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An extensive study on restricted and extended symmetric difference operations of soft sets

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ABSTRACT

Soft set theory, first introduced by Molodtsov, is a flexible approach for handling uncertainty-related problems and modeling uncertain information. Since soft set operations form the core of the theory, their algebraic properties and related structures have attracted considerable research interest. Several forms of symmetric difference operations have been proposed, including restricted and extended symmetric difference operations. Although restricted symmetric difference has already been defined, its definition is not fully consistent with the general formula of restricted soft set operations. In this paper, we first provide an alternative definition of restricted symmetric difference that follows the general form of restricted soft set operations. We then investigate its algebraic properties together with the extended symmetric difference operation, both for soft sets with a fixed parameter set and for soft sets over U. We also establish new properties analogous to the symmetric difference operation in classical set theory, including the case where parameter sets may be disjoint. By deriving distributive rules, we show that important algebraic structures arise when restricted or extended symmetric difference operations are combined with other soft set operations. This study fills a gap in the literature, guides readers new to the theory, and presents a comprehensive analysis of restricted and extended symmetric difference operations, including corrected theorems and proofs from earlier studies.

Keywords: soft sets, restricted symmetric difference, extended symmetric difference, decision-making, uncertainty modeling

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1. Introduction

We are unable to successfully solve difficulties in many disciplines, including engineering, environmental and health sciences, and economics, since certain forms of uncertainty exist. In 1999, Molodtsov [33] presented Soft Set ($\mathcal{S}$) Theory as a mathematical way to cope with these concerns. Since then, this theory has been employed in a range of domains, including information systems, decision-making [12, 66, 21, 67], optimization theory, game theory, operations research, measurement theory, and various algebraic structures. Moreover, the semantic of N-soft sets, which are a multinary version of the parameterized descriptions that define a soft set was presented in [2] together with their applications, and a coda about three-way decision. Cheng et al [12], Maji et al. [32] as well as Pei and Miao [37] made the first contributions to soft set operations. Following that, Ali et al. [4] presented and discussed many soft set operations, including restricted and extended soft set operations. In [50], Sezgin and Atagün studied the fundamental features of soft set operations and their relationships. They also explored and established the concept of restricted symmetric difference in soft sets. Sezgin et al. [49] proposed a novel soft set operation termed "extended difference of soft sets," while Stojanovic [62] presented the notion "extended symmetric difference of soft sets" and explored its properties. The operations of soft set theory are classified into two categories: restricted soft set operations and extended soft set operations. Eren [13] presented a novel sort of soft difference operation, which we refer to as the "soft binary piecewise difference operation," and extensively examined its essential properties. Yavuz [68] introduced more soft binary piecewise operations and thoroughly examined their key features. Since 2003, there has been a great deal of research done on soft set operations since they are the core ideas of soft set theory. To learn more about soft set operations, see [56, 53, 52, 54, 55, 59, 26, 5, 35, 31, 14, 61, 51, 60, 58, 69, 25, 36, 23]. Vandiver [63] gave the first description of semirings in 1934. A range of hypotheses and conclusions on semirings have been presented by many academics, including [64, 28], and some researchers have investigated semirings with additive inverse [27, 18, 38, 39].

More recently, semirings have been the subject of much research, especially with regard to applications (see [16]). Semirings are highly important in geometry, but they are also important in pure mathematics and are necessary for addressing problems in many applications of practical mathematics and information sciences [30, 22, 11, 10, 15, 65, 17, 20, 34]. A specific semiring with a zero and commutative addition is referred to as a hemiring. Hemirings play a critical role in theoretical computer science as well. Hemirings naturally arise in a number of applications to automata, computer sciences, and formal language theory [20, 34]. When studying soft sets as algebraic structures, we see that there are two main types of soft set collections. The first one is the collection of soft sets with a fixed parameter set, and the second one is the soft sets with a different set of parameters. These two collection types with soft set operations may sometimes behave the same, or they may behave differently. When operations are defined on the soft set, it is imperative to examine what kinds of algebraic structures are involved in these operations. For more about the analysis of algebraic operations on soft sets, we refer to [1, 41], and for the updated, authoritative presentation and a comprehensive survey of soft set theory, encompassing its foundational concepts, developments, and applications we refer to [3].

In soft set literature, there are four types of symmetric difference operations: the first one is the restricted symmetric difference of soft sets defined first in [50], the second one is the extended symmetric difference of soft sets defined first in [62], the third and fourth one is the (complementary) soft binary piecewise symmetric difference operations. In [50], the basic properties of the restricted symmetric difference operation and in [62], the basic properties of the extended symmetric difference operation were analyzed, and Eren [13] investigated more on the extended symmetric difference operation, especially with the help of the soft binary piecewise difference operation. In [13], it was shown that the set of all the soft sets over U with a fixed parameter together with restricted intersection and extended symmetric difference is a Boolean ring. However, in the aforementioned papers, a full analysis of restricted symmetric difference operation and extended symmetric difference operation was not handled. Moreover, we explore the properties by taking into account the specific case where the soft sets' parameter sets may be disjoint. In this respect, this paper can be regarded as a following study of [50, 62, 13], where we explore more about restricted and extended symmetric difference soft set operations and complete all the missing parts regarding these soft set operations studied in [50, 62, 13].

In this study, we examine the overall algebraic properties of the restricted and extended symmetric difference operations, including closure, associativity, unit, inverse element, and Abelian property, not only in the collections of soft sets with a fixed parameter set but also in the collections of soft sets over U. In particular, we consider the properties of these soft set operations in comparison with the basic properties of the symmetric difference operation existing in classical set theory and obtain many interesting analogies. We also examine the distribution rules to determine the relationships between these soft set operations and the restricted intersection operation to reveal which algebraic structures they form. And we prove that the set of all the soft sets over U, together with the restricted intersection operation and extended symmetric difference operation, is a commutative hemiring with identity. Also, the definition of restricted symmetric difference defined in [50] is not in the general formula of restricted soft set operations (for the general formula, see Definition 2.5).

In this paper, we give an alternative definition for this definition by staying faithful to the general formula of restricted soft set operations. For working with uncertain objects, soft sets, and soft operations are useful parametric tools. To address problems involving parametric data, it is necessary to enhance the algebraic properties of the previously initiated soft set operations. This work offers a comprehensive analysis of restricted and extended symmetric difference operations in this respect. In addition, since studying the algebraic structure of soft sets from the standpoint of soft set operations provides a comprehensive understanding of their application as well as an appreciation of how soft set algebra can be applied to classical and nonclassical logic, this paper hopes to contribute to the soft set literature from this standpoint.

The structure of this document is as follows: We review the foundational ideas of soft set theory in Section 2, along with semirings and hemirings. Section 3 presents novel and alternative definitions for the restricted symmetric difference operation. It also discusses the properties of this soft set operation in comparison to the classical set theory's symmetric difference operation. We argue that the set of all the soft sets over U, along with the restricted symmetric difference operation and the restricted intersection, constitute a commutative hemiring with identity. Section 4 serves as a reminder of the extended symmetric difference operation. Furthermore, by comparing the characteristics of extended symmetric difference operation to the symmetric difference operation in set theory, we are able to draw some insightful similarities. Additionally, it is demonstrated that the restricted intersection, the extended symmetric difference operation, and the set of all the soft sets over U form a commutative hemiring with identity. We focused on the significance of the study's results and their possible impact on the field in the conclusion section.

2. Preliminaries

Definition 2.1. [2] Let U be the universal set, E be the parameter set, and $P(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a set-valued function such that $F\colon A \to P(U)$.

Throughout this paper, the collection of all the soft sets defined over U is designated by $S_E(U)$. Let A be a fixed subset of E, and let $S_A(U)$ be the collection of all those soft sets over U with the fixed parameter set A. That is, while in the set $S_A(U)$, there are only soft sets whose parameter sets are A; in the set $S_E(U)$, there are soft sets whose parameter sets may be any set. $S_A(U)$ is a subset of $S_E(U)$. From now on, while soft set will be designated by $\mathcal{S}$ and parameter set by $\mathcal{P}$; soft sets will be designated by $\mathcal{S}$s and parameter sets by $\mathcal{P}$s for the sake of ease.

Definition 2.2. [4] (F, A) is called a relative null $\mathcal{S}$ (with respect to A), denoted by, $\varnothing_A$, if $F(o) = \varnothing$ for all $o \in A$, and (F, A) is called a relative whole $\mathcal{S}$ (with respect to A), denoted by U_A if $F(o) = U$ for all $o \in A$. The relative whole $\mathcal{S}$ U_E with respect to E is called the absolute $\mathcal{S}$ over U. We shall denote by $\varnothing_\varnothing$ the unique $\mathcal{S}$ over U with an empty $\mathcal{P}$, which is called the empty $\mathcal{S}$ over U.

Note that $\varnothing_\varnothing$ and by $\varnothing_A$ are different $\mathcal{S}$s over U [5].

Definition 2.3. [37] For two $\mathcal{S}$s (F, A) and (T, B), (F, A) is a soft subset of (T, B) and it is denoted by $(F, A)\widetilde{\subseteq}(T, B)$, if $A \subseteq B$ and $F(o) \subseteq T(o)$, for all $o \in A$. Two $\mathcal{S}$s (F, A) and (T, B) are said to be soft equal if (F, A) is a soft subset of (T, B) and (T, B) is a soft subset of (F, A).

Definition 2.4. [4] The relative complement of an $\mathcal{S}$ (F, A), denoted by $(F, A)^r$, is defined by, $(F, A)^r = (F^r, A)$, where $F^r\colon A \to P(U)$ is a mapping given by, $(F, A)^r = U \setminus F(o)$ for all $o \in A$.

From now on, $U \setminus F(o) = [F(o)]'$ will be designated by $F'(o)$ for the sake of ease. Let "$\star$"

be used to denote the set operations, then the following type of $\mathcal{S}$ operations are defined as follows:

Definition 2.5. [4, 50] Let (F, A) and (T, B) be $\mathcal{S}$s over U. The restricted $\star$ operation of (F, A) and (T, B) is the $\mathcal{S}$ (B, K), denoted by, $(F, A) \star_R (T, B) = (B, K)$, where $K = A \cap B \neq \varnothing$ and for all $o \in K$, $B(o) = F(o) \star T(o)$. Here, note that if $A \cap B = \varnothing$, then $(F, A) \star_R (T, B) = \varnothing_\varnothing$.

Here, “R” refers to the term “restricted”.

Definition 2.6. [32, 4, 49, 62] Let (F, A) and (T, B) be $\mathcal{S}$s over U. The extended $\star$ operation of (F, A) and (T, B) is the $\mathcal{S}$ (B, K), denoted by, $(F, A) \star_\varepsilon (T, B) = (B, K)$, where $K = A \cup B$ and for all $o \in K$,

$$B(o) = \begin{cases} F(o), & o \in A \setminus B, \\ T(o), & o \in B \setminus A, \\ F(o) \star T(o), & o \in A \cap B. \end{cases}$$

Here, “ε” refers to the term “extended”.

