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A QUBO formulation for nowhere-zero k-flows

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ABSTRACT

We consider the encoding of graph problems as Quadratic Unconstrained Binary Optimization (QUBO) problems, solvable by quantum or classical annealers. However, nowhere-zero flows have not previously been included among graph problems encoded as QUBO problems. Nowhere-zero flows are related to Tutte's 5-flow conjecture and occur in many contexts in graph theory. We provide a QUBO Hamiltonian encoding of nowhere-zero flows and prove the correctness of the construction. The resulting Hamiltonian $H_{\mathrm{mod},k}$ has zero ground-state energy if and only if the graph G has a nowhere-zero $\mathbb{Z}_k$-flow. By Tutte's equivalence theorem, zero ground energy is equivalent to $\varphi(G) \leq k$, and the zero-energy degeneracy is given by the flow polynomial $F(G; k)$. The construction uses one-hot variables for edge flow residues modulo k and auxiliary variables for the per-vertex modular quotient. We prove that correctness is independent of the choice of orientation, root vertex, and positive penalty weights. We verify the construction on 59 graph and k examples, including both yes-instances and no-instances. We also sweep orientations and root choices on selected robustness instances and test a finite suite of positive penalty weights. The Hamiltonian is implemented using the `dimod.BinaryQuadraticModel` class, compatible with the D-Wave Ocean SDK. Quantum-hardware runs and claims about potential speedup are left to future work.

Keywords: nowhere-zero flows, QUBO, graph optimization

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1. Introduction

Nowhere-zero flows are among the oldest objects in algebraic graph theory. Tutte [14] introduced nowhere-zero flows as a generalisation of map colourings and conjectured that every bridgeless multigraph admits a 5-flow. Whilst this conjecture is still open after seventy years, the best general result is Seymour's 6-flow theorem [13], with a recent short proof by [3]. Jaeger showed that whenever G is 4-edge-connected, it admits a 4-flow [9]. More recently, [11] showed that every 6-edge-connected graph admits a nowhere-zero 3-flow. Integer flows and cycle covers were treated by [16]; we follow the notation of [4] throughout. Fix an orientation of the edges of G. A nowhere-zero $\mathbb{Z}_k$-flow assigns each edge a nonzero residue in $\{1, \ldots, k-1\}$ such that the signed sum at every vertex is 0 (mod k). By Tutte's equivalence theorem, this is equivalent to the integer k-flow defined in Definition 2.5.

Despite its importance in graph theory, nowhere-zero flows has so far remained outside of the quantum-optimisation toolkit. Lucas et al. [12] provided QUBO encodings of a variety of NP-hard problems on graphs, such as the Hamiltonian cycle, graph colouring, vertex cover, and set packing problems. Nowhere-zero flows is one that is absent from this catalogue. To our knowledge, no QUBO encoding of the problem has been published.

[2] consider minimum-cost nowhere-zero flows and cut-balanced orientations, proving hardness and inapproximability results for this optimisation problem. The feasibility problem represents the decision problem at the heart of the proof. Thus, these results motivate the study of binary encodings of nowhere-zero flows, which can be used with QUBO-compatible optimisation methods. Such methods include quantum annealing as proposed by [10], adiabatic optimisation as proposed by [7], the Quantum Approximate Optimization Algorithm as proposed by [6], and related digital or classical annealing methods.

[5] recently introduced the flow reconfiguration graph $\mathcal{F}(G, k)$ and asked when it is connected. Their questions depend computationally on the ability to enumerate or sample nowhere-zero $\mathbb{Z}_k$-flows, for which a QUBO formulation provides a search space whose zero-energy states correspond exactly to the flows. The closest existing quantum-algorithmic result in the Tutte setting is the work of [1], who give polynomial-time quantum algorithms for additive approximation of the Tutte polynomial of planar graphs. Their setting is approximation on planar graphs, whereas the present paper concerns exact feasibility on arbitrary nonempty loopless multigraphs.

The difficulty with modular conservation is that it is not itself a quadratic constraint: the congruence $R_v \equiv 0 \pmod{k}$ does not have a native form in the space of QUBOs. Instead, at each non-root vertex, we introduce an auxiliary block of binary variables representing the signed quotient, which allows the conservation constraint to be encoded as an exact integer equality and the whole penalty to be quadratic. Theorem 3.6 establishes a bijection between nowhere-zero $\mathbb{Z}_k$-flows on G and the zero-energy states of $H_{\text{mod},k}$, relates zero ground energy to the decision problem $\varphi(G) \leq k$, and proves that the emptiness and cardinality of the set of zero-energy states are invariant under the choice of orientation, root vertex, and positive penalty weights. Furthermore, Corollary 3.7 shows that the size of the set of zero-energy states is equal to the flow polynomial $F(G; k)$, and Proposition 3.8 gives a lower bound $\min(A, B)$ on positive energies.

We verify the construction on a base test set of 59 (G,k) configurations, which include simple graphs, multigraphs with parallel edges, and both yes- and no-instances for the existence of a k-flow, ranging from graphs as small as the complete graph K_3 up to the Petersen graph. Furthermore, we test robustness under the three encoding choices appearing in the theorem, including exhaustive tests of all possible orientations of K_4, $K_{3,3}$, and Θ_3, exhaustive tests of all possible roots of $K_{3,3}$ and the Petersen graph, and a test of different penalty weights on K_4.

The claim of the paper is limited to the reduction itself. A zero-energy state that any solver (quantum or classical) returns is a witness to the fact that $\varphi(G) \leq k$. A solver that only returns positive energies is not a certificate of nonexistence of solutions. The Hamiltonian is not run on any quantum hardware in this work. No claims are made about the speedup enabled by using quantum hardware to evaluate the Hamiltonian. The experiments performed in this work are implementation checks for correctness of the construction; the exact checks are supplemented by a randomized non-flow test. The Tutte conjecture and the 4-flow conjecture of [15] remain open problems in graph theory, and are not addressed in this paper.

The construction extends the QUBO catalogue of [12] and [8] by one entry. The one-hot scheme is shared with Lucas's QUBO for graph k-colouring. The novelty is the auxiliary signed-quotient block encoding modular conservation. The bridge obstruction (Proposition 2.11) is a classical result; we include a short proof for completeness because the same summation trick reappears in the QUBO correctness argument. In this paper, Section 2 fixes notation, introduces orientations and signed incidence, proves the bridge obstruction, and reduces the integer formulation to its modular version via Tutte's theorem. Section 3 constructs $H_{\mathrm{mod},k}$, proves the bijection between zero-energy states and nowhere-zero $\mathbb{Z}_k$-flows, derives the flow-polynomial corollary, and gives a sharp lower bound on positive energies. Section 4 reports the verification protocol, the correctness and robustness suite, and the empirical variable counts on representative graph families.

2. Preliminaries

2.1. Graphs and oriented edges

This section fixes notation and records the facts from flow theory used later. We follow Diestel [4] throughout. The objects of study are multigraphs without loops.

Definition 2.1 (Loopless multigraph)**.** A loopless multigraph $G = (V, E)$ consists of a finite vertex set V and a finite edge set E together with an incidence map assigning to each $e \in E$ an unordered pair of distinct vertices, its endpoints. Parallel edges (distinct $e, e' \in E$ with the same endpoints) are permitted; loops (edges with equal endpoints) are not.

A flow assigns a signed value to each edge, and the sign depends on which end we measure from. To make this precise, we work with oriented edges. Each undirected edge has two oriented copies.

Definition 2.2 (Oriented edge)**.** If $e = xy$ is an undirected edge, its two oriented copies are (e, x, y) and (e, y, x). We write $\vec{E}$ for the set of all oriented edges of G.

We can now define the central object: a function on oriented edges that is consistent with reversing direction and balances at every vertex.

