mathrm{ev}}$, which eventually completes the (simple) proof of Theorem 1.2.

We note in passing that we have actually proved the following two properties.

Corollary 2.3. *Let T be a non-trivial tree. If D_{pr} is a unique γ_{pr}-set of T, then T has a unique γ_{ev}-set which is the (unique) perfect matching of $V_T(D_{\mathrm{ev}}) = D_{\mathrm{pr}}$. Analogously, if D_{ev} is a unique γ_{ev}-set of T, then T has a unique γ_{pr}-set which is $V_T(D_{\mathrm{ev}})$.*

Corollary 2.4. *Let G be a simple graph without isolated vertices. Then G has a unique minimum paired dominating set, say D, if and only if all its minimum edge-vertex dominating sets are spanned on all and only vertices.*

References

[1] C. Berge. *Théorie des Graphes et ses Applications.* Dunod, Paris, 1958. https://doi.org/10.1007/BF03020407.

[2] M. Blidia, M. Chellali, R. Lounes, and F. Maffray. Characterizations of trees with unique minimum locating-dominating sets. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 76:225–232, 2011.

[3] M. Bouzefrane, I. Bouchemakh, M. Zamime, and N. Ikhlef-Eschouf. Trees with unique minimum global offensive alliance. *RAIRO – Operations Research*, 55:863–872, 2021. https://doi.org/10.1051/ro/2020017.

[4] M. Chellali and T. W. Haynes. Trees with unique minimum paired-dominating sets. *Ars Combinatoria*, 73:3–12, 2004.

[5] M. Chellali and T. W. Haynes. A characterization of trees with unique minimum double dominating sets. *Utilitas Mathematica*, 83:233–242, 2010.

[6] M. Chellali and N. J. Rad. Trees with unique roman dominating functions of minimum weight. *Discrete Mathematics, Algorithms and Applications*, 6(3):1450038, 2014. https://doi.org/10.1142/S1793830914500384.

[7] M. Fischermann and L. Volkmann. Unique minimum domination in trees. *The Australasian Journal of Combinatorics*, 25:117–124, 2002.

[8] G. Gunther, B. Hartnell, L. R. Markus, and D. Rall. Graphs with unique minimum dominating sets. *Congressus Numerantium*, 101:55–63, 1994.

[9] T. W. Haynes and M. A. Henning. Trees with unique minimum total dominating sets. *Discussiones Mathematicae Graph Theory*, 22(2):233–246, 2002. https://doi.org/10.7151/dmgt.1172.

[10] T. W. Haynes and M. A. Henning. Trees with unique minimum semi-total dominating sets. *Graphs and Combinatorics*, 36:689–702, 2020. https://doi.org/10.1007/s00373-020-02145-0.

[11] T. W. Haynes and M. A. Henning. Construction of trees with unique minimum semipaired dominating sets. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 116:1–12, 2021.

[12] T. W. Haynes and P. J. Slater. Paired domination and the paired domatic number. *Congressus Numerantium*, 109:65–72, 1995.

[13] T. W. Haynes and P. J. Slater. Paired-domination in graphs. *Networks*, 32:199–206, 1998. https://doi.org/10.1002/(SICI)1097-0037(199810)32:3<199::AID-NET4>3.0.CO;2-F.

[14] J. Hedetniemi. On graphs having a unique minimum independent dominating set. *Australasian Journal of Combinatorics*, 68(3):357–370, 2017.

[15] S. M. Hedetniemi, S. T. Hedetniemi, and T. W. Haynes. Two parameters equivalent to paired-domination. *Graph Theory Notes of New York*, 66:16–19, 2014.

[16] S. M. Lane. *Trees with unique minimum locating-dominating sets*. Master's thesis, East Tennessee State University, 2006, page 72.

[17] R. Laskar and K. Peters. Vertex and edge domination parameters in graphs. *Congressus Numerantium*, 48:291–305, 1985.

[18] J. R. Lewis. *Vertex-Edge and Edge-Vertex Domination in Graphs*. PhD thesis, Clemson University, 2007.

[19] Y. J. Lu, X. Hou, J.-M. Xu, and N. Li. Trees with unique minimum p-dominating sets. *Utilitas Mathematica*, 86:193–205, 2011.

[20] O. Ore. *Theory of Graphs*, volume 38 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, 1962.

[21] B. Senthilkumar, M. Chellali, H. N. Kumar, and Y. B. Venkatakrishnan. Graphs with unique minimum vertex-edge dominating sets. *RAIRO – Operations Research*, 57:1785–1795, 2023. https://doi.org/10.1051/ro/2023074.

[22] B. Senthilkumar, M. Chellali, H. N. Kumar, and V. B. Yanamandram. Graphs with unique minimum edge-vertex dominating sets. *Communications in Combinatorics and Optimization*, 10(1):99–109, 2025. https://doi.org/10.22049/cco.2023.28605.1631.

[23] B. Sharada. Trees with unique minimum dominating sets. *International Journal of Soft Computing, Mathematics and Control*, 4(1):13–17, 2015. https://doi.org/10.14810/ijscmc.2015.4102.

[24] J. Topp. Graphs with unique minimum edge dominating sets and graphs with unique maximum independent sets of vertices. *Discrete Mathematics*, 121(1–3):199–210, 1993. https://doi.org/10.1016/0012-365X(93)90553-6.

[25] W. Zhao, F. Wang, and H. Zhang. Construction for trees with unique minimum dominating sets. *International Journal of Computer Mathematics: Computer Systems Theory*, 3(3):204–213, 2018. https://doi.org/10.1080/23799927.2018.1531930.

Mateusz Miotk
University of Gdańsk, Poland
E-mail mateusz.miotk@ug.edu.pl

Michał Zakrzewski
University of Gdańsk, Poland
E-mail michal.zakrzewski@ug.edu.pl

Paweł Żyliński
University of Gdańsk, Poland
E-mail pawel.zylinski@ug.edu.pl