Definition 2.7. [13, 68, 59] Let (F, A) and (T, B) be $\mathcal{S}$s over U. The soft binary piecewise $\star$ operation of (F, A) and (T, B) is the $\mathcal{S}$, (B, A), denoted by, $(F, A)\widetilde{\star}(T, B) = (B, A)$, where for all $o \in A$,

$$B(o) = \begin{cases} F(o), & o \in A \setminus B, \\ F(o) \star T(o), & o \in A \cap B. \end{cases}$$

Definition 2.8. [56, 53, 52, 54, 55] Let (F, A) and (T, B) be $\mathcal{S}$s over U. The complementary soft binary piecewise $\star$ operation of (F, A) and (T, B) is the $\mathcal{S}$, (B, A), denoted by, $(F, A)\underset{\star}{\overset{*}{\sim}}(T, B) = (B, A)$, where for all $o \in A$,

$$B(o) = \begin{cases} F'(o), & o \in A \setminus B, \\ F(o) \star T(o), & o \in A \cap B. \end{cases}$$

A semiring is a more general algebraic structure than a ring in mathematics, and it is utilized in abstract algebra. Semirings $(R, +, \cdot)$ are algebraic structures made composed of a non-empty set R and two binary operations (addition and multiplication, respectively), such that multiplication is distributive over addition from both sides and $(R, +)$ is a semigroup. It is termed a semiring with identity if it has identity with multiplication, and it is considered a commutative semiring if it has commutative multiplication. An element 0 in R is referred to as the zero of R if it exists such that $0 \cdot a = a \cdot 0 = 0$ and $0 + a = a + 0 = a$ for every a in R. The term "hemiring" refers to a semiring with commutative addition and zero element. Please see [63, 64, 28, 27, 18, 38, 39, 16, 30, 22, 11, 10, 15, 65, 17, 20, 34] for further information regarding semirings and hemirings. We consult [29] with respect to the possible uses of graph applications and network analysis on soft sets and for more about algebraic structure of soft sets, we refer to [48, 43, 44, 24, 42, 47, 57, 19, 7, 6, 46, 40, 8, 45, 9].

3. On restricted symmetric difference

The set of elements that are in either of the sets but not their intersection is known as the symmetric difference of two sets. In [50], with the inspiration of this definition, the restricted symmetric difference of $\mathcal{S}$s was defined as follows: Let (F, A) and (G, B) be $\mathcal{S}$s over U. Then, the restricted symmetric difference of (F, A) and (G, B) is defined as follows: $[(F, A) \cup_R (G, B)] \backslash_R [(F,A) \cap_R (G, B)]$. However, this definition is not in the general formula of restricted $\mathcal{S}$ operations (for the general formula, see Definition 2.5). In this section, we give an alternative definition for restricted symmetric difference by staying faithful to the general formula of restricted $\mathcal{S}$ operations. We also explore the full algebraic properties of this operation, not only in the collection of $\mathcal{S}$s with a fixed $\mathcal{P}$ but also in the collection of $\mathcal{S}$s over U.

Definition 3.1. Let (F, A) and (G, B) be $\mathcal{S}$s over U. The restricted symmetric difference operation of (F, A) and (G, B), denoted by, $(F, A)\Delta_R(G, B)$, is defined as $(F, A)\Delta_R(G, B) = (H, K)$, where $K = A \cap B$, and if $K = A \cap B \neq \varnothing$, then for all $e \in K$, $H(e) = F(e)\Delta G(e)$ if $K = A \cap B = \varnothing$, then $(F, A)\Delta_R(G, B) = \varnothing_\varnothing$.

Thus, we can conclude that for deriving the restricted symmetric difference of two $\mathcal{S}$s, it is not a necessary condition that the intersection of their parameter sets is not an empty set. In [50], Sezgin and Atagün used "$\widetilde{\Delta}$" for restricted symmetric difference; however, we prefer to use "Δ_R" for restricted symmetric difference as the general designation of restricted $\mathcal{S}$ operations, where "R" refers to "restricted".

Example 3.2. Let $E = \{e_1, e_2, e_3, e_4\}$ be the $\mathcal{P}$, $A = \{e_1, e_3\}$ and $B = \{e_2, e_3, e_4\}$ be the subsets of E and $U = \{h_1, h_2, h_3, h_4, h_5\}$ be the universe set. Let (F, A) and (G, B) be $\mathcal{S}$s over U, defined as follows: $(F, A) = \{(e_1, \{h_2, h_5\}), (e_3, \{h_1, h_2, h_5\})\}$ $(G, B) = \{(e_2, \{h_1, h_4, h_5\}), (e_3, \{h_2, h_3, h_4\}), (e_4, \{h_3, h_5\})\}$. Let $(F, A)\Delta_R(G, B) = (H, A \cap B)$, where $H(e) = F(e)\Delta G(e)$ for all $e \in A \cap B$. Since $A \cap B = \{e_3\}$, so $H(e_3) = F(e_3)\Delta G(e_3) = \{h_1, h_3, h_4, h_5\}$. Thus, $(F, A)\Delta_R(G, B) = \{(e_3, \{h_1, h_3, h_4, h_5\})\}$.

It is well-known that $A\Delta B = (A \cup B) \setminus (A \cap B)$. Now, we have the following:

Theorem 3.3. *Let (F, A) and (G, B) be $\mathcal{S}$s over U. Then,* $(F, A)\Delta_R(G, B) = [(F, A) \cup_R (G, B)] \setminus_R [(F, A) \cap_R (G, B)]$.

Proof. Since the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A \cap B$, soft equality's initial requirement is met. Now let $(F, A) \cup_R (G, B) = (H, A \cap B)$, where for all $e \in A \cap B$; $H(e) = F(e) \cup G(e)$. Let $(F, A) \cap_R (G, B) = (M, A \cap B)$, where for all $e \in A \cap B$; $M(e) = F(e) \cap G(e)$. Let

$(H, A\cap B)\setminus_R (M, A\cap B) = (S, A\cap B)$, where for all $e\in A\cap B$; $S(e) = H(e)\setminus M(e)$. Thus, $S(e) = [F(e)\cup G(e)]\setminus[F(e)\cap G(e)] = F(e)\Delta G(e)$, for all $e\in A\cap B$. Hence, $(S, A\cap B) = (F,A)\Delta_R(G,B)$. Here, it is obvious that if $A\cap B=\varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Theorem 3.3 shows us that the restricted symmetric difference defined in [50] coincides with the definition of the restricted symmetric difference of $\mathcal{S}$s defined in this paper. Moreover, we have the following:

Theorem 3.4. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_R(G,B) = [(F,A)\cup_\varepsilon (G,B)]\setminus_R[(F,A)\cap_R(G,B)]$.*

Proof. Since $(A\cup B)\cap(A\cap B) = A\cap B$, the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A\cap B$. And thus, soft equality's initial requirement is met. Now let $(F,A)\cup_\varepsilon(G,B) = (H, A\cup B)$, where for all $e\in A\cup B$,

$$H(e) = \begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\cup G(e), & e\in A\cap B.\end{cases}$$

Let $(F,A)\cap_R(G,B) = (M, A\cap B)$, where for all $e\in A\cap B$, $M(e) = F(e)\cap G(e)$. Let $(H, A\cup B)\setminus_R(M, A\cap B) = (S, A\cap B)$, where for all $e\in A\cap B$, $S(e) = H(e)\setminus M(e)$. Thus,

$$S(e) = \begin{cases} F(e)\setminus(F(e)\cap G(e)), & e\in (A\setminus B)\cap(A\cap B)=\emptyset,\\ G(e)\setminus(F(e)\cap G(e)), & e\in (B\setminus A)\cap(A\cap B)=\emptyset,\\ (F(e)\cup G(e))\setminus(F(e)\cap G(e)), & e\in (A\cap B)\cap(A\cap B) = A\cap B.\end{cases}$$

Thereby, $S(e) = [F(e)\cup G(e)]\setminus[F(e)\cap G(e)] = F(e)\Delta G(e)$ for all $e\in A\cap B$. Hence, $(S, A\cap B) = (F,A)\Delta_R(G,B)$. Here, it is obvious that if $A\cap B=\varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Theorem 3.5. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_R(G,B) = [(F,A)\setminus_R (G,B)]\cup_R[(G,B)\setminus_R(F,A)]$* [50].

Theorem 3.6. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then,*

$$(F,A)\Delta_R(G,B) = [(F,A)\widetilde{\setminus}(G,B)]\cup_R[(G,B)\widetilde{\setminus}(F,A)].$$

Proof. Since the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A\cap B$, soft equality's initial requirement is met. Now let $(F,A)\widetilde{\setminus}(G,B) = (H,A)$, where for all $e\in A$,

$$H(e) = \begin{cases} F(e), & e\in A\setminus B,\\ F(e)\setminus G(e), & e\in A\cap B.\end{cases}$$

Let $(G,B)\widetilde{\setminus}(F,A)=(K,B)$, where for all $e\in B$,

$$K(e)=\begin{cases}G(e), & e\in B\setminus A,\\ G(e)\setminus F(e), & e\in B\cap A.\end{cases}$$

Let $(H,A)\cup_R(K,B)=(S,A\cap B)$, where for all $e\in A\cap B$, $S(e)=H(e)\cup K(e)$. Hence,

$$S(e)=\begin{cases}F(e)\cup G(e), & e\in (A\setminus B)\cap(B\setminus A)=\emptyset,\\ F(e)\cup(G(e)\setminus F(e)), & e\in (A\setminus B)\cap(B\cap A)=\emptyset,\\ (F(e)\setminus G(e))\cup G(e) & e\in (A\cap B)\cap(B\setminus A)=\emptyset,\\ (F(e)\setminus G(e))\cup(G(e)\setminus F(e)) & e\in (A\cap B)\cap(B\cap A)=A\cap B.\end{cases}$$

Hence, $S(e)=(F(e)\setminus G(e))\cup(G(e)\setminus F(e))=F(e)\Delta G(e)$ for all $e\in A\cap B$, and hence $(S,A\cap B)=(F,A)\Delta_R(G,B)$. Here, it is obvious that if $A\cap B=\varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Theorem 3.7. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_R(G,B)=[(F,A)\setminus_\varepsilon (G,B)]\cup_R[(G,B)\setminus_R(F,A)]$.*

Proof. Since $(A\cup B)\cap(A\cap B)=A\cap B$, the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A\cap B$. Thus, soft equality's initial requirement is met. Now let $(F,A)\setminus_\varepsilon(G,B)=(H,A\cup B)$, where for all $e\in A\cup B$,

$$H(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\setminus G(e), & e\in A\cap B.\end{cases}$$

Let $(G,B)\setminus_R(F,A)=(M,A\cap B)$, where for all $e\in A\cap B$, $M(e)=G(e)\setminus F(e)$. Let $(H,A\cup B)\cup_R(M,A\cap B)=(S,A\cap B)$, where for all $e\in A\cap B$, $S(e)=H(e)\cup M(e)$. Thus,

$$S(e)=\begin{cases}F(e)\cup(G(e)\setminus F(e)), & e\in (A\setminus B)\cap(A\cap B)=\emptyset,\\ G(e)\cup(G(e)\setminus F(e)), & e\in (B\setminus A)\cap(A\cap B)=\emptyset,\\ (F(e)\setminus G(e))\cup(G(e)\setminus F(e)), & e\in (A\cap B)\cap(A\cap B)=A\cap B.\end{cases}$$

Thus, $S(e)=(F(e)\setminus G(e))\cup(G(e)\setminus F(e))=F(e)\Delta G(e)$ for all $e\in A\cap B$. Hence, $(S,A\cap B)=(F,A)\Delta_R(G,B)$. Here, it is obvious that if $A\cap B=\varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.8. *The sets $S_E(U)$ and $S_A(U)$ are closed under the operation Δ_R.*

Proof. It is clear that Δ_R is a well-defined binary operation in $S_E(U)$ and $S_A(U)$, where A is a fixed $\mathcal{P}$ of U. □

Proposition 3.9. *Let (F,A), (G,B), and (N,C) be $\mathcal{S}$s over U. Then,*

$$[(F,A)\Delta_R(G,B)]\Delta_R(H,C) = (F,A)\Delta_R[(G,B)\Delta_R(H,C)].$$

Proof. Let $(F,A)\Delta_R(G,B) = (T, A\cap B)$, where $T(e) = F(e)\Delta G(e)$ for all $e \in A\cap B$. Let $(T, A\cap B)\Delta_R(H,C) = (M, (A\cap B)\cap C)$, where $M(e) = T(e)\Delta H(e)$ for all $e \in (A\cap B)\cap C$. Thus, $M(e) = [F(e)\Delta G(e)]\Delta H(e)$ for all $e \in (A\cap B)\cap C$.