Definition 2.3 (Circulation)**.** Let H be an abelian group written additively. A function $f : \vec{E} \to H$ is a *circulation* if it satisfies:
F1) $f(e, x, y) = -f(e, y, x)$ for every oriented edge (e, x, y);
F2) for every $v \in V$, $\sum_{(e,v,w)\in\vec{E}} f(e, v, w) = 0$.

The first condition is *antisymmetry*: reversing the orientation of an edge negates its value. The second is *Kirchhoff conservation*: the signed sum of flow values at every vertex is zero. These are conditions F1 and F2 of Diestel [4, Ch. 6].

A circulation may vanish on some edges. The nowhere-zero flows studied here do not.

Definition 2.4 (H-flow)**.** An *H-flow* is a circulation $f : \vec{E} \to H$ with $f(\vec{e}) \neq 0$ for every $\vec{e} \in \vec{E}$.

The group H can be anything, but the classical case takes $H = \mathbb{Z}$ and bounds the magnitude of f.

Definition 2.5 (k-flow)**.** For an integer $k > 1$, a *k-flow* on G is a circulation $f : \vec{E} \to \mathbb{Z}$ satisfying

$$0 < |f(\vec{e})| < k \qquad \text{for every } \vec{e} \in \vec{E}.$$

The *flow number* $\varphi(G)$ is the least k for which G has a k-flow, and $\varphi(G) = \infty$ if no such k exists.

2.2. Orientations and signed incidence

The circulation $f : \vec{E} \to H$ is defined on both oriented copies of every edge, but by (F1) its values on the two copies determine each other: knowing f on one copy determines it on the other. It is therefore wasteful to store both, and for the QUBO constructions in later sections we want one value per edge. We now make this reduction precise.

First, we need a name for a choice of one copy per edge.

Definition 2.6 (Orientation)**.** An *orientation* of G is a function $D : E \to \vec{E}$ with $D(e) \in \{(e, x, y), (e, y, x)\}$ for every edge $e = xy$.

Equivalently, an orientation is a subset of $\vec{E}$ containing exactly one of the two oriented copies of each edge.

From now on, fix an orientation D of G. For each edge $e \in E$, write $t(e)$ and $h(e)$ for the tail and head of $D(e)$; that is, $D(e) = (e, t(e), h(e))$. Storing one value per edge means working with the single triple $D(e)$ rather than both oriented copies, and we need to rewrite Kirchhoff conservation in those terms. The bookkeeping is done by the sign of how each edge meets a vertex.

Definition 2.7 (Signed incidence)**.** The *signed incidence* of edge $e \in E$ at vertex $v \in V$ is

$$\sigma_{v,e} = \begin{cases} +1, & v = t(e), \\ -1, & v = h(e), \\ 0, & \text{otherwise.} \end{cases}$$

We say e is *incident to* v if $v \in \{t(e), h(e)\}$, equivalently if $\sigma_{v,e} \neq 0$. So $\sigma_{v,e}$ is $+1$ if e leaves v, -1 if e enters v, and 0 if e is not incident to v. We can now rewrite Kirchhoff.

Proposition 2.8 (Fixed-orientation form of Kirchhoff)**.** *Let $f : \vec{E} \to H$ be a circulation and D an orientation of G. Then*

$$\sum_{e \in E} \sigma_{v,e}\, f(D(e)) \;=\; 0 \qquad \textit{for every } v \in V.$$

Proof. Fix $v \in V$. Partition

$$E = T_v \sqcup H_v \sqcup O_v,\ T_v := \{e : t(e) = v\},\ H_v := \{e : h(e) = v\},\ O_v := \{e : v \notin \{t(e), h(e)\}\},$$

so that

$$\sigma_{v,e} = \begin{cases} +1, & e \in T_v, \\ -1, & e \in H_v, \\ 0, & e \in O_v. \end{cases}$$

For each edge e incident to v, exactly one triple in $\vec{E}$ has v in the tail slot:

$$e \in T_v \implies (e, v, h(e)) = D(e), \qquad e \in H_v \implies (e, v, t(e)) = \text{reverse of } D(e).$$

Therefore

$$\begin{aligned} 0 &\overset{F2}{=} \sum_{(e,v,w) \in \vec{E}} f(e, v, w) \\ &= \sum_{e \in T_v} f(e, v, h(e)) \;+\; \sum_{e \in H_v} f(e, v, t(e)) \\ &\overset{F1}{=} \sum_{e \in T_v} f(e, v, h(e)) \;-\; \sum_{e \in H_v} f(e, t(e), v) \\ &= \sum_{e \in T_v} f(D(e)) \;-\; \sum_{e \in H_v} f(D(e)) \end{aligned}$$

$$= \sum_{e\in T_v} \sigma_{v,e}\, f(D(e)) \;+\; \sum_{e\in H_v} \sigma_{v,e}\, f(D(e)) \;+\; \sum_{e\in O_v} \sigma_{v,e}\, f(D(e))$$
$$= \sum_{e\in E} \sigma_{v,e}\, f(D(e)).$$

□

Proposition 2.8 is the form of Kirchhoff we will use throughout the rest of the paper. To keep the notation light, we write $f(e)$ for $f(D(e))$ from now on; the two-orientation form $f(e,x,y)$ will not appear again. For the fixed orientation D, we call the function $e \mapsto f(D(e))$ the *circulation f relative to D*. When this function is nowhere zero on E, we call it the *H-flow f relative to D*. Likewise, a nowhere-zero $\mathbb{Z}_k$-flow or a k-flow relative to D is a function on E obtained from a circulation on $\vec{E}$ by restricting to the chosen orientation. For nowhere-zero $\mathbb{Z}_k$-flows we always represent the nonzero residues by the set $\{1,\dots,k-1\} \subset \mathbb{Z}_k$. Equivalently, by Proposition 2.8, a circulation relative to D is a function $f : E \to H$ satisfying

$$\sum_{e\in E} \sigma_{v,e}\, f(e) = 0 \qquad \text{for every } v \in V.$$

Having fixed D, one should ask whether the choice matters. Could reversing some edges change whether G admits a nowhere-zero flow? The next lemma shows it cannot: flow existence is invariant under any subset of edge reversals. This is a standard fact, but we give the proof in detail because the bookkeeping it establishes will reappear in later arguments.

Lemma 2.9 (Orientation independence)**.** *Let G be a loopless multigraph and let D, D' be two orientations of G. Let*

$$S := \{e \in E : D'(e) \text{ is the reverse of } D(e)\}.$$

For any abelian group H and any circulation f on G relative to D, define

$$(\Phi_{D\to D'} f)(e) = \begin{cases} -f(e), & e \in S, \\ f(e), & e \notin S. \end{cases}$$

Then $\Phi_{D\to D'}$ is a bijection from the circulations of G relative to D onto the circulations of G relative to D'. It restricts to bijections on nowhere-zero H-flows and on nowhere-zero k-flows. In particular, the existence of either kind of flow is independent of the chosen orientation.

Proof. Write σ and σ' for the signed-incidence functions of D and D', respectively. If $e \notin S$, then

$$\sigma'_{u,e} = \sigma_{u,e} \qquad \text{for all } u \in V.$$

If $e \in S$, then

$$\sigma'_{u,e} = -\sigma_{u,e} \qquad \text{for all } u \in V.$$

Let f be a circulation relative to D and set $f' := \Phi_{D\to D'} f$. For every vertex $u \in V$,

$$\begin{aligned}\sum_{e\in E} \sigma'_{u,e} f'(e) &= \sum_{e\notin S} \sigma'_{u,e} f'(e) + \sum_{e\in S} \sigma'_{u,e} f'(e) \\ &= \sum_{e\notin S} \sigma_{u,e} f(e) + \sum_{e\in S} (-\sigma_{u,e})(-f(e)) \\ &= \sum_{e\in E} \sigma_{u,e} f(e) \\ &= 0,\end{aligned}$$

so f' is a circulation relative to D'.