Let $(G,B)\Delta_R(H,C) = (K, B\cap C)$, where $K(e) = G(e)\Delta H(e)$, for all $e \in B\cap C$. Let $(F,A)\ \Delta_R(K, B\cap C) = (N, A\cap (B\cap C))$, $N(e) = F(e)\ \Delta K(e)$, for all $e \in A\cap (B\cap C)$. Thus, $N(e) = F(e)\Delta[G(e)\Delta H(e)]$ for all $e \in A\cap(B\cap C)$. It is seen that $(T, (A\cap B)\cap C) = (N, A\cap(B\cap C))$. Namely, Δ_R is associative in $S_E(U)$. Here, it is obvious that if $A\cap B = \varnothing$ or $B\cap C = \varnothing$ or $A\cap C = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.10. [50] *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then,*

$$(F,A)\Delta_R(G,B) = (G,B)\Delta_R(F,A).$$

Proof. Let $(F,A)\Delta_R(G,B) = (H, A\cap B)$. Then, for all $e \in A\cap B$, $H(e) = F(e)\Delta G(e)$. Let $(G,B)\Delta_R(F,A) = (T, B\cap A)$. Then, for all $e \in B\cap A$, $T(e) = G(e)\Delta F(e)$. Since for all $e \in B\cap A$, $F(e)\Delta G(e) = G(e)\Delta F(e)$, $(F,A)\Delta_R(G,B) = (G,B)\Delta_R(F,A)$. That is to say, Δ_R has commutative property in the set $S_E(U)$.

Here, it is obvious that if $A\cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.11. *Let (F,A) be an $\mathcal{S}$ over U. Then, $(F,A)\Delta_R\varnothing_A = \varnothing_A\Delta_R(F,A) = (F,A)$.*

Proof. Let $\varnothing_A = (S,A)$. Then, for all $e \in A$, $S(e) = \varnothing$. Let $(F,A)\Delta_R(S,A) = (H,A)$, where for all $e \in A$, $H(e) = F(e)\Delta S(e)$. Hence, for all $e \in A$, $H(e) = F(e)\Delta S(e) = F(e)\Delta\varnothing = F(e)$. Thus, $(H,A) = (F,A)$ and $\varnothing_A$ is the identity element for Δ_R in $S_A(U)$. □

Proposition 3.12. *Let (F,A) be an $\mathcal{S}$ over U. Then, $(F,A)\Delta_R(F,A) = \varnothing_A$.*

Proof. Let $(F,A)\Delta_R(F,A) = (H,A)$, where $H(e) = F(e)\Delta F(e)$ for all $e \in A$. Here for all $e \in A$, $H(e) = F(e)\Delta F(e) = \varnothing$, thus $(H,A) = \varnothing_A$. This feature shows us that every $\mathcal{S}$ is its own inverse for Δ_R in $S_A(U)$ and also Δ_R is not idempotent on $S_E(U)$. □

Proposition 3.13. *Let (F,A) be an $\mathcal{S}$ over U. Then, $(F,A)\Delta_R\varnothing_E = \varnothing_E\Delta_R(F,A) = (F,A)$.*

Proof. Let $\varnothing_E = (S, E)$. Hence, for all $e \in E$, $S(e) = \varnothing$. Let $(F, A)\Delta_R(S, E) = (H, A \cap E) = (H, A)$. Thus, for all $e \in A$, $H(e) = F(e)\Delta S(e)$ for all $e \in A$. Hence, for all $e \in A$, $H(e) = F(e)\Delta S(e) = F(e)\Delta\varnothing = F(e)$, so $(H, A) = (F, A)$. Thus, $\varnothing_E$ is the identity element for Δ_R in $S_E(U)$.

Here note that we cannot observe an element (G, B) in $S_E(U)$ such that $(F, A)\Delta_R(G, B) = (G, B)\Delta_R(F, A) = \varnothing_E$, as this requires $A \cap B = E$, and thus this requires $A = E$ and $B = E$. Here we can also conclude that in the set in $S_E(U)$, only the identity element $\varnothing_E$ has an inverse (as usual, its inverse is its own), all the other elements do not have an inverse element. □

Remark 3.14. $(S_A(U), \Delta_R)$ is an abelian group with identity $\varnothing_A$.

Proof. By Proposition 3.8, Proposition 3.9, Proposition 3.10, Proposition 3.11, and Proposition 3.12, the set of all $\mathcal{S}$s with a fixed $\mathcal{P}$ is an abelian group together with the operation Δ_R. Let the fixed $\mathcal{P}$ be A. Then, $(S_A(U), \Delta_R)$ is an abelian group with identity $\varnothing_A$. □

Remark 3.15. $(S_E(U), \Delta_R)$ is a commutative monoid with identity $\varnothing_E$.

Proof. By Proposition 3.8, Proposition 3.9, Proposition 3.10, and Proposition 3.13, $(S_E(U), \Delta_R)$ is a commutative monoid with identity $\varnothing_E$. □

Proposition 3.16. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_R\varnothing_\varnothing = \varnothing_\varnothing\Delta_R(F, A) = \varnothing_\varnothing$.*

Proof. Let $\varnothing_\varnothing = (S, \varnothing)$ and $(F, A)\Delta_R(S, \varnothing) = (H, A \cap \varnothing) = (H, \varnothing)$. Since, $\varnothing_\varnothing$ is the unique $\mathcal{S}$ with empty $\mathcal{P}$, $(H, \varnothing) = \varnothing_\varnothing$. Note that, $\varnothing_\varnothing$ is the absorbing element for Δ_R in $S_E(U)$. □

Proposition 3.17. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_R U_A = U_A\Delta_R(F, A) = (F, A)^r$.*

Proof. Let $U_A = (T, A)$. Then, for all $e \in A$, $T(e) = U$. Let $(F, A)\Delta_R(T, A) = (H, A)$, where $H(e) = F(e)\Delta T(e)$, for all $e \in A$. Thus, $H(e) = F(e)\ \Delta T(e) = F(e)\Delta U = F'(e)$ for all $e \in A$, hence $(H, A) = (F, A)^r$. □

Proposition 3.18. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_R U_E = U_E\Delta_R(F, A) = (F, A)^r$.*

Proof. Let $U_E = (T, E)$. Hence, for all $e \in E$, $T(e) = U$. Let $(F, A)\Delta_R(T, E) = (H, A \cap E) = (H, A)$, where $H(e) = F(e)\ \Delta T(e)$ for all $e \in A$. Hence, for all $e \in A$, $H(e) = F(e)\ \Delta T(e) = F(e)\Delta U = F'(e)$, so $(H, A) = (F, A)^r$. □

Proposition 3.19. *Let (F,A) be an $\mathcal{S}$ over U. Then, $(F,A)\Delta_R(F,A)^r = (F,A)^r\Delta_R(F,A) = U_A$.*

Proof. Let $(F,A)^r = (H,A)$. Hence, for all $e \in A$, $H(e) = F'(e)$. Let $(F,A)\Delta_R(H,A) = (T,A)$, where $T(e) = F(e)\Delta H(e)$ for all $e \in A$. Hence, $T(e) = F(e)\,\Delta H(e) = F(e)\Delta F'(e) = U$ for all $e \in A$. Thus, $(T,A) = U_A$. □

Proposition 3.20. *Let (F,A), (G,B), (H,A) and (H,B) be $\mathcal{S}$s over U.*

$$[(F,A)\Delta_R(G,B)]\Delta_R[(G,B)\Delta_R(H,A)] = (F,A)\Delta_R(H,B).$$

Proof. Since the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A \cap B$, soft equality's initial requirement is met. Now let $(F,A)\Delta_R(G,B) = (H, A\cap B)$, where $H(e) = F(e)\Delta G(e)$ for all $e \in A \cap B$. Let $(G,B)\Delta_R(H,A) = (K, B\cap A)$, where $K(e) = G(e)\Delta H(e)$ for all $e \in B\cap A$ and let $(H, A\cap B)\ \Delta_R(K, B\cap A) = (S, A\cap B)$, where $S(e) = H(e)\Delta K(e)$ for all $e \in A\cap B$. Hence, $S(e) = [F(e)\Delta G(e)]\Delta[G(e)\Delta H(e)] = F(e)\Delta H(e)$ for all $e \in A\cap B$. Thus, $(S, A\cap B) = (F,A)\Delta_R(H,B)$.

Here, it is obvious that if $A \cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.21. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)^r\Delta_R(G,B)^r = (F,A)\Delta_R(G,B)$.*

Proof. Let $(F,A)^r\Delta_R(G,B)^r = (H, A\cap B)$. Then, $H(e) = F'(e)\Delta G'(e)$ for all $e \in A\cap B$. Since $H(e) = F'(e)\ \Delta G'(e) = F(e)\Delta G(e)$ for all $e \in A\cap B$, $(H, A\cap B) = (F,A)\Delta_R(G,B)$. Here, it is obvious that if $A\cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.22. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $\varnothing_{A\cap B}\widetilde{\subseteq}(F,A)\Delta_R(G,B)$, Also, $(F,A)\Delta_R(G,B)\widetilde{\subseteq}U_A$, $(F,A)\Delta_R(G,B)\widetilde{\subseteq}U_B$ and $(F,A)\Delta_R(G,B)\widetilde{\subseteq}U_{A\cap B}$, where $A\cap B \neq \varnothing$.*

Proposition 3.23. *Let (F,A), (G,A) and (H,A) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_R(G,A) = (F,A)\Delta_R(H,A) \Longrightarrow (G,A) = (H,A)$.*

Proof. Let $(F,A)\Delta_R(G,A) = (H,A)$. Then, $H(e) = F(e)\Delta G(e)$ for all $e \in A$. Let, $(F,A)\Delta_R(H,A) = (T,A)$, where $T(e) = F(e)\Delta H(e)$ for all $e \in A$. Since $(H,A) = (T,A)$, then for all $e \in A$, $F(e)\Delta G(e) = F(e)\Delta H(e)$. Thus, $G(e) = H(e)$, for all $e \in A$. Hence, $(G,A) = (H,A)$. □

Here note that $(F,A)\Delta_R(G,K) = (F,A)\Delta_R(H,B)$ does not imply that $(G,K) = (H,B)$, as the elements do not have an inverse element in the algebraic structure $(S_E(U),\Delta_R)$ as shown in Proposition 3.13. In fact, when we subject (F,A) with the operation Δ_R to both sides of equality from the left side, it yields the following: $[(F,A)\,\Delta_R\,(F,A)]\Delta_R(G,K) = [(F,A)\,\Delta_R(F,A)]\Delta_R(H,B) \Longrightarrow \varnothing_A\Delta_R(G,K) = \varnothing_A\Delta_R(H,B)$. Here, since $\varnothing_A$ is not the identity element, $\varnothing_A\Delta_R(G,K) = \varnothing_A\Delta_R(H,B)$ does not imply that $(G,K) = (H,B)$. However, it is easy to see that $(F,A)\Delta_R(G,B) = (F,A)\Delta_R(H,B) \Longrightarrow (G,A\cap B) = (H,A\cap B)$.

Proposition 3.24. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_R(G,B)\widetilde{\subseteq}(F,A)\cup_R (G,B)$.*

Proof. Let $(F,A)\Delta_R(G,B) = (H,A\cap B)$, where $H(e) = F(e)\Delta G(e)$ for all $e \in A\cap B$. Now let $(F,A)\cup_R(G,B) = (T,A\cap B)$, where $T(e) = F(e)\cup G(e)$ for all $e \in A\cap B$. Since $F(e)\Delta G(e) \subseteq F(e)\cup G(e)$ for all $e \in A\cap B$, thus $H(e) \subseteq T(e)$. Hence, $(H,A\cap B)\widetilde{\subseteq}(T,A\cap B)$.

Here, it is obvious that if $A\cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.25. *Let (F,A) and (G,A) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_R(G,A) = \varnothing_A \Leftrightarrow (F,A) = (G,A)$.*

Proof. *Necessity.* Let $(F,A)\Delta_R(G,A) = (T,A)$. Hence, $T(e) = F(e)\Delta G(e)$ for all $e \in A$. Since $(T,A) = \varnothing_A$, $T(e) = \varnothing$ for all $e \in A$. Thus, $F(e)\Delta G(e) = \varnothing$ for all $e \in A$. Hence, $F(e) = G(e)$ for all $e \in A$. So, $(F,A) = (G,A)$.

Sufficiency. Let $(F,A) = (G,A)$. Then, $(F,A)\Delta_R(G,A) = \varnothing_A$. □

Proposition 3.26. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_R(G,B) = (F,A)\cup_R (G,B) \Leftrightarrow (F,A)\cap_R(G,B) = \varnothing_{A\cap B}$.*

Proof. Let $(F,A)\Delta_R(G,B) = (H,A\cap B)$ and $(F,A)\cup_R(G,B) = (T,A\cap B)$. Then, $H(e) = F(e)\Delta G(e)$ for all $e \in A\cap B$ and $T(e) = F(e)\cup G(e)$ for all $e \in A\cap B$. Since $(H,A\cap B) = (T,A\cap B)$, then $F(e)\Delta G(e) = F(e)\cup G(e)$ for all $e \in A\cap B$. Thus, $F(e)\cap G(e) = \varnothing$ for all $e \in A\cap B$. Hence, $(F,A)\cap_r(G,B) = \varnothing_{A\cap B}$.

Now let, $(F,A)\cap_R(G,B) = \varnothing_{A\cap B}$. By Theorem 3.1. $(F,A)\Delta_R(G,B) = [(F,A)\cup_R (G,B)]\setminus_R((F,A)\cap_R(G,B))$. Thus, $[(F,A)\cup_R(G,B)]\setminus_R\varnothing_{A\cap B} = (F,A)\cup_R(G,B)$. Hence, $(F,A)\Delta_R(G,B) = (F,A)\cup_R(G,B)$. □

Proposition 3.27. *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,*

$$(F,A)\widetilde{\subseteq}(G,B) \Longrightarrow (F,A)\Delta_R(G,B) = (G,B)\setminus_R (F,A).$$

Proof. Let $(F,A)\widetilde{\subseteq}(G,B)$. Then, $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Let $(F,A)\Delta_R(G,B) = (H, A\cap B) = (H,A)$. Then, $H(e) = F(e)\Delta G(e)$ for all $e \in A$. Let $(G,B)\setminus_R (F,A) = (T, B\cap A) = (T,A)$. Then, $T(e) = G(e)\setminus F(e)$ for all $e \in A$. Since $F(e) \subseteq G(e)$ for all $e \in A$, $H(e) = F(e)\ \Delta G(e) = G(e)\setminus F(e)$. Therefore, $(H,A) = (T,A)$.

Here, it is obvious that if $A \cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.28. *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,* $(F,A)\Delta_R[(F,A)\cap_R (G,B)] = (F,A)\setminus_R (G,B)$, *where* $A\cap B \neq \varnothing$.

Proof. Let $(F,A)\cap_R (G,B) = (H, A\cap B)$. Then, $H(e) = F(e)\cap G(e)$ for all $e \in A\cap B$. Let, $(F,A)\Delta_R(H, A\cap B) = (T, A\cap B)$, where $T(e) = F(e)\Delta H(e)$ for all $e \in A\cap B$. Hence, $T(e) = F(e)\Delta[F(e)\cap G(e)] = F(e)\setminus G(e)$ for all $e \in A\cap B$. Thus, $(T, A\cap B) = (F,A)\setminus_R (G,B)$. □

Proposition 3.29. *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,* $(F,A)\cup_R (G,B) = [(F,A)\Delta_R(G,B)]\cup_R [(F,A)\cap_R (G,B)]$.

Proof. First let's consider the right side. Let $(F,A)\cup_R(G,B) = (H, A\cap B)$. Then, $H(e) = F(e)\cup G(e)$ for all $e \in A\cap B$. Now let's consider the left side. Let $(F,A)\Delta_R(G,B) = (K, A\cap B)$. Then, $K(e) = F(e)\Delta G(e)$ for all $e \in A\cap B$. Let, $(F,A)\cap_R(G,B) = (T, A\cap B)$, where $T(e) = F(e)\cap G(e)$ for all $e \in A\cap B$. Now, let $(K, A\cap B)\cup_R(T, A\cap B) = (S, A\cap B)$, where $S(e) = K(e)\cup T(e)$ for all $e \in A\cap B$. Thus, $S(e) = [F(e)\Delta G(e)]\cup[F(e)\cap G(e)] = F(e)\cup G(e)$ for all $e \in A\cap B$. Therefore, $(H, A\cap B) = (S, A\cap B)$. This completes the proof.

Here, it is obvious that if $A \cap B = \varnothing$, then both sides of the equality is $\varnothing_\varnothing$, implying that the equality is satisfied in all circumstances. □

Proposition 3.30. *Let* (F,A), (G,B), (H,C) *be* $\mathcal{S}s$ *over* U. *Then,*

$$(F,A)\cap_R [(G,B)\Delta_R(H,C)] = [(F,A)\cap_R (G,B)]\Delta_R[(F,A)\cap_R (H,C)]$$

Proof. First evaluate the left side. Let $(G,B)\Delta_R(H,C) = (M, B\cap C)$, where $M(e) = G(e)\Delta H(e)$ for all $e \in B\cap C$. Assume that $(F,A)\cap_r (M, B\cap C) = (N, A\cap(B\cap C))$, where $N(e) = F(e)\cap M(e)$ for all $e \in A\cap(B\cap C)$. Hence, $N(e) = F(e)\cap[G(e)\Delta H(e)]$ for all $e \in A\cap(B\cap C)$.

Now evaluate the right side. Let $(F,A)\cap_R(G,B)=(K,A\cap B)$, where $K(e)=F(e)\cap G(e)$ for all $e\in A\cap B$. Let $(F,A)\cap_R(H,C)=(T,A\cap C)$, where $T(e)=F(e)\cap H(e)$ for all $e\in A\cap C$. Thus, $(K,A\cap B)\Delta_R(T,A\cap C)=(P,(A\cap B)\cap C)$, where $P(e)=K(e)\Delta T(e)$ for all $e\in(A\cap B)\cap C$. Thus, $P(e)=[F(e)\cap G(e)]\Delta[F(e)\cap H(e)]$ for all $e\in A\cap(B\cap C)$. Hence, $N(e)=P(e)$ for all $e\in(A\cap B)\cap C$ implying that $(N,(A\cap B)\cap C)=(P,(A\cap B)\cap C)$. □

Proposition 3.31. *Let (F,A), (G,B), (H,C) be $\mathcal{S}s$ over U. Then, $[(F,A)\Delta_R(G,B)]\cap_R(H,C)=[(F,A)\cap_R(H,C)]\Delta_R[(G,B)\cap_r(H,C)]$.*

Proof. First evaluate the left side. Let $(F,A)\Delta_R(G,B)=(M,A\cap B)$, where $M(e)=F(e)\Delta G(e)$ for all $e\in A\cap B$. Now, let $(M,A\cap B)\cap_R(H,C)=(W,(A\cap B)\cap C)$, where $W(e)=M(e)\cap H(e)$ for all $e\in(A\cap B)\cap C$. Thus, $W(e)=[F(e)\Delta G(e)]\cap H(e)$ for all $e\in(A\cap B)\cap C$.

Now evaluate the right side. Let $(F,A)\cap_R(H,C)=(K,A\cap C)$, where $K(e)=F(e)\cap H(e)$ for all $e\in A\cap C$. Let $(G,B)\cap_R(H,C)=(T,B\cap C)$, where $T(e)=G(e)\cap H(e)$ for all $e\in B\cap C$. Thus, $(K,A\cap C)\Delta_R(T,B\cap C)=(R,(A\cap C)\cap(B\cap C))=(R,A\cap B\cap C)$, where $R(e)=K(e)\Delta T(e)=[F(e)\cap H(e)]\Delta[G(e)\cap H(e)]$ for all $e\in A\cap B\cap C$. Hence, $W(e)=R(e)$ for all $e\in(A\cap B)\cap C$, implying that $(W,A\cap B\cap C)=(R,A\cap B\cap C)$.

Here, note that to satisfy Proposition 3.30 and Proposition 3.31, we do not need to have $B\cap C\neq\varnothing$ or $A\cap B\neq\varnothing$ or $A\cap C\neq\varnothing$. Anyway, if $B\cap C=\varnothing$ or $A\cap B=\varnothing$ or $A\cap C=\varnothing$, then both sides are equal to $\varnothing_\varnothing$, and thus the equality is satisfied in every circumstances. □

Remark 3.32. $(S_A(U),\Delta_R,\cap_R)$ is a commutative hemiring with identity U_A.