If f is nowhere zero, then $f'(e)$ is again nowhere zero on every edge, so $\Phi_{D\to D'}$ restricts to a map on nowhere-zero H-flows. In the case $H = \mathbb{Z}_k$, used throughout the rest of the paper, a reversed value a in the representative set $\{1, \dots, k-1\}$ is written as $k - a$. For integer k-flows, the same edgewise sign change preserves the bound

$$0 < |f'(e)| = |f(e)| < k \qquad \text{for all } e \in E,$$

so $\Phi_{D\to D'}$ also maps nowhere-zero k-flows to nowhere-zero k-flows.

Finally, the set S is the same whether we pass from D to D' or from D' to D, so applying the construction twice restores the original values. Thus

$$\Phi_{D'\to D} \circ \Phi_{D\to D'} = \mathrm{id} \qquad \text{and} \qquad \Phi_{D\to D'} \circ \Phi_{D'\to D} = \mathrm{id}.$$

Hence $\Phi_{D\to D'}$ and $\Phi_{D'\to D}$ are inverse bijections. □

2.3. *The bridge obstruction*

Before building any algorithm to find nowhere-zero flows, we should ask when they exist at all. There is one local obstruction so simple it rules the problem out instantly: a *bridge*.

Definition 2.10 (Bridge)**.** A *component* of G is a maximal connected subgraph. An edge $e \in E$ is a *bridge* if deleting e increases the number of components of G.

Equivalently, a bridge is an edge whose removal splits G into two nonempty vertex sets with no other edges between them. The following proposition shows that such an edge forbids every nowhere-zero flow: no matter which group we pick, a bridge must carry value zero, which is exactly what nowhere-zero flows disallow.

Proposition 2.11 (Bridge obstruction)**.** *If G has a bridge, then G admits no nowhere-zero H-flow for any abelian group H. In particular $\varphi(G) = \infty$.*

Proof. The strategy is to sum Kirchhoff's equation over one side of the cut induced by the bridge, then show that every term cancels except the one coming from the bridge itself.

Let $e_0 = xy$ be a bridge, and write X for the vertex set of the component of $G - e_0$ containing x and $\bar{X} = V \setminus X$ for its complement. Since e_0 is a bridge, it is the only edge of G with one endpoint in X and the other in $\bar{X}$.

Let f be any circulation on G. We sum Kirchhoff's equation (Proposition 2.8) over all vertices in X. Since each individual Kirchhoff sum is zero, so is the total:

$$S := \sum_{v \in X} \sum_{e \in E} \sigma_{v,e}\, f(e) \;=\; 0.$$

We evaluate S by swapping the order of summation and grouping edges by how many endpoints they have in X:

$$S \;=\; \sum_{e \in E} f(e) \underbrace{\sum_{v \in X} \sigma_{v,e}}_{=:\, c_e}.$$

The coefficient c_e depends only on the edge:

$$c_e \;=\; \begin{cases} (+1) + (-1) \;=\; 0, & \text{if } t(e), h(e) \in X, \\ 0, & \text{if } t(e), h(e) \in \bar{X}, \\ +1, & \text{if } t(e) \in X,\; h(e) \in \bar{X}, \\ -1, & \text{if } t(e) \in \bar{X},\; h(e) \in X. \end{cases}$$

The only edge with an endpoint in each of X and $\bar{X}$ is the bridge e_0. Every other c_e is zero, so the sum collapses to a single term:

$$S \;=\; c_{e_0}\, f(e_0) \;=\; \pm f(e_0).$$

Combined with $S = 0$, this forces $f(e_0) = 0$. Thus every circulation on G vanishes on the bridge e_0, and no nowhere-zero H-flow can exist. □

A nowhere-zero flow can exist only on a bridgeless graph, so $\varphi(G) = \infty$ whenever G has a bridge. The QUBO construction in Section 3, however, is well-defined for every loopless multigraph, including disconnected and bridged graphs. On a no-instance, its ground-state energy is strictly positive; see Remark 3.9.

2.4. *The modular version*

The definition of k-flow requires searching over signed integers $\{\pm 1, \ldots, \pm(k-1)\}$ with conservation in $\mathbb{Z}$. For a QUBO encoding it is more convenient to drop signs altogether and work modulo k: a smaller search space and a cleaner match for binary variables. The reduction to modular form is classical, due to Tutte.

Write $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ for the cyclic group of integers modulo k.

Definition 2.12 (Nowhere-zero $\mathbb{Z}_k$-flow)**.** A *nowhere-zero $\mathbb{Z}_k$-flow* on G is a function $f : E \to \{1, \ldots, k-1\} \subset \mathbb{Z}_k$ satisfying

$$\sum_{e \in E} \sigma_{v,e}\, f(e) \;\equiv\; 0 \pmod{k} \qquad \text{for every } v \in V.$$

Comparing with Definition 2.5, the edge values now live in $\{1, \ldots, k-1\}$ rather than $\{\pm 1, \ldots, \pm(k-1)\}$, and Kirchhoff conservation is required only modulo k. This reduction loses nothing:

Theorem 2.13 (Tutte equivalence). *A multigraph G admits a k-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$-flow.*

Proof. See [4, Theorem 6.3.3]. □

Theorem 2.13 is the bridge from graph theory to QUBO. The integer formulation has signed variables and integer conservation; the modular formulation has nonnegative residue variables and conservation modulo k. Our QUBO constructions in Section 3 encode the modular form directly.

3. A modular QUBO for nowhere-zero k-flows

Here, we will construct a Quadratic Unconstrained Binary Optimization (QUBO) Hamiltonian $H_{\mathrm{mod},k}$ whose ground-state energy is zero if and only if G admits a nowhere-zero $\mathbb{Z}_k$-flow (Theorem 3.6); Theorem 3.6(i) identifies the zero-energy states explicitly with the flows. By Theorem 2.13, this is equivalent to $\varphi(G) \le k$. In the rest of the paper we focus on numerical evaluation.

3.1. Encoding the two constraints

A nowhere-zero $\mathbb{Z}_k$-flow is specified by two requirements (Definition 2.12), each edge carries a residue in $\{1, \ldots, k-1\}$, and the signed sum of those residues at every vertex is zero modulo k. A QUBO minimizes a quadratic polynomial in binary variables, so we need to express both requirements as vanishing quadratic penalties.

The first requirement is a one-hot condition: for each edge e, exactly one of $k-1$ indicator variables is on. This is standard in QUBO, and a squared penalty of the form $(\sum_a x_{e,a} - 1)^2$ vanishes precisely when the sum is 1.

The second requirement is modular conservation, which QUBO cannot express directly: binary variables range over integers, not residues. The standard route is to replace the congruence $R_v \equiv 0 \pmod{k}$ with the integer equality $R_v = kq_v$ for an auxiliary integer quotient q_v, then encode q_v in binary. A squared penalty $(R_v - kq_v)^2$ vanishes exactly when the congruence holds, provided q_v is allowed to take every integer value in a range wide enough to contain the true quotient.

3.2. Binary variables and decoded values

The QUBO uses two blocks of binary variables: one-hot indicators for edge values and binary encodings of per-vertex quotients. We use the orientation D of G fixed in Section 2.2, with signed incidences $\sigma_{v,e}$ as in Definition 2.7. The Hamiltonian's polynomial form depends on D through these signs; by Lemma 2.9 and the correctness theorem below, the existence of a zero-energy state will not.

Definition 3.1 (One-hot edge variables)**.** For each edge $e \in E$ and each residue $a \in \{1, \dots, k-1\}$, let

$$x_{e,a} \in \{0, 1\},$$

be a binary variable. Write

$$x \;:=\; \{\, x_{e,a} \;:\; e \in E,\; 1 \le a \le k-1 \,\},$$

for the collection of all such variables.