Proof. In Remark 3.14, we show that $(S_A(U),\Delta_R)$ is an abelian group with identity $\varnothing_A$, hence $(S_A(U),\Delta_R)$ is a semigroup. Moreover, in [37, 50, 5], it was proved that $(S_A(U),\cap_R)$ is a commutative monoid with identity U_A, hence $(S_A(U),\cap_R)$ is a semigroup. Moreover, by Proposition 3.31, $\cap_R$ distributes over Δ_R from both sides. Therefore, $(S_A(U),\Delta_R,\cap_R)$ is a semiring. Further, by Proposition 3.30 and Proposition 3.10, $(F,A)\Delta_R(G,A)=(G,A)\Delta_R(F,A)$. That is to say, Δ_R is commutative in $S_A(U)$ and $(F,A)\Delta_R\varnothing_A=\varnothing_A\Delta_R(F,A)=(F,A)$ by Proposition 3.11 and from [37, 50, 5], $(F,A)\cap_R\varnothing_A=\varnothing_A\cap_R(F,A)=\varnothing_A$. That is to say, $\varnothing_A$ is the zero element of $(S_A(U),\Delta_R,\cap_R)$. Therefore, $(S_A(U),\Delta_R,\cap_R)$ is a hemiring. Besides, since $(F,A)\cap_R(G,A)=(G,A)\cap_R(F,A)$ and $(F,A)\cap_R U_A=U_A\cap_R(F,A)=(F,A)$ (see [37, 50, 5]), $(S_A(U),\Delta_R,\cap_R)$ is a commutative hemiring with identity U_A. □

Remark 3.33. $(S_A(U),\Delta_R,\cap_R)$ is a Boolean ring.

Proof. $(S_A(U),\Delta_R)$ is an abelian group by Remark 3.14, $(S_A(U),\cap_R)$ is a semigroup by [37, 50, 5] and $\cap_R$ distributes over Δ_R from both sides by Proposition 3.30 and Proposition 3.31. Thus, we can also deduce that $(S_A(U), \Delta_R, \cap_R)$ is a ring. Also, since $(F,A) \cap_R (G,A) = (G,A) \cap_R (F,A)$ and $(F,A) \cap_R U_A = U_A \cap_R (F,A) = (F,A)$, (see [37, 50, 5]), $(S_A(U), \Delta_R, \cap_R)$ is a commutative ring with identity U_A. Moreover, $(F,A)^2 = (F,A) \cap_R (F,A) = (F,A)$ for all $(F,A) \in S_A(U)$. Thus, $(S_A(U), \Delta_R, \cap_R)$ is a Boolean ring and $(F,A)\Delta_R(F,A) = \varnothing_A$ and $(F,A) \cap_R (G,A) = (G,A) \cap_R (F,A)$ provides naturally as a result of being a Boolean ring. □

4. On extended symmetric difference

Definition 4.1. [62] Let (F,A) and (G,B) be $\mathcal{S}$s over U. The extended symmetric difference operation of (F,A) and (G,B) is the $\mathcal{S}$ (H,K), denoted by, $(F,A)\Delta_\varepsilon(G,B) = (H,K)$, where $K = A \cup B$ and for all $e \in A \cup B$,

$$H(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e)\Delta G(e), & e \in A \cap B. \end{cases}$$

In the designation of extended symmetric difference operation, "ε" refers to the term "extended".

Note 4.2. Let (F,A) and (G,B) be $\mathcal{S}$s over U such that $A = B$. Then, it is easy to see that the extended symmetric difference operation turns out to be the restricted symmetric difference operation. Thus, in the collection $S_A(U)$, the extended symmetric difference operation coincides with the restricted symmetric difference operation.

Example 4.3. Let (F,A) and (G,B) be the $\mathcal{S}$s in Example 3.2 and let $(F,A)\Delta_\varepsilon(G,B) = (H, A \cup B)$. Then, since $A \cup B = \{e_1, e_2, e_3, e_4\}$ and $A \setminus B = \{e_1\}$, so $H(e_1) = F(e_1) = \{h_2, h_5\}$, since $B \setminus A = \{e_2, e_4\}$, so $H(e_2) = G(e_2) = \{h_1, h_4, h_5\}$ and $H(e_4) = G(e_4) = \{h_3, h_5\}$. And since $A \cap B = \{e_3\}$, so $H(e_3) = F(e_3)\Delta G(e_3) = \{h_1, h_3, h_4, h_5\}$. That is,

$$H(e) = \begin{cases} F(e), & e \in A \setminus B = \{e_1\}, \\ G(e), & e \in B \setminus A = \{e_2, e_4\}, \\ F(e)\triangle G(e), & e \in A \cap B = \{e_3\}. \end{cases}$$

Thus, $(F,A)\Delta_\varepsilon(G,B) = \{(e_1, \{h_2, h_5\}), (e_2, \{h_1, h_4, h_5\}), (e_3, \{h_1, h_3, h_4, h_5\}), (e_4, \{h_3, h_5\})\}$.

Theorem 4.4. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_\varepsilon(G,B) = [(F,A) \cup_\varepsilon (G,B)] \setminus_\varepsilon [(F,A) \cap_R (G,B)]$* [62].

Theorem 4.5. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_\varepsilon(G,B) = [(F,A) \cup_R (G,B)] \setminus_\varepsilon [(F,A) \cap_\varepsilon (G,B)]$* [62].

Theorem 4.6. *Let (F,A) and (G,B) be $\mathcal{S}$s over U. Then, $(F,A)\Delta_\varepsilon(G,B) = [(F,A) \cup_\varepsilon (G,B)] \widetilde{\setminus} [(F,A) \cap_R (G,B)]$* [13].

Theorem 4.7. *Let* (F,A) *and* (G,B) *be* $\mathcal{S}$*s over* U. *Then,* $(F,A)\Delta_\varepsilon(G,B) = [(F,A)\widetilde{\cup}(G,B)]\setminus_\varepsilon [(G,B)\widetilde{\cap}(F,A)]$.

Proof. Since the $\mathcal{P}$ of the $\mathcal{S}$s of both sides is $A\cup B$, soft equality's initial requirement is met. Now let $(F,A)\widetilde{\cup}(G,B) = (H,A)$, where for all $e\in A$,

$$H(e) = \begin{cases} F(e), & e\in A\setminus B,\\ F(e)\cup G(e), & e\in A\cap B.\end{cases}$$

Let $(G,B)\widetilde{\cap}(F,A) = (K,B)$, where for all $e\in B$,

$$K(e) = \begin{cases} G(e), & e\in B\setminus A,\\ G(e)\cap F(e), & e\in B\cap A.\end{cases}$$

Let $(H,A)\setminus_\varepsilon (K,B) = (S,A\cup B)$, where for all $e\in A\cup B$,

$$S(e) = \begin{cases} H(e), & e\in A\setminus B,\\ K(e), & e\in B\setminus A,\\ H(e)\setminus K(e), & e\in A\cap B.\end{cases}$$

Hence,

$$S(e) = \begin{cases} F(e), & e\in (A\setminus B)\setminus B = A\setminus B,\\ F(e)\cup G(e), & e\in (A\cap B)\setminus B = \emptyset,\\ G(e), & e\in (B\setminus A)\setminus A = B\setminus A,\\ G(e)\cap F(e), & e\in (B\cap A)\setminus A = \emptyset,\\ F(e)\setminus G(e), & e\in (A\setminus B)\cap(B\setminus A) = \emptyset,\\ F(e)\setminus (G(e)\cap F(e)), & e\in (A\cap B)\cap (B\setminus A) = \emptyset,\\ (F(e)\cup G(e))\setminus G(e), & e\in (A\cap B)\cap(B\setminus A) = \emptyset,\\ (F(e)\cup G(e))\setminus (G(e)\cap F(e)), & e\in (A\cap B)\cap (B\cap A) = A\cap B.\end{cases}$$

Therefore,

$$S(e) = \begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ [F(e)\cup G(e)]\setminus [G(e)\cap F(e)], & e\in A\cap B.\end{cases}$$

Hence, $(S,A\cup B) = (F,A)\Delta_\varepsilon(G,B)$. □

Here, note that to Here, note that since $A\cap B = B\cap A$, then $A\Delta B = (A\cup B)\setminus (A\cap B) = (A\cup B)\setminus(B\cap A)$. But since $(F,A)\widetilde{\cap}(G,B)\neq (G,B)\widetilde{\cap}(F,A)$ (as they have different $\mathcal{P}$s), although $(F,A)\Delta_\varepsilon(G,B) = [(F,A)\widetilde{\cup}(G,B)]\setminus_\varepsilon [(G,B)\widetilde{\cap}(F,A)]$ is satisfied, $(F,A)\Delta_\varepsilon(G,B)\neq [(F,A)\widetilde{\cup}(G,B)]\setminus_\varepsilon [(\widetilde{\cap}(G,B)]$ as they have different $\mathcal{P}$s (While the left side's $\mathcal{P}$ is $A\cup B$, the right side's $\mathcal{P}$ is A, so they cannot be soft equal.)

Theorem 4.8. [62] *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,*

$$(F,A)\Delta_{\varepsilon}(G,B) = [(F,A)\Delta_{\varepsilon}(G,B)] \cup_R [(G,B)\Delta_{\varepsilon}(F,A)].$$

Theorem 4.9. [13] *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,*

$$(F,A)\Delta_{\varepsilon}(G,B) = [(F,A)\widetilde{\setminus}(G,B)] \cup_{\varepsilon} [(G,B)\widetilde{\setminus}(F,A)].$$

Theorem 4.10. *Let* (F,A) *and* (G,B) *be* $\mathcal{S}s$ *over* U. *Then,*

$$(F,A)\Delta_{\varepsilon}(G,B) = [(F,A) \setminus_{\varepsilon} (G,B)]\widetilde{\cup}[(G,B) \setminus_{\varepsilon} (F,A)].$$

Proof. Let $(F,A)\Delta_{\varepsilon}(G,B) = (S, A\cup B)$, where for all $e \in A\cup B$,

$$S(e) = \begin{cases} F(e), & e \in A\setminus B, \\ G(e), & e \in B\setminus A, \\ F(e)\setminus G(e), & e \in A\cap B. \end{cases}$$

Let $(G,B)\Delta_{\varepsilon}(F,A) = (L, B\cup A)$, where for all $e \in B\cup A$,

$$L(e) = \begin{cases} G(e), & e \in B\setminus A, \\ F(e), & e \in A\setminus B, \\ G(e)\setminus F(e), & e \in A\cap B. \end{cases}$$

Let $(S, A\cup B)\widetilde{\cup}(L, B\cup A) = (W, A\cup B)$, where for all $e \in A\cup B$,

$$W(e) = \begin{cases} S(e), & e \in (A\cup B)\setminus(B\cup A) = \emptyset, \\ S(e)\cup L(e), & e \in (A\cup B)\cap(B\cup A) = A\cup B. \end{cases}$$