The variable $x_{e,a}$ records whether the flow on edge e takes value a. The one-hot penalty below enforces that exactly one such variable is on for each edge. Once the one-hot penalty is satisfied, the decoded flow value on edge e is

$$F_e(x) \;=\; \sum_{a=1}^{k-1} a\, x_{e,a}.$$

Summing F_e with signed incidence gives the per-vertex residual.

Definition 3.2 (Vertex residual)**.** For a vertex $v \in V$, the *signed residual* at v is

$$R_v(x) \;=\; \sum_{e \in E} \sigma_{v,e}\, F_e(x).$$

If the one-hot penalties are satisfied, $R_v(x)$ equals the Kirchhoff sum of the decoded flow at v. Our goal is to require $R_v(x) \equiv 0 \pmod{k}$ at every vertex.

We need a range for the auxiliary integer q_v. For any x satisfying the one-hot constraints, each decoded value $F_e(x)$ lies in $\{1, \dots, k-1\}$; writing $d(v)$, the *degree* of v, for the number of edges incident to v, we then have

$$-(k-1)\, d(v) \;\le\; R_v(x) \;\le\; (k-1)\, d(v).$$

Hence if $R_v(x)$ is divisible by k, the quotient q_v lies in the integer interval $[L_v, U_v]$, where

$$L_v := \left\lceil -\frac{(k-1)d(v)}{k} \right\rceil, \qquad U_v := \left\lfloor \frac{(k-1)d(v)}{k} \right\rfloor.$$

One conservation constraint is redundant. Summing the signed residuals over all vertices gives

$$\sum_{v \in V} R_v(x) \;=\; \sum_{v \in V} \sum_{e \in E} \sigma_{v,e}\, F_e(x) \;=\; \sum_{e \in E} F_e(x) \sum_{v \in V} \sigma_{v,e} \;=\; 0,$$

since each edge contributes $+1$ at its tail and -1 at its head, and this identity holds for every assignment x. Assume $V \neq \varnothing$ for the construction, choose an arbitrary root $r \in V$, and impose conservation only on

$$V^\star := V \setminus \{r\}.$$

Then $R_v \equiv 0 \pmod{k}$ for every $v \in V^\star$ implies $R_r \equiv 0 \pmod{k}$. If G is disconnected, one could instead omit one conservation equation in each connected component; we use the single-root convention to keep the notation uniform.

We encode each non-root quotient in binary. For each $v \in V^\star$, let

$$B_v \;:=\; \lceil \log_2(U_v - L_v + 1) \rceil,$$

so that B_v is the smallest number of bits needed to represent every integer in $[L_v, U_v]$.

Definition 3.3 (Quotient variables)**.** For each non-root vertex $v \in V^\star$, introduce binary variables $p_{v,0}, \ldots, p_{v,B_v-1} \in \{0,1\}$ and define

$$M_v(p) \;=\; L_v + \sum_{b=0}^{B_v-1} 2^b\, p_{v,b}.$$

Write

$$p \;:=\; \{\, p_{v,b} \;:\; v \in V^\star,\; 0 \le b \le B_v - 1 \,\},$$

for the collection of all such variables.

The map $(p_{v,0}, \ldots, p_{v,B_v-1}) \mapsto M_v(p)$ is a bijection onto $[L_v, L_v+2^{B_v}-1]$, which contains $[L_v, U_v]$ but may extend slightly beyond U_v when $U_v - L_v + 1$ is not a power of two.

This overshoot is harmless. If $R_v(x) = k\, M_v(p)$ for any integer M_v in the representable range, then $R_v(x) \equiv 0 \pmod{k}$.

Remark 3.4 (Isolated vertices)**.** If $d(v) = 0$, then $L_v = U_v = 0$, so

$$B_v = \lceil \log_2(U_v - L_v + 1) \rceil = \lceil \log_2 1 \rceil = 0.$$

Thus no quotient bits are introduced at v. In this case the quantity $M_v(p)$ is interpreted as the constant $L_v = 0$, and

$$R_v(x) = \sum_{e \in E} \sigma_{v,e} F_e(x) = 0,$$

because no edge is incident to v. Hence the conservation term at v vanishes identically:

$$(R_v(x) - kM_v(p))^2 = (0 - 0)^2 = 0.$$

The construction therefore extends without modification to loopless multigraphs with isolated vertices.

3.3. The Hamiltonian

We can now define the Hamiltonian.

Definition 3.5 (Modular flow Hamiltonian)**.** Fix an orientation D, a root r, and positive constants A, B. Define

$$H^{D,r,A,B}_{\mathrm{mod},k}(x,p) \;:=\; A \sum_{e \in E} \Big(\sum_{a=1}^{k-1} x_{e,a} - 1 \Big)^2 \;+\; B \sum_{v \in V^\star} (R_v(x) - k\, M_v(p))^2. \tag{1}$$

When D, r, A, and B are fixed, we suppress the superscript and write

$$H_{\mathrm{mod},k}(x,p).$$

The first sum is the one-hot penalty, one term per edge. The second is the modular conservation penalty, one term per non-root vertex. Every term is a nonnegative square scaled by a positive constant, so

$$H^{D,r,A,B}_{\mathrm{mod},k}(x,p) \geq 0,$$

and zero energy requires every term to vanish simultaneously.

We are now ready to state the main result of the paper, Theorem 3.6. It describes the zero-energy set of $H_{\mathrm{mod},k}$ completely: part (i) exhibits an explicit bijection with the nowhere-zero $\mathbb{Z}_k$-flows of G, part (ii) reads that bijection as a decision procedure for $\varphi(G) \leq k$, and part (iii) records that the emptiness and cardinality of the zero-energy set depend only on G and k, not on the encoding choices D, r, A, and B.

Theorem 3.6 (Structure of the modular QUBO)**.** *Let G be a nonempty loopless multigraph and let $k > 1$. For the chosen orientation D, root r, and positive penalty weights A, B, define*

$$\mathcal{F}^D_k(G) := \{f : E \to \{1,\dots,k-1\} : f \text{ is a nowhere-zero } \mathbb{Z}_k\text{-flow on } G \text{ relative to } D\},$$

and

$$\mathcal{Z}^{D,r,A,B}_k(G) := \{(x,p) : H^{D,r,A,B}_{\mathrm{mod},k}(x,p) = 0\}.$$

Then:

(i) *Bijection. The map*

$$\Psi_{D,r,A,B} : \mathcal{F}^D_k(G) \to \mathcal{Z}^{D,r,A,B}_k(G), \qquad f \mapsto (x,p),$$

defined by

$$x_{e,a} = \mathbf{1}[a = f(e)] \quad \text{for } e \in E,\ 1 \leq a \leq k-1,$$

and by

$$M_v(p) = \frac{R_v(x)}{k} \quad \text{for } v \in V^\star,$$

is a bijection.

(ii) *Decision equivalence. The ground-state energy of $H^{D,r,A,B}_{\mathrm{mod},k}$ is zero if and only if*

$$\mathcal{F}^D_k(G) \neq \varnothing,$$

equivalently if and only if

$$\varphi(G) \leq k.$$

(iii) *Parameter independence of emptiness and cardinality. For every admissible choice of D, r, A, and B, the corresponding zero-energy set is in bijection with $\mathcal{F}^D_k(G)$. Consequently, the emptiness and cardinality of*

$$\mathcal{Z}^{D,r,A,B}_k(G)$$

depend only on G and k.

Proof. For part (i), let $f \in \mathcal{F}_k^D(G)$. Define the edge variables by

$$x_{e,a} := \mathbf{1}[a = f(e)].$$

Then every one-hot term vanishes. Because f is a nowhere-zero $\mathbb{Z}_k$-flow, each non-root residual $R_v(x)$ is divisible by k. For each $v \in V^\star$, set

$$q_v := R_v(x)/k.$$

By the residual bound we showed above, each q_v lies in $[L_v, U_v]$. By Definition 3.3, there is a unique quotient-bit pattern p_v with $M_v(p) = q_v$. Hence

$$(x, p) \in \mathcal{Z}_k^{D,r,A,B}(G),$$

so $\Psi_{D,r,A,B}$ is well defined.

Conversely, let $(x, p) \in \mathcal{Z}_k^{D,r,A,B}(G)$. Since the one-hot block vanishes, each edge e selects a unique residue $f(e) \in \{1, \ldots, k-1\}$. Thus

$$f : E \to \{1, \ldots, k-1\}$$

is well defined and nowhere zero. Since every conservation term vanishes,

$$R_v(x) = k\, M_v(p) \qquad (v \in V^\star),$$

so

$$R_v(x) \equiv 0 \pmod{k} \qquad (v \in V^\star).$$

Since

$$\sum_{v \in V} R_v(x) = 0,$$

for every binary assignment x, and since

$$R_v(x) \equiv 0 \pmod{k} \qquad \text{for all } v \in V^\star,$$

it follows that

$$R_r(x) \equiv 0 \pmod{k}.$$

Hence $f \in \mathcal{F}_k^D(G)$.

Moreover, for each $v \in V^\star$, the integer $R_v(x)/k$ lies in $[L_v, U_v]$, and the binary map of Definition 3.3 assigns a unique bit pattern to that value. Therefore the quotient bits are uniquely determined by f, so

$$(x, p) = \Psi_{D,r,A,B}(f).$$

This proves part (i).

Part (ii) is immediate from part (i) together with Theorem 2.13.

For part (iii), fix an orientation D. By part (i), every admissible choice of root r and positive penalty weights A, B yields a zero-energy set in bijection with the same set $\mathcal{F}_k^D(G)$. It therefore remains only to compare different orientations. By Lemma 2.9, for any other orientation D', negating the value on each edge where D and D' disagree gives a bijection

$$\mathcal{F}_k^D(G) \longrightarrow \mathcal{F}_k^{D'}(G).$$

Hence the emptiness and cardinality of the flow set are independent of D, and so the emptiness and cardinality of the corresponding zero-energy sets depend only on G and k. □

Part (i) is a bijection, so it transfers any count of nowhere-zero $\mathbb{Z}_k$-flows directly to a count of zero-energy states. Applying the classical flow-polynomial theorem on the flow side gives the following.

Corollary 3.7 (Flow polynomial)**.** *For every admissible choice of D, r, A, and B,*

$$\left|\mathcal{Z}_k^{D,r,A,B}(G)\right| = F(G;k) = (-1)^{|E|-|V|+c(G)}\, T_G(0,1-k),$$

where $c(G)$ is the number of connected components of G and T_G is the Tutte polynomial of G.

Proof. By Theorem 3.6(i),

$$\left|\mathcal{Z}_k^{D,r,A,B}(G)\right| = \left|\mathcal{F}_k^D(G)\right|.$$

The classical flow-polynomial theorem identifies $\left|\mathcal{F}_k^D(G)\right|$ with $F(G;k)$; see [4, Theorem 6.3.1 and Corollary 6.3.2]. The Tutte-polynomial specialization is standard. □

Consequently, any classical lower bound on the number of nowhere-zero group flows immediately becomes a lower bound on the zero-energy degeneracy of $H_{\mathrm{mod},k}$. Theorem 3.6 and Corollary 3.7 describe the zero-energy set. We now turn to the energies above it: the next result shows that a positive energy cannot be arbitrarily small, but is bounded below by $\min(A,B)$.

Proposition 3.8 (Energy gap)**.** *For any assignment (x,p) with*

$$H_{\mathrm{mod},k}^{D,r,A,B}(x,p) > 0,$$

one has

$$H_{\mathrm{mod},k}^{D,r,A,B}(x,p) \geq \min(A,B).$$

The bound is best possible in general, but not attained on every instance.

Proof. In Eq. (1) each squared factor

$$\Big(\sum_{a=1}^{k-1} x_{e,a} - 1\Big)^2 \qquad \text{and} \qquad \big(R_v(x) - kM_v(p)\big)^2,$$

is the square of an integer, hence a nonnegative integer, and therefore at least 1 whenever it is nonzero. If $H_{\mathrm{mod},k}^{D,r,A,B}(x,p) > 0$, at least one squared factor is nonzero. If it is a one-hot factor, the corresponding summand is at least A; if it is a conservation factor, the corresponding summand is at least B. All remaining summands are nonnegative, so in either case

$$H_{\mathrm{mod},k}^{D,r,A,B}(x,p) \geq \min(A,B).$$

The bound is attained, for example, on Θ_3 at $k=2$: if $A \leq B$, choose an assignment with exactly one violated one-hot factor and all conservation factors zero; if $B \leq A$, choose an assignment with all one-hot factors zero and exactly one violated conservation factor. It is not attained on every instance; on K_3 at $k=2$ with $A=B=1$, exact enumeration gives smallest positive energy

$$2 > 1 = \min(A,B).$$

□

Remark 3.9 (Scope of Theorem 3.6)**.** The hypotheses of Theorem 3.6 require only that G be a nonempty loopless multigraph; connectedness and bridgelessness are not assumed. The identity

$$\sum_{v\in V} R_v(x) = 0,$$

holds for every orientation and every binary assignment x, because each edge contributes $(+1)+(-1)=0$ to the total. Hence omitting one root conservation equation is valid whether or not G is connected. A disconnected graph may still admit a nowhere-zero $\mathbb{Z}_k$-flow if each component does. If G has a bridge, or more generally if G is a no-instance for the chosen value of k, the Hamiltonian remains well-defined and its ground-state energy is strictly positive.

Theorem 3.6 reduces the question $\varphi(G) \leq k$ to a QUBO ground-state problem. The construction is uniform in k: each edge contributes $k-1$ one-hot variables, while each quotient block grows only logarithmically through B_v. In particular, doubling k increases each B_v by at most one bit. The next section validates Theorem 3.6 by exact computation and quantifies the size of the resulting QUBO on representative graph families.

4. Numerical experiments

We validate Theorem 3.6 numerically and characterize the Hamiltonian's size on benchmark graphs. Section 4.1 defines the verification protocol: three implementation checks (C1)–(C3) covering forward encoding, sampled non-flow labellings, and direct enumeration within budget. Section 4.2 applies the protocol to a base test set of 59 (G,k) configurations across 16 graphs, spanning simple graphs, multigraphs with parallel edges, yes-instances, and no-instances. Section 4.3 then tests the parameter-independence of the emptiness and cardinality of the zero-energy set under the three theorem parameters (A,B), r, and D through exhaustive sweeps on K_4, $K_{3,3}$, Θ_3, and the Petersen graph.

Section 4.4 reports variable counts, coupler counts, density, offset, and coefficient magnitudes on named benchmarks, on 30 random cubic and 33 random $G(n,m)$ samples, and on a dedicated snark family; see Table 7. Section 4.5 states the scope of these checks. Finally, Section 4.6 separates theorem-level invariance from heuristic-solver behaviour, which can depend on (A, B) and on the instance. No quantum hardware was tested; a reference implementation is available at https://github.com/alilotfi90/nzflow-qubo.

4.1. Verification protocol

We implemented Theorem 3.6 as a Python 3 package built on the D-Wave Ocean SDK, with deterministic random seeds throughout. The Hamiltonian is exposed as a `dimod.BinaryQuadraticModel`, which makes the construction directly compatible with D-Wave's exact, simulated-annealing, and hardware backends. We verified that the `neal` simulated-annealing sampler recovers zero-energy flows on K_4 at $k = 4$ and $K_{3,3}$ at $k = 3$ with default hyperparameters. On the Petersen graph at $k = 5$, `neal` does not consistently locate zero-energy states at default hyperparameters; sampler performance on that instance is discussed separately in Section 4.6.