Hence,

$$W(e) = \begin{cases} F(e)\cup G(e), & e \in (A\setminus B)\cap(B\setminus A) = \emptyset, \\ F(e)\cup F(e), & e \in (A\setminus B)\cap(A\setminus B) = A\setminus B, \\ F(e)\cup(G(e)\setminus F(e)), & e \in (A\setminus B)\cap(A\cap B) = \emptyset, \\ G(e)\cup G(e), & e \in (B\setminus A)\cap(B\setminus A) = B\setminus A, \\ G(e)\cup F(e), & e \in (B\setminus A)\cap(A\setminus B) = \emptyset, \\ G(e)\cup(G(e)\setminus F(e)), & e \in (B\setminus A)\cap(A\cap B) = \emptyset, \\ (F(e)\setminus G(e))\cup G(e), & e \in (A\cap B)\cap(B\setminus A) = \emptyset, \\ (F(e)\setminus G(e))\cup F(e), & e \in (A\cap B)\cap(A\setminus B) = \emptyset, \\ (F(e)\setminus G(e))\cup(G(e)\setminus F(e)), & e \in (A\cap B)\cap(A\cap B) = A\cap B, \end{cases}$$

Thus,

$$W(e) = \begin{cases} F(e), & e \in A\setminus B, \\ G(e), & e \in B\setminus A, \\ [F(e)\setminus G(e)]\cup[G(e)\setminus F(e)], & e \in A\cap B. \end{cases}$$

Since $[F(e) \setminus G(e)] \cup [G(e) \setminus F(e)] = F(e)\Delta G(e)$, thus,

$$W(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e)\Delta G(e), & e \in A \cap B. \end{cases}$$

Therefore, $(W, A \cup B) = (F, A)\Delta_\varepsilon(G, B)$. □

Proposition 4.11. *The sets $S_E(U)$ and $S_A(U)$ are closed under the operation Δ_ε.*

Proof. It is clear that Δ_ε is a well-defined binary operation in $S_E(U)$ and $S_A(U)$, where A is a fixed $\mathcal{P}$ of U. □

Proposition 4.12. [62] *Let (F, A), (G, B), (H, C) be $\mathcal{S}$s over U. Then,*

$$[(F, A)\Delta_\varepsilon(G, B)]\Delta_\varepsilon(H, C) = (F, A)\Delta_\varepsilon[(G, B)\Delta_\varepsilon(H, C)].$$

That is to say, Δ_ε is associative in the set $S_E(U)$.

Proposition 4.13. [62] *Let (F, A) and (G, B) be $\mathcal{S}$s over U. Then, $(F, A)\Delta_\varepsilon(G, B) = (G, B)\Delta_\varepsilon(F, A)$. That is to say, Δ_ε is commutative in the set $S_E(U)$.*

Proposition 4.14. [62] *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_\varepsilon\varnothing_A = \varnothing_A\Delta_\varepsilon(F, A) = (F, A)$.*

Proof. The proof follows from Proposition 3.11 and Note 4.2. This feature shows us that $\varnothing_A$ is the identity element for Δ_ε in $S_A(U)$. □

Proposition 4.15. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_\varepsilon(F, A) = \varnothing_A$* [62].

Proof. The proof follows from Proposition 3.12 and Note 4.2. This feature shows us that every $\mathcal{S}$ is its own inverse for Δ_ε in $S_A(U)$ and also Δ_ε is not idempotent on $S_E(U)$. □

Proposition 4.16. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_\varepsilon\varnothing_\varnothing = (F, A)$.*

Proof. Let $\varnothing_\varnothing = (S, \varnothing)$ and $(F, A)\Delta_\varepsilon(S, \varnothing) = (H, A \cup \varnothing) = (H, A)$, where for all $e \in A$,

$$H(e) = \begin{cases} F(e), & e \in A \setminus \varnothing = A, \\ S(e), & e \in \varnothing \setminus A = \varnothing, \\ F(e)\Delta S(e), & e \in A \cap \varnothing = \varnothing. \end{cases}$$

Hence, $H(e) = F(e)$ for all $e \in A$. Thus, $(H, A) = (F, A)$. That is to say, $\varnothing_\varnothing$ is the identity element for Δ_ε in $S_E(U)$. It is well known that $A \cup B = \varnothing \iff A = \varnothing$

and $B = \varnothing$. Here note that we cannot find an element (G, B) in $S_E(U)$ such that $(F, A)\Delta_\varepsilon(G, B) = (G, B)\Delta_\varepsilon(F, A) = \varnothing_\varnothing$ as this requires $A \cup B = \varnothing$, and thus this requires $A = \varnothing$ and $B = \varnothing$. Here we can conclude that in the set in $S_E(U)$, only the identity $\varnothing_\varnothing$ element has an inverse (as usual, its inverse is its own), all the other elements do not have an inverse element. □

Remark 4.17. By Proposition 4.11, Proposition 4.12, Proposition 4.3, Proposition 4.14, and Proposition 4.15, the set of all $\mathcal{S}$s with a fixed $\mathcal{P}$ is an abelian group together with the operation Δ_ε. Let the fixed $\mathcal{P}$ be A. Then, $S_A(U), \Delta_\varepsilon)$ is an abelian group with identity $\varnothing_A$.

Remark 4.18. By Proposition 4.11, Proposition 4.12, Proposition 4.3, Proposition 4.14 and Proposition 4.16, $S_E(U)$, $\Delta_\varepsilon)$ is a commutative monoid with identity $\varnothing_\varnothing$.

Proposition 4.19. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_\varepsilon U_A = U_A\Delta_\varepsilon(F, A) = (F, A)^r$.*

Proof. The proof follows from Proposition 3.17 and Note 4.2. □

Proposition 4.20. *Let (F, A) be an $\mathcal{S}$ over U. Then, $(F, A)\Delta_\varepsilon(F, A)^r = (F, A)^r\Delta_\varepsilon(F, A) = U_A$*

Proof. The proof follows from Proposition 3.19 and Note 4.2. □

Proposition 4.21. *Let (F, A), (G, A) and (H, A) be $\mathcal{S}$s over U. Then,*

$$[(F, A)\Delta_\varepsilon(G, A)]\Delta_\varepsilon[(G, A)\Delta_\varepsilon(H, A)] = (F, A)\Delta_\varepsilon(H, A).$$

Proof. The proof follows from Proposition 3.20 and Note 4.2. □

Proposition 4.22. *Let (F, A) and (G, A) be $\mathcal{S}$s over U. Then, $(F, A)^r\Delta_\varepsilon(G, A)^r = (F, A)\Delta_\varepsilon(G, A)$.*

Proof. The proof follows from Proposition 3.21 and Note 4.2. □

Proposition 4.23. *Let (F, A) and (G, B) be $\mathcal{S}$s over U. Then $\varnothing_A\widetilde{\subseteq}(F, A)\Delta_\varepsilon(G, B)$, $\varnothing_B\widetilde{\subseteq}(F, A)\Delta_\varepsilon(G, B)$ and $\varnothing_{A\cup B}\widetilde{\subseteq}(F, A)\Delta_\varepsilon(G, B)$ and $(F, A)\Delta_\varepsilon(G, B)\widetilde{\subseteq}U_{A\cup B}$.*

Proposition 4.24. *Let (F, A), (G, A) and (H, A) be $\mathcal{S}$s over U. Then, $(F, A)\Delta_\varepsilon(G, A) = (F, A)\Delta_\varepsilon(H, A) \Longrightarrow (G, A) = (H, A)$.*

Proof. The proof follows from Proposition 3.23 and Note 4.2. □

Here, note that $(F,A)\Delta_\varepsilon(G,K) = (F,A)\Delta_\varepsilon(H,B)$ does not imply that $(G,K) = (H,B)$, as the elements do not have inverse elements in the algebraic structure $(S_E(U),\Delta_\varepsilon)$. In fact, when we subject (F,A) with the operation Δ_ε to both sides of equality from the left side, it yields the following: $[(F,A)\,\Delta_\varepsilon\,(F,A)]\Delta_\varepsilon(G,K) = [(F,A)\,\Delta_\varepsilon(F,A)]\Delta_\varepsilon(H,B) \Longrightarrow \varnothing_A\Delta_\varepsilon(G,K) = \varnothing_A\Delta_\varepsilon(H,B)$ Here, since $\varnothing_A$ is not the identity element, $\varnothing_A\Delta_\varepsilon(G,K) = \varnothing_A\Delta_\varepsilon(H,B)$ does not imply that $(G,K) = (H,B)$. However, it is easy to see that $(F,A)\Delta_\varepsilon(G,B) = (F,A)\Delta_\varepsilon(H,B) \Longrightarrow (G,B) = (H,B)$.

Proposition 4.25. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_\varepsilon(G,B)\widetilde{\subseteq}(F,A)\cup_\varepsilon (G,B)$.*

Proof. Let $(F,A)\Delta_\varepsilon(G,B) = (H,A\cup B)$, where for all $e\in A\cup B$,

$$H(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\Delta G(e), & e\in A\cap B.\end{cases}$$

Now let $(F,A)\cup_\varepsilon (G,B) = (T,A\cup B)$, where for all $e\in A\cup B$,

$$T(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\cup G(e), & e\in A\cap B.\end{cases}$$

Since for all $e\in A\setminus B$, $F(e)\subseteq F(e)$, for all $e\in B\setminus A$, $G(e)\subseteq G(e)$, and for all $e\in A\cap B$, $F(e)\Delta G(e)\subseteq F(e)\cup G(e)$, thus for all $e\in A\cup B$, $H(e)\subseteq T(e)$. Hence, $(H,A\cup B)\widetilde{\subseteq}(T,A\cup B)$. □

Proposition 4.26. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_\varepsilon(G,B) = \varnothing_{A\cup B} \Leftrightarrow (F,A) = (G,B)$.*

Proof. The proof follows from Proposition 3.25 and Note 4.2. □

Proposition 4.27. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_\varepsilon(G,B) = (F,A)\cup_\varepsilon (G,B) \Leftrightarrow (F,A)\cap_R (G,B) = \varnothing_{A\cap B}$.*

Proof. Necessity: Let $(F,A)\Delta_\varepsilon(G,B) = (H,A\cup B)$ and $(F,A)\cup_\varepsilon (G,B) = (T,A\cup B)$, where for all $e\in A\cup B$,

$$H(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\Delta G(e), & e\in A\cap B,\end{cases}$$

and

$$T(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

Since $(H, A \cup B) = (T, A \cup B)$, then for all $e \in A \cap B$, $F(e)\Delta G(e) = F(e) \cup G(e)$. Thus, $F(e) \cap G(e) = \varnothing$ for all $e \in A \cap B$. Hence, $(F, A) \cap_R (G, B) = \varnothing_{A\cap B}$.

Sufficiency: Let $(F, A) \cap_R (G, B) = \varnothing_{A\cap B}$. By Theorem 4.4. $(F, A)\Delta_\varepsilon(G, B) = [(F, A) \cup_\varepsilon (G, B)]\Delta_\varepsilon[(F, A) \cap_R (G, B)]$. Let $(F, A) \cup_\varepsilon (G, B) = (W, A \cup B)$, where for all $e \in A \cup B$,

$$W(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

and let $\varnothing_{A\cap B} = (L, A\cap B)$, where $L(e) = \varnothing$ for all $e \in A\cap B$. Let $(W, A\cup B)\Delta_\varepsilon(L, A\cap B) = (Y, A \cup B)$, where for all $e \in A \cup B$,

$$Y(e) = \begin{cases} W(e), & e \in (A \cup B) \setminus (A \cap B), \\ W(e) \setminus L(e), & e \in (A \cup B) \cap (A \cap B). \end{cases}$$

Thus,

$$Y(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

Therefore, $(Y, A \cup B) = (F, A) \cup_\varepsilon (G, B) = (F, A)\Delta_\varepsilon(G, B)$. As a special case of Proposition 4.27, we have $(F, A)\Delta_\varepsilon(G, A) = (F, A)\cup_\varepsilon(G, A) \Leftrightarrow (F, A)\cap_\varepsilon(G, A) = \varnothing_A$. $\square$

Proposition 4.28. *Let (F, A) and (G, B) be $\mathcal{S}s$ over U. Then, $(F, A)\tilde{\subseteq}(G, B) \Longrightarrow (G, B)\Delta_\varepsilon(F, A) = (G, B)\tilde{\setminus}(F, A)$.*

Proof. Let $(F, A)\tilde{\subseteq}(G, B)$. Then, $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Let $(G, B)\Delta_\varepsilon(F, A) = (H, A \cup B = B)$. Then, for all $e \in A \cup B = B$,

$$H(e) = \begin{cases} G(e), & e \in B \setminus A, \\ F(e), & e \in A \setminus B = \emptyset, \\ F(e)\Delta G(e), & e \in A \cap B = A. \end{cases}$$

Since for all $e \in A$, $F(e) \subseteq G(e)$, $F(e)\ \Delta G(e) = G(e) \setminus F(e)$. Thus, for all $e \in B$,

$$H(e) = \begin{cases} G(e), & e \in B \setminus A, \\ G(e) \setminus F(e), & e \in A \cap B = A. \end{cases}$$

Hence, $(H, B) = (G, B)\tilde{\setminus}(F, A)$. $\square$

Proposition 4.29. *Let (F,A) and (G,A) be $\mathcal{S}s$ over U. Then, $(F,A)\Delta_\varepsilon[(F,A)\cap_R (G,A)] = (F,A)\Delta_\varepsilon(G,A)$.*

Proof. The proof follows from Proposition 3.28 and Note 4.2. □

Proposition 4.30. *Let (F,A) and (G,B) be $\mathcal{S}s$ over U. Then, $(F,A)\cup_\varepsilon (G,B) = [(F,A)\Delta_\varepsilon(G,B)]\cup_\varepsilon [(F,A)\cap_\varepsilon (G,B)]$.*

Proof. First, let's consider the right side. Let $(F,A)\cup_\varepsilon (G,B) = (H,A\cup B)$. Then, for all $e\in A\cup B$,

$$H(e)=\begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\cup G(e), & e\in A\cap B.\end{cases}$$

Now let's consider the left side. Let $(F,A)\Delta_\varepsilon(G,B) = (K,A\cup B)$. Then, for all $e\in A\cup B$,

$$K(e)=\begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\Delta G(e), & e\in A\cap B.\end{cases}$$

Let, $(F,A)\cap_\varepsilon (G,B) = (T,A\cup B)$, where for all $e\in A\cup B$,

$$T(e)=\begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\cap G(e), & e\in A\cap B.\end{cases}$$

Now, let $(K,A\cup B)\cup_\varepsilon (T,A\cup B) = (S,A\cup B)$, where for all $e\in A\cup B$,

$$S(e)=\begin{cases} K(e), & e\in (A\cup B)\setminus(A\cup B)=\emptyset,\\ T(e), & e\in (A\cup B)\setminus(A\cup B)=\emptyset,\\ K(e)\cup T(e), & e\in (A\cup B)\cap(A\cup B)=A\cup B.\end{cases}$$

Thus,

$$S(e)=\begin{cases} F(e)\cup F(e), & e\in (A\setminus B)\cap(A\setminus B)=A\setminus B,\\ F(e)\cup G(e), & e\in (A\setminus B)\cap(B\setminus A)=\emptyset,\\ F(e)\cup(F(e)\cap G(e)), & e(A\setminus B)\cap(A\cap B)=\emptyset,\\ G(e)\cup F(e), & e\in (B\setminus A)\cap(A\setminus B)=\emptyset,\\ G(e)\cup G(e), & e\in (B\setminus A)\cap(B\setminus A)=B\setminus A,\\ G(e)\cup(F(e)\cap G(e)), & e\in (B\setminus A)\cap(A\cap B)=\emptyset,\\ .(F(e)\Delta G(e))\cup F(e), & e\in (A\cap B)\cap(A\setminus B)=\emptyset,\\ (F(e)\Delta G(e)\cup G(e), & e\in (A\cap B)\cap(B\setminus A)=\emptyset,\\ (F(e)\Delta G(e)\cup(F(e)\cap G(e)), & e\in (A\cap B)\cap(A\cap B)=A\cap B.\end{cases}$$

Thus,

$$S(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ [F(e)\Delta G(e)]\cup[F(e)\cap G(e)], & e\in A\cap B.\end{cases}$$

Since, $[F(e)\Delta G(e)]\cup[F(e)\cap G(e)]=F(e)\cup G(e)$, we have

$$S(e)=\begin{cases}F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\cup G(e), & e\in A\cap B.\end{cases}$$

Thus, $(H,A\cup B)=(S,A\cup B)$. This completes the proof. □

It is well-known that intersection distributes over the symmetric difference from both left and right sides, that is, $K\cap(Q\Delta N)=(K\cap Q)\Delta(K\cap N)$ and $(K\Delta Q)\cap N=(K\cap N)\Delta(Q\cap N)$ for all K, Q, N. In [13], Eren showed that, as an analogy, restricted intersection distributes over the extended symmetric difference from both sides but under certain conditions. However, their proofs are based on Theorem 4.8. So, from a different aspect, we give the following proofs and we show that for providing the distribution laws, extra conditions are not needed.

Proposition 4.31. [13] *Let (F,A), (G,B), and (H,C) be $\mathcal{S}$s over U. Then, $(F,A)\cap_R[(G,B)\Delta_\varepsilon(H,C)]=[(F,A)\cap_R(G,B)]\Delta_\varepsilon[(F,A)\cap_R(H,C)]$.*

Proof. First evaluate the left side. Let $(G,B)\Delta_\varepsilon(H,C)=(M,B\cup C)$, where for all $e\in B\cup C$,

$$M(e)=\begin{cases}G(e), & e\in B\setminus C,\\ H(e), & e\in C\setminus B,\\ G(e)\Delta H(e), & e\in B\cap C.\end{cases}$$

Assume that $(F,A)\cap_R(M,B\cup C)=(N,A\cap(B\cup C))$, where for all $e\in A\cap(B\cup C)$, $N(e)=F(e)\cap M(e)$. Hence,

$$N(e)=\begin{cases}F(e)\cap G(e), & e\in A\cap(B\setminus C),\\ F(e)\cap H(e), & e\in A\cap(C\setminus B),\\ F(e)\cap[G(e)\Delta H(e)], & e\in A\cap(B\cap C).\end{cases}$$

Now evaluate the right side: $[(F,A)\cap_R(G,B)]\Delta_\varepsilon[(F,A)\cap_R(H,C)]$. Let $(F,A)\cap_R(G,B)=(K,A\cap B)$, where for all $e\in A\cap B$, $K(e)=F(e)\cap G(e)$. Let $(F,A)\cap_R(H,C)=(T,A\cap C)$, where for all $e\in A\cap C$, $T(e)=F(e)\cap H(e)$. Thus, $(K,A\cap B)\Delta_\varepsilon(T,A\cap C)=(L,(A\cap B)\cup(A\cap C))$, where for all $e\in(A\cap B)\cup(A\cap C)=A\cap(B\cup C)$,

$$L(e)=\begin{cases}K(e), & e\in(A\cap B)\setminus(A\cap C),\\ T(e), & e\in(A\cap C)\setminus(A\cap B),\\ K(e)\Delta T(e), & e\in(A\cap B)\cap(A\cap C).\end{cases}$$

Thus,

$$L(e)=\begin{cases}F(e)\cap G(e), & e\in A\cap(B\setminus C),\\ F(e)\cap H(e), & e\in A\cap(C\setminus B),\\ [F(e)\cap G(e)]\Delta[F(e)\cap H(e)], & e\in A\cap(B\cap C).\end{cases}$$

Since $F(e)\cap[G(e)\Delta H(e)]=[F(e)\cap G(e)]\Delta[F(e)\cap H(e)]$, hence $(N,A\cap(B\cup C))=(L,(A\cap B)\cup(A\cap C))=(L,A\cap(B\cup C))$. □

Note 4.32. In [13], it was shown that the above distribution is satisfied when $A\cap B\neq\varnothing$ and $A\cap C\neq\varnothing$. However, $A\cap B\neq\varnothing$ and $A\cap C\neq\varnothing$ are not necessary conditions. In fact,

i) Let $A\cap B=\varnothing$. Then, $[(F,A)\cap_R(G,B)]\Delta_\varepsilon[(F,A)\cap_R(H,C)]=\varnothing_\varnothing\Delta_\varepsilon[(F,A)\cap_R(H,C)]=(F,A)\cap_R(H,C)$ by Proposition 4.19. Now let's consider $(F,A)\cap_R[(G,B)\Delta_\varepsilon(H,C)]$. Let $(G,B)\Delta_\varepsilon(H,C)=(M,B\cup C)$, where for all $e\in B\cup C$,

$$M(e)=\begin{cases}G(e), & e\in B\setminus C,\\ H(e), & e\in C\setminus B,\\ G(e)\Delta H(e), & e\in B\cap C.\end{cases}$$

Assume that $(F,A)\cap_R(M,B\cup C)=(N,A\cap(B\cup C))$, where for all $e\in A\cap(B\cup C)$, $N(e)=F(e)\cap M(e)$. Since $A\cap B=\varnothing$ by assumption, hence,

$$N(e)=\begin{cases}\varnothing, & e\in A\cap(B\setminus C),\\ F(e)\cap H(e), & e\in A\cap(C\setminus B),\\ \varnothing, & e\in A\cap(B\cap C).\end{cases}$$

Thus, $N(e)=F(e)\cap H(e)$ for all $e\in A\cap C$. Therefore, $(N,A\cap C)=(F,A)\cap_R(H,C)$.

ii) Let $A\cap C=\varnothing$. Then, $[(F,A)\cap_R(G,B)]\Delta_\varepsilon[(F,A)\cap_R(H,C)]=[(F,A)\cap_R(G,B)]\Delta_\varepsilon\varnothing_\varnothing=(F,A)\cap_R(G,B)$ by Proposition 4.19. Now let's consider $(F,A)\cap_R[(G,B)\Delta_\varepsilon(H,C)]$. Let $(G,B)\Delta_\varepsilon(H,C)=(M,B\cup C)$, where for all $e\in B\cup C$,

$$M(e)=\begin{cases}G(e), & e\in B\setminus C,\\ H(e), & e\in C\setminus B,\\ G(e)\Delta H(e), & e\in B\cap C.\end{cases}$$

Assume that $(F,A)\cap_R(M,B\cup C)=(N,A\cap(B\cup C))$, where for all $e\in A\cap(B\cup C)$, $N(e)=F(e)\cap M(e)$. Since $A\cap C=\varnothing$ by assumption, hence,

$$N(e)=\begin{cases}F(e)\cap G(e), & e\in A\cap(B\setminus C),\\ \varnothing, & e\in A\cap(C\setminus B),\\ \varnothing, & e\in A\cap(B\cap C).\end{cases}$$

Thus, $N(e)=F(e)\cap G(e)$ for all $e\in A\cap B$. Therefore, $(N,A\cap B)=(F,A)\cap_R(G,B)$.