The code builds $H_{\text{mod},k}$ from the explicit formula in Eq. (1) and encodes a given $\mathbb{Z}_k$-flow as the binary assignment (x, p) predicted by the proof of Theorem 3.6. It also enumerates nowhere-zero $\mathbb{Z}_k$-flows by two independent methods. The first is a spanning-tree extension following [4, Ch. 6, Exer. 6]: iterate over the k^β assignments of residues in $\mathbb{Z}_k$ to the non-tree edges and propagate Kirchhoff conservation to the tree. Here $\beta := |E|-|V|+c(G)$ is the cycle rank of G. The second is a direct enumerator that scans every labelling $f : E \to \{1, \dots, k-1\}$ and tests modular conservation at every non-root vertex.

For each (G, k) configuration the script performs three checks: **(C1)** it encodes every nowhere-zero $\mathbb{Z}_k$-flow on G via the proof of Theorem 3.6 and verifies $H_{\text{mod},k} = 0$; **(C2)** it samples random non-flow labellings and verifies the minimum energy over the quotient bits is positive; and (C3) when $(k-1)^{|E|} \leq 2\times 10^7$, it directly enumerates all edge labellings and verifies that the number of feasible labellings equals the number of nowhere-zero $\mathbb{Z}_k$-flows reported by the spanning-tree verifier. By Theorem 3.6, these feasible labellings correspond bijectively to zero-energy states. Any failure of these checks would indicate a mismatch between the implementation and Theorem 3.6 and would require investigation.

4.2. Theorem-correctness results

The principal test set comprises 59 (G, k) configurations across 16 graphs (complete and complete bipartite graphs, cycles, the cube Q_3, the Petersen graph, and several multigraph families with parallel edges), spanning 36 simple-graph and 23 multigraph configurations, and including both yes-instances and no-instances. For each graph we tested $k \in \{2, \dots, k_{\max}\}$ where $k_{\max} \leq 6$ is constrained by the (C3) budget.

Outcomes. All 59 pairs passed all applicable checks. C1 passed on the 49 configurations with at least one enumerated flow and was vacuous on the 10 no-flow configurations. C2 passed on all configurations with sampled non-flow labellings. Among the 58 pairs for which C3 was within budget, the direct enumeration count matched the spanning-tree count in every case. The single C3 skip is the Petersen graph at $k = 5$, where $4^{15} \approx 10^9$ exceeds the budget; all 240 nowhere-zero $\mathbb{Z}_5$-flows still encoded to $H_{\text{mod},k} = 0$. The largest direct edge-labelling enumeration completed was Q_3 at $k = 5$, examining $4^{12} \approx 1.7 \times 10^7$ labellings and finding exactly 156 feasible labellings, matching the verifier's 156 nowhere-zero $\mathbb{Z}_5$-flows. By Theorem 3.6(i), these correspond to 156 zero-energy states. Table 1 reports representative cases.

Table 1. Representative exact flow counts and threshold instances. The flow counts are exact and use the cycle-space extension enumerator. Column C1 is marked for yes-instances where every exact flow encodes to zero energy under $H_{\text{mod},k}$. Column *C3 count* records the number of feasible edge labellings $f : E \to \{1, \ldots, k-1\}$ found by direct enumeration when within the budget $(k-1)^{|E|} \le 2 \times 10^7$; by Theorem 3.6, this equals the zero-energy-state count because the quotient bits are uniquely determined. A dash (–) in the C3 count column indicates that direct enumeration was outside budget; a dash in C1 indicates that there were no enumerated flows to check.

Graph	k	$\lvert E\rvert$	β	#flows	C1	vars	C3 count
K_3	2	3	1	1	yes	7	1
K_4	3	6	3	0	–	21	0
K_4	4	6	3	6	yes	27	6
$K_{3,3}$	3	9	4	2	yes	33	2
Θ_3	3	3	2	2	yes	9	2
Triangular prism	3	9	4	0	–	33	0
Petersen	3	15	6	0	–	57	0
Petersen	4	15	6	0	–	72	0
Petersen	5	15	6	240	yes	87	–
K_4 doubled	3	12	9	176	yes	36	176
Q_3	5	12	5	156	yes	69	156

4.3. *Robustness*

Theorem 3.6(i) gives a bijection between nowhere-zero $\mathbb{Z}_k$-flows and zero-energy states, and Theorem 3.6(iii) shows that the emptiness and cardinality of the zero-energy set are independent of the choice of orientation, root, and positive penalty weights. We test those three encoding choices here. On K_4 at $k = 4$, six pairs of penalty weights spanning two orders of magnitude in each direction ($(A, B) \in \{(1, 1), (10, 1), (1, 10), (0.1, 1), (1, 0.1), (5, 0.5)\}$) all produced exactly 6 zero-energy states matching the 6 nowhere-zero $\mathbb{Z}_4$-flows of K_4. On $K_{3,3}$ at $k = 3$, the suite was rerun with each of the 6 vertices in turn as the chosen root, with identical outcomes.

We exhaustively swept all $2^6 = 64$ orientations of K_4 at $k = 4$, all $2^9 = 512$ orientations of $K_{3,3}$ at $k = 3$, and all $2^3 = 8$ orientations of Θ_3 at $k = 3$; in every one of these 584 orientations the spanning-tree verifier returned the predicted flow count exactly, confirming the orientation-independence guaranteed by Lemma 2.9. All 10 root choices on Petersen at $k = 5$ produced 240 nowhere-zero $\mathbb{Z}_5$-flows encoding to $H_{\text{mod},k} = 0$. Together these sweeps test the three parameter choices appearing in the theorem at the level reported in Table 2, weight and root sweeps compare zero-energy counts for the constructed BQM, while orientation sweeps compare the induced flow counts. No deviation was found.

Table 2. Exact robustness sweeps. Orientation sweeps preserve the exact nowhere-zero flow count. Root and weight sweeps preserve the exact zero-energy count of the constructed BQM, consistent with the theorem-level invariance of $H_{\mathrm{mod},k}$.

Sweep	Graph	k	cases	invariant count
orientations	K_4	4	64	6 (yes)
orientations	$K_{3,3}$	3	512	2 (yes)
orientations	Θ_3	3	8	2 (yes)
roots	$K_{3,3}$	3	6	2 (yes)
roots	Petersen	5	10	240 (yes)
weights	K_4	4	6	6 (yes)

4.4. Empirical size and structure of the BQM

This subsection records the size of $H_{\mathrm{mod},k}$ on the benchmark suite. We give the variable-count formula, expand the coupler bound, then read off measured values for named and random graph families. The total binary-variable count for $H_{\mathrm{mod},k}$ is

$$N_k(G) \;=\; (k-1)|E| + \sum_{v \in V^\star} B_v, \qquad B_v = \lceil \log_2(U_v - L_v + 1) \rceil. \tag{2}$$

The first term grows linearly in $|E|$ and proportionally to $k-1$. The second term depends on the degree distribution of G in a way that is not captured by $|E|$ and $|V|$ alone. For irregular graphs $N_k(G)$ depends on the choice of root r, since $V^\star = V \setminus \{r\}$ excludes a different vertex for each choice and B_v depends on $d(v)$. Unless stated otherwise, all reported totals use r as the highest-indexed vertex; on the irregular $G(n,m)$ samples below, changing r alters the total by at most two bits.

Coupler counts, offset, and density. Variable count alone does not describe the size of a QUBO. Expanding Eq. (1), the one-hot block contributes at most

$$|E|\binom{k-1}{2},$$

quadratic couplers, since each edge contributes a clique on its $k-1$ residue variables. At a non-root vertex v, the conservation block couples the $(k-1)d(v)$ incident edge-residue variables among themselves, couples those edge-residue variables to the B_v quotient bits, and couples the quotient bits among themselves. Hence the number of nonzero quadratic couplers is bounded by

$$\#\text{couplers} \;\le\; |E|\binom{k-1}{2} + \sum_{v \in V^\star}\left[\binom{(k-1)d(v)}{2} + (k-1)d(v)\,B_v + \binom{B_v}{2}\right], \tag{3}$$

with repeated couplers merging after expansion. The constant offset of the binary quadratic model is

$$A|E| + Bk^2 \sum_{v \in V^\star} L_v^2. \tag{4}$$

Here the term $A|E|$ comes from the constant $+1$ in each $\left(\sum_a x_{e,a} - 1\right)^2$, and the term $Bk^2 \sum_{v \in V^\star} L_v^2$ comes from the constant part of $\left(R_v(x) - kM_v(p)\right)^2$. We report the measured variable count, quadratic-coupler count, density, and offset for representative benchmarks in Table 3 at $A = B = 1$.

Table 3. Structural statistics of the binary quadratic models $H_{\mathrm{mod},k}$ at $A = B = 1$. Density is the fraction of present quadratic couplers among all possible couplers on the logical graph. The column max$|Q|$ reports the largest absolute final BQM coefficient after merging identical variable pairs across the one-hot and conservation blocks; the dynamic range divides this by the smallest nonzero quadratic coefficient.

Graph	k	vars	couplers	density	offset	max$\|Q\|$	dyn. range
Θ_3	3	9	36	1.000	39	144	72.0:1
K_4	4	27	189	0.538	198	256	128.0:1
$K_{3,3}$	3	33	174	0.330	189	144	72.0:1
Triangular prism	3	33	174	0.330	189	144	72.0:1
K_4 doubled	3	36	342	0.543	444	576	288.0:1
Petersen	4	72	558	0.218	591	256	128.0:1
Petersen	5	87	873	0.233	915	400	200.0:1
Q_3	5	69	681	0.290	712	400	200.0:1

Coefficient magnitudes. The one-hot block contributes quadratic coefficients of magnitude

$$2A,$$

between distinct residue variables on the same edge. Expanding the conservation block

$$B(R_v(x) - kM_v(p))^2,$$

produces three further families of quadratic coefficients. For distinct one-hot variables at the same vertex, the coefficient magnitude is at most

$$2B\,aa' \le 2B(k-1)^2.$$

For a one-hot variable and a quotient bit, the magnitude is at most

$$2Bk\,a\,2^b \le 2Bk(k-1)2^{B_v-1}.$$

For two distinct quotient bits, the magnitude is at most

$$2Bk^2\,2^{b+c} \le 2Bk^2\,2^{2(B_v-1)}.$$

These three bounds are contributions from a single conservation block. The final BQM coefficient on a pair is the sum of contributions from every block in which the pair appears; in particular, a same-edge residue pair $(x_{e,a}, x_{e,a'})$ receives the one-hot contribution $2A$ together with both endpoints' conservation contributions when those endpoints are non-root, giving a final coefficient up to $2A + 4B(k-1)^2$. In the benchmark set the maximum $|Q|$ is dominated by the two-quotient-bit contribution at a single vertex, which is unaffected by such merging. Thus the conservation-block coefficient magnitudes grow quadratically in k, exponentially in B_v, and linearly in B. The measured values reported below are for $A = B = 1$. In this benchmark set, the largest measured coefficient occurs on the densest tested graph, K_4-doubled at $k = 3$, where $\max|Q|= 576$ and the smallest nonzero quadratic coefficient is 2, giving a dynamic range of 288:1. The Petersen graph at $k = 5$ is next, with $\max|Q|= 400$ and a dynamic range of 200:1. For fixed-precision annealing hardware the BQM would have to be normalized before solving, and alternative quotient encodings remain a reasonable direction for future work.

Table 4. Exact flow numbers and variable counts for named benchmark graphs.

Graph	n	m	β	φ	vars (k=4)	vars (k=5)
K_3	3	3	1	2	13	16
K_4	4	6	3	4	27	33
K_5	5	10	6	2	42	52
$K_{3,3}$	6	9	4	3	42	51
$K_{3,4}$	7	12	6	3	54	66
C_5	5	5	1	2	23	28
C_6	6	6	1	2	28	34
Q_3	8	12	5	3	57	69
Petersen	10	15	6	5	72	87
GP(7, 2)	14	21	8	4	102	123
GP(8, 3)	16	24	9	3	117	141

The named-benchmark family (Table 4) ranges from K_3 ($n = 3, |E|= 3$) to the generalised Petersen graph GP(8, 3) ($n = 16, |E|= 24$). Variable counts at $k = 5$ range from 16 to 141, with the Petersen graph itself contributing 87.

On random connected bridgeless cubic graphs (Table 5), 30 samples spanning $n \in \{6, \ldots, 16\}$, every vertex has $d(v) = 3$, so $L_v = -2$, $U_v = 2$, $B_v = 3$ at both $k = 4$ and $k = 5$. The variable count reduces to

$$N_4(G) = 3|E|+3(|V|-1), \qquad N_5(G) = 4|E|+3(|V|-1).$$

Flow numbers in the sample fall in $\{3, 4\}$. The threshold no-instances in the present revision include Petersen at $k = 3$ and $k = 4$, together with the triangular prism at $k = 3$.

Snarks, the bridgeless cubic graphs that are not 3-edge-colorable, are no-instances for $k \leq 4$: for cubic graphs, 3-edge-colorability is equivalent to admitting a nowhere-zero 4-flow, and a nowhere-zero 3-flow would induce a 4-flow. They are exercised directly in Table 7. The suite comprises the Petersen graph and the Isaacs flower snarks J_5 (20 vertices) and J_7 (28 vertices), the latter members of an infinite parametric family; `flower_snark(5)` is verified isomorphic to SageMath's canonical `graphs.FlowerSnark()`. These graphs are confirmed to be no-instances at $k = 3$ and $k = 4$ by a combination of enumeration and structural reasoning: Petersen and J_5 are checked by exact enumeration, J_7 at $k = 3$ is checked by exact enumeration, and J_7 at $k = 4$ follows from non-3-edge-colorability. The flower snark J_5 admits 16,200 nowhere-zero $\mathbb{Z}_5$-flows, every one of which encodes to $H_{\text{mod},k} = 0$ (check C1). A complete finite snark catalogue is not required for the correctness claim, and a systematic snark hardness survey remains future work.

Table 5. Random connected bridgeless cubic graphs, summarized by vertex count. Five samples were drawn at each n. Variable counts are constant within each row because every cubic vertex contributes the same number of auxiliary bits. No snarks occurred in this random cubic family; threshold no-instances such as Petersen at $k = 3, 4$ and the triangular prism at $k = 3$ are listed separately in Table 1.

n	$\|E\|$	β	samples	φ values	vars ($k = 4$)	vars ($k = 5$)
6	9	4	5	$\varphi = 3$ ($\times 3$), $\varphi = 4$ ($\times 2$)	42	51
8	12	5	5	$\varphi = 3$ ($\times 1$), $\varphi = 4$ ($\times 4$)	57	69
10	15	6	5	$\varphi = 3$ ($\times 2$), $\varphi = 4$ ($\times 3$)	72	87
12	18	7	5	$\varphi = 4$ ($\times 5$)	87	105
14	21	8	5	$\varphi = 4$ ($\times 5$)	102	123
16	24	9	5	$\varphi = 4$ ($\times 5$)	117	141

On 33 random $G(n, m)$ samples at controlled cycle-rank (Table 6), the variable count varies within each (n, m) pair. For instance, the 5 samples at $(n, m) = (10, 14)$ gave variable counts ranging from 64 to 66 at $k = 4$. This illustrates that the auxiliary block depends on the degree distribution, not on (n, m) alone. Across the 33 samples, 26 have $\varphi = 3$ and the remaining 7 have $\varphi = 4$. Across both random families and the named benchmarks, the measured variable counts agree with Eq. (2); variation within a fixed (n, m) pair comes from the degree distribution through the values B_v.