Proposition 4.33. *Let (F,A), (G,B), and (H,C) be $\mathcal{S}s$ over U. Then,* $[(F,A)\Delta_\varepsilon(G,B)]\cap_R (H,C) = [(F,A)\cap_R (H,C)]\Delta_\varepsilon[(G,B)\cap_R (H,C)]$.

Proof. First evaluate first the left side. Let $(F,A)\Delta_\varepsilon(G,B) = (M, A\cup B)$, where for all $e \in A\cup B$,

$$M(e) = \begin{cases} F(e), & e \in A\setminus B, \\ G(e), & e \in B\setminus A, \\ F(e)\Delta G(e), & e \in A\cap B. \end{cases}$$

Now, let $(M, A\cup B)\cap_R (H,C) = (W, (A\cup B)\cap C)$, where for all $e \in (A\cup B)\cap C$, $W(e) = M(e)\cap H(e)$. Thus,

$$W(e) = \begin{cases} F(e)\cap H(e), & e \in (A\setminus B)\cap C, \\ G(e)\cap H(e), & e \in (B\setminus A)\cap C, \\ [F(e)\Delta G(e)]\cap H(e), & e \in (A\cap B)\cap C. \end{cases}$$

Now evaluate the right side: $[(F,A)\cap_R (H,C)]\Delta_\varepsilon[(G,B)\cap_R (H,C)]$. Let $(F,A)\cap_R (H,C) = (K, A\cap C)$, where for all $e \in A\cap C$, $K(e) = F(e)\cap H(e)$. Let $(G,B)\cap_R (H,C) = (T, B\cap C)$, where for all $e \in B\cap C$, $T(e) = G(e)\cap H(e)$. Thus, $(K, A\cap C)\Delta_\varepsilon(T, B\cap C) = (R, (A\cap C)\cup(B\cap C))$, where for all $e \in (A\cap C)\cup(B\cap C)$,

$$R(e) = \begin{cases} K(e), & e \in (A\cap C)\setminus(B\cap C), \\ T(e), & e \in (B\cap C)\setminus(A\cap C), \\ K(e)\Delta T(e), & e \in (A\cap C)\cap(B\cap C). \end{cases}$$

Thus,

$$R(e) = \begin{cases} F(e)\cap H(e), & e \in (A\setminus B)\cap C, \\ G(e)\cap H(e), & e \in (B\setminus A)\cap C, \\ [F(e)\cap H(e)]\Delta[G(e)\cap H(e)], & e \in (A\cap B)\cap C. \end{cases}$$

Hence, $(W, (A\cup B)\cap C) = (R, (A\cup B)\cap C)$. □

Note 4.34. In [13], it was shown that the above distribution is satisfied when $A\cap C \neq \varnothing$ and $B\cap C \neq \varnothing$. However, $A\cap C \neq \varnothing$ and $B\cap C \neq \varnothing$ are not necessary conditions. In fact,

i) Let $A\cap C = \varnothing$. Then, $[(F,A)\cap_R (H,C)]\Delta_\varepsilon[(G,B)\cap_R (H,C)] = \varnothing_\varnothing\Delta_\varepsilon[(G,B)\cap_R (H,C)] = (G,B)\cap_R (H,C)$ by Proposition 4.19. Let $(F,A)\Delta_\varepsilon(G,B) = (M, A\cup B)$, where for all $e \in A\cup B$,

$$M(e) = \begin{cases} F(e), & e \in A\setminus B, \\ G(e), & e \in B\setminus A, \\ F(e)\Delta G(e), & e \in A\cap B. \end{cases}$$

And let $(M, A\cup B)\cap_R (H,C) = (W,(A\cup B)\cap C)$, where for all $e\in(A\cup B)\cap C$, $W(e) = M(e)\cap H(e)$. Thus,

$$W(e)=\begin{cases} F(e)\cap H(e), & e\in(A\setminus B)\cap C,\\ G(e)\cap H(e), & e\in(B\setminus A)\cap C,\\ [F(e)\Delta G(e)]\cap H(e), & e\in(A\cap B)\cap C.\end{cases}$$

Since, $A\cap C=\varnothing$ by assumption,

$$W(e)=\begin{cases} \varnothing, & e\in(A\setminus B)\cap C,\\ G(e)\cap H(e), & e\in(B\setminus A)\cap C,\\ \varnothing, & e\in(A\cap B)\cap C.\end{cases}$$

$W(e)=G(e)\cap H(e)$ for all $e\in B\cap C$. Therefore, $(W,B\cap C)=(G,B)\cap_R (H,C)$.

ii) Let $B\cap C=\varnothing$. Then, $[(F,A)\cap_R (H,C)]\Delta_\varepsilon[(G,B)\cap_R (H,C)] = [(F,A)\cap_R (H,C)]\Delta_\varepsilon\varnothing_\varnothing = (F,A)\cap_R (H,C)$ by Proposition 4.19. Let $(F,A)\Delta_\varepsilon(G,B)=(M,A\cup B)$, where for all $e\in A\cup B$,

$$M(e)=\begin{cases} F(e), & e\in A\setminus B,\\ G(e), & e\in B\setminus A,\\ F(e)\Delta G(e), & e\in A\cap B.\end{cases}$$

And let $(M, A\cup B)\cap_R (H,C) = (W,(A\cup B)\cap C)$, where for all $e\in(A\cup B)\cap C$, $W(e) = M(e)\cap H(e)$. Thus,

$$W(e)=\begin{cases} F(e)\cap H(e), & e\in(A\setminus B)\cap C,\\ G(e)\cap H(e), & e\in(B\setminus A)\cap C,\\ [F(e)\Delta G(e)]\cap H(e), & e\in(A\cap B)\cap C.\end{cases}$$

Since $B\cap C=\varnothing$ by assumption,

$$W(e)=\begin{cases} F(e)\cap H(e), & e\in(A\setminus B)\cap C,\\ \varnothing, & e\in(B\setminus A)\cap C,\\ \varnothing, & e\in(A\cap B)\cap C.\end{cases}$$

$W(e)=F(e)\cap H(e)$ for all $e\in A\cap C$. Therefore, $(W,A\cap C)=(F,A)\cap_R (H,C)$. By Note 4.32 and Note 4.34, we can conclude that $\cap_R$ distributes over Δ_ε from both sides without any condition or necessity. In [13], it was shown that $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a Boolean ring. In fact, since $S_A(U)$, $\Delta_\varepsilon)$ is an abelian group by Remark 4.17, $(S_A(U),\cap_R)$ is a semigroup by [37, 50, 5] and $\cap_R$ distributes over Δ_ε from both sides by Proposition 4.31 and Proposition 4.33, we can also deduce that $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a ring. Also, since $(F,A)\cap_R (G,A) = (G,A)\cap_R (F,A)$ and $(F,A)\cap_R U_A = U_A\cap_R (F,A) = (F,A)$, (see [37, 50, 5], $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a commutative ring with identity U_A. Moreover, $(F,A)^2 =$

$(F,A)\cap_R(F,A)=(F,A)$ for all $(F,A)\in S_A(U)$. Thus, $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a Boolean ring and $(F,A)\Delta_\varepsilon(F,A)=\varnothing_A$ and $(F,A)\cap_R(G,A)=(G,A)\cap_R(F,A)$ satisfy naturally as a result of being a Boolean ring. Also, $(S_A(U),\Delta_R,\cap_R)$ and $(S_A(U),\Delta_\varepsilon,\cap_R)$ are the same Boolean ring, since Δ_R and Δ_ε coincide in the collection $S_A(U)$ by Note 4.2. Now, we have the following theorem for the collection of $S_E(U)$.

Theorem 4.35. *$(S_E(U),\Delta_\varepsilon,\cap_R)$ is a commutative hemiring with identity U_E.*

Proof. In Remark 4.18, we show that $(S_E(U),\Delta_\varepsilon)$ is a commutative monoid with identity $\varnothing_\varnothing$. Hence, we can deduce that $S_E(U),\Delta_\varepsilon)$ is a semigroup. Moreover, in [37, 50, 5], it was proved that $S_E(U),\cap_R)$ is a commutative monoid with identity U_E. Hence, we can deduce that $(S_E(U),\cap_R)$ is a semigroup. Moreover, by Proposition 4.31, and Proposition 4.33, $\cap_R$ distributes over Δ_ε from both sides. Therefore, $S_E(U),\Delta_\varepsilon,\cap_R)$ is a semiring. Further, by Proposition 4.13, $(F,A)\Delta_\varepsilon(G,B)=(G,B)\Delta_\varepsilon(F,A)$. That is to say, Δ_ε is commutative in $S_E(U)$ and $(F,A)\Delta_\varepsilon\varnothing_\varnothing=\varnothing_\varnothing\Delta_\varepsilon(F,A)=(F,A)$ and $(F,A)\cap_R\varnothing_\varnothing=\varnothing_\varnothing\cap_R(F,A)=\varnothing_\varnothing$ (as null $\mathcal{S}$ is the only $\mathcal{S}$ with an empty $\mathcal{P}$). That is to say, $\varnothing_\varnothing$ is the zero element of $(S_E(U),\Delta_\varepsilon,\cap_R)$. Therefore, $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a hemiring. Besides, since $(F,A)\cap_R U_E=U_E\cap_R(F,A)=(F,A)$ and $(F,A)\cap_R(G,B)=(G,B)\cap_R(F,A)$ (see [37, 50, 5]), $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a commutative hemiring with identity U_E. □

5. Conclusion

Soft sets and soft set operations are effective parametric tools for dealing with uncertain objects. To tackle problems that involve parametric data, it is important to propose new soft operations and develop the algebraic properties of soft sets defined before. In this study, we have handled the restricted and extended symmetric difference operations in detail and examined their full algebraic properties, including closure, associativity, unit, inverse element, and abelian property, in the collection of soft sets over U and the collections of soft sets with a fixed parameter set of soft sets. We have also looked through the distribution rules to establish the connections between these soft set operations and the restricted intersection operation to reveal which algebraic structures they form. Also, we have proposed new and alternative definitions for restricted symmetric difference operation. Besides, the properties of these soft set operations are handled comparatively with the symmetric difference operation in classical set theory, and we obtain stunning analogies. We have demonstrated that the set of all soft sets over U, together with the restricted intersection operation and extended symmetric difference operation, forms a commutative hemiring with identity. In this regard, this paper is a complete study of restricted and extended symmetric difference operation. As the study of the algebraic structure of soft sets from the perspective of soft set operations offers a comprehensive understanding of their application as well as an appreciation of how soft set algebra can be applied to classical and nonclassical logic, this paper hopes to add to the literature on soft sets on this topic. Moreover, since soft sets are a powerful mathematical tool for detecting uncertain objects, this study will serve as the foundation for various applications, including new decision-making methods and novel cryptography techniques based on soft sets. In the future studies, the researchers may apply these operations in decision making problems.

Declarations

The authors declare no conflict of interest.

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