Table 6. Random $G(n, m)$ samples, summarized by (n, m). Up to five samples were drawn at each pair and then filtered for connectivity and bridgelessness; some pairs retain fewer than five graphs after filtering. Variable-count ranges document that the auxiliary block depends on degree distribution, not just (n, m).

n	$\|E\|$	β	samples	φ values	vars ($k = 4$)	vars ($k = 5$)
10	14	5	5	$\varphi = 3$ ($\times 4$), $\varphi = 4$ ($\times 1$)	64–66	79–80
10	16	7	5	$\varphi = 3$ ($\times 5$)	71–73	88–90
10	18	9	5	$\varphi = 3$ ($\times 4$), $\varphi = 4$ ($\times 1$)	79–81	99–100
15	19	5	3	$\varphi = 3$ ($\times 3$)	89–91	109–110
15	21	7	5	$\varphi = 3$ ($\times 5$)	98–99	119–121
15	23	9	5	$\varphi = 3$ ($\times 5$)	104–106	129–130
20	28	9	5	$\varphi = 4$ ($\times 5$)	133–134	161–163

4.5. Scope and limitations

The theorem is proved analytically. The verification protocol provides exact implementation checks for (C1) and, within the enumeration budget, for (C3), while (C2) is a randomized non-flow check; several questions remain outside its scope. We mark them here. In the base 59-configuration theorem-correctness table, the C3 budget of 2×10^7 edge labellings was exceeded only by Petersen at $k = 5$, where C1 and C2 still passed. The snark table uses a separate budget of 5×10^6 cycle-space assignments; the corresponding out-of-budget rows are marked by dashes in Table 7. No quantum hardware was tested: all energies reported are computed exactly from Eq. (1). The behaviour of $H_{\mathrm{mod},k}$ under quantum or classical annealing, including embedding overhead on D-Wave Pegasus or Zephyr topologies and chain-strength selection, is left for future work. The random-graph samples reach cycle-rank $\beta = 9$ and vertex count $n = 20$. These are exact correctness checks and formulation-size measurements, not scaling claims. A reference implementation is available at https://github.com/alilotfi90/nzflow-qubo.

Table 7. Snark benchmarks. The Petersen graph and the Isaacs flower snarks J_5 (20 vertices) and J_7 (28 vertices) are connected, bridgeless, cubic, girth at least 5, and not 3-edge-colorable; `flower_snark(5)` is verified isomorphic to SageMath's canonical `graphs.FlowerSnark()`. Since for cubic graphs 3-edge-colorability is equivalent to admitting a nowhere-zero 4-flow, and a nowhere-zero 3-flow would imply a nowhere-zero 4-flow, the $k \in \{3, 4\}$ snark rows are no-instances. The $k = 5$ rows probe the near-threshold case: Petersen and J_5 are confirmed yes-instances by the enumerated counts in the table, while the $J_7, k = 5$ enumeration is outside the stated budget. The column #flows is the exact cycle-space enumerator count when the enumeration is within budget; a dash (–) in the #flows column marks configurations whose enumeration $(k-1)^\beta$ exceeds the budget 5×10^6. The C1 column is marked yes when enumerated flows were checked to encode to $H_{\mathrm{mod},k} = 0$, and is marked – when C1 is vacuous or outside budget. The structural columns are exact for every row. For $J_7, k = 4$, the no-instance conclusion follows from non-3-edge-colorability rather than from enumeration.

Graph	k	n	$\lvert E\rvert$	β	#flows	C1	vars	couplers
Petersen	3	10	15	6	0	–	57	312
Petersen	4	10	15	6	0	–	72	558
Petersen	5	10	15	6	240	yes	87	873
Flower snark J_5	3	20	30	11	0	–	117	657
Flower snark J_5	4	20	30	11	0	–	147	1173
Flower snark J_5	5	20	30	11	16200	yes	177	1833
Flower snark J_7	3	28	42	15	0	–	165	933
Flower snark J_7	4	28	42	15	–	–	207	1665
Flower snark J_7	5	28	42	15	–	–	249	2601

4.6. Theorem-level versus heuristic-level parameter sensitivity

The parameter-independence in Theorem 3.6(iii) is a statement about the exact zero-energy set, not about the behaviour of a particular heuristic solver. Our exact penalty-weight sweep on K_4 at $k = 4$ confirms the theorem-level statement: the six tested pairs

$$(A, B) \in \{(1, 1), (10, 1), (1, 10), (0.1, 1), (1, 0.1), (5, 0.5)\},$$

all produce the same ground-state manifold of six zero-energy states.

Heuristic samplers behave differently. Their success probability and time-to-solution can depend strongly on the ratio A/B and on the instance itself. In our `neal` runs, K_4 at $k = 4$ and $K_{3,3}$ at $k = 3$ reached zero energy readily across all tested weight pairs. On the Petersen graph at $k = 5$, by contrast, `neal` stalled at strictly positive energies across the tested grid

$$(A, B) \in \{1, 3, 5, 10, 50\} \times \{1, 3, 10, 50\},$$

with up to 5000 reads and 50,000 sweeps per run across all tested random seeds. This behaviour does not contradict Theorem 3.6; it reflects heuristic search difficulty on a hard near-threshold instance. Determining the best-performing weight ratios, annealing schedules, and hardware embeddings is a separate solver-study question and is left for future work.

5. Conclusion

In this work, we construct a QUBO Hamiltonian $H_{\text{mod},k}$ representing the problem of finding nowhere-zero $\mathbb{Z}_k$-flows on nonempty loopless multigraphs. We establish a bijection between the zero-energy states of $H_{\text{mod},k}$ and flows on G (Theorem 3.6(i)), identify the number of zero-energy states with the flow polynomial $F(G;k)$ (Corollary 3.7), and show a lower bound of $\min(A,B)$ on the positive energies of $H_{\text{mod},k}$ (Proposition 3.8). We verify $H_{\text{mod},k}$ on a test set of 59 instances of (G,k), covering yes-instances and no-instances, simple graphs, and multigraphs with parallel edges, and we separately test orientation, root, and penalty-weight choices on the robustness instances described in Section 4.3. We additionally benchmark the construction on snark families, namely the Petersen graph and the Isaacs flower snarks J_5 and J_7, confirming the expected no-instance behaviour for $k \leq 4$ (Table 7).

This result should be read as a reduction, not as a statement about the performance of solvers. A zero-energy state of any solver whatsoever, regardless of the type of solver, indicates that $\varphi(G) \leq k$. If a solver returns only positive energies, this is not evidence that $\varphi(G) > k$; establishing nonexistence requires the exact verifier described in Section 4.1 or an exhaustive ground-state enumeration of $H_{\text{mod},k}$. Thus, within these stated limits, the Hamiltonian is independent of any particular solver, and the analyses performed here are entirely analytical and independent of any particular sampler.

Three follow-up directions are immediate. First, one can deploy the algorithm on an annealer to evaluate the success probability and time-to-solution of $H_{\text{mod},k}$ on hard cubic instances, as compared to simulated annealing on the same Hamiltonian. Second, it would be of interest to study alternative encodings of $H_{\text{mod},k}$, such as a signed-integer Hamiltonian $H_{\text{int},k}$, or a specialised Hamiltonian H_4 for the case $k=4$. Third, the Hamiltonian $H_{\text{mod},k}$ may be of interest to study in relation to the work of [5] on flow-reconfiguration; its samples of states with zero energy are candidates for the endpoints of a reconfiguration sequence within $\mathcal{F}(G,k)$. We leave these interesting directions for future work.

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