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Hall numbers of complete multipartite graphs, 2-trees, and wheels

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ABSTRACT

The *Hall number* $h(G)$ of a graph G is the minimum integer k such that every k-list assignment satisfying Hall's condition on all induced subgraphs of G admits a proper coloring. In this paper, we investigate graphs for which the Hall number strictly captures list colorability, satisfying the equality $h(G) = \mathrm{ch}(G)$. We confirm a conjecture of Allagan by proving that this equality holds for every complete multipartite graph without singleton parts. For complete k-partite graphs of the form $K(m, n, 1, \ldots, 1)$, we establish that $h(G) = \mathrm{ch}(G)$ for all sufficiently large n. Furthermore, we also determine $h(G)$ for 2-trees and wheel graphs W_n. We show that for a 2-tree G, $h(G) \in \{1, 2, 3\}$ for $|V(G)| = 3, 4$, and ≥ 5, respectively. For wheel graphs, we demonstrate that $h(W_n)$ is dictated by the parity of the rim: $h(W_n) = 3$ for odd $n \geq 5$, and $h(W_n) = 4$ for even $n \geq 6$.

Keywords: Hall number, list coloring, complete multipartite graphs

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1. Introduction

All graphs considered are finite and simple. For a graph G we denote by $V(G)$ and $E(G)$ its vertex and edge sets, by $\chi(G)$ its chromatic number, and by $\alpha(G)$ its independence number. Recall that a graph is d-degenerate if every induced subgraph has a vertex of degree at most d; equivalently, G admits an ordering of its vertices in which each vertex has at most d later neighbors. In particular, every d-degenerate graph satisfies $\mathrm{ch}(G) \leq d + 1$.

In list coloring, each vertex v is assigned a finite set $L(v)$ of admissible colors. An L-coloring is a proper coloring φ with $\varphi(v) \in L(v)$ for all $v \in V(G)$. The *choice number* (or list chromatic number) $\mathrm{ch}(G)$ is the least integer k such that every assignment with

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$|L(v)| \geq k$ for all v admits a proper coloring.

Let L be a list assignment and $H \subseteq G$ an induced subgraph. For a color σ, define

$$H_\sigma \;:=\; H[\{v \in V(H) : \sigma \in L(v)\}],$$

the subgraph of H induced by the vertices whose lists contain σ. Any proper L-coloring of H partitions $V(H)$ into independent color classes C_σ. Since $|C_\sigma| \leq \alpha(H_\sigma)$ for each σ and $\sum_\sigma |C_\sigma| = |V(H)|$, every proper L-coloring of H satisfies

$$\sum_\sigma \alpha(H_\sigma) \;\geq\; |V(H)|. \tag{1}$$

Following Hilton and Johnson [8, 9], we call (1) *Hall's condition.* Crucially, in the definition of $h(G)$ below, Hall's condition is required to hold for *every* induced subgraph $H \subseteq G$, not merely for G itself. When G is complete, Hall's condition is also sufficient and reduces to Hall's classical theorem on systems of distinct representatives [7].

The *Hall number* $h(G)$ is the least integer $k \geq 1$ such that every k-list assignment satisfying Hall's condition on all induced subgraphs of G is colorable. Thus $h(G)$ measures when the global obstruction encoded in (1) already guarantees list colorability, independent of local list-size constraints.

Hilton and Johnson [9] established the following basic facts:

(H1) $h(G) \leq \mathrm{ch}(G)$ for every graph G;

(H2) if H is an induced subgraph of G, then $h(H) \leq h(G)$;

(H3) if $\mathrm{ch}(G) > \chi(G)$, then $h(G) = \mathrm{ch}(G)$;

(H4) $h(G) = 1$ if and only if every block of G is complete.

In particular, strict inequality $h(G) < \mathrm{ch}(G)$ can occur only in the borderline case $\mathrm{ch}(G) = \chi(G)$. Classical examples illustrate the range: $h(K_n) = 1$ and $h(T) = 1$ for every tree, whereas $h(C_n) = 2$ for $n \geq 4$ [9]. At the other extreme, Tuza [11] proved the general bound $h(G) \leq \lfloor |V(G)|/2 \rfloor$ and showed that deleting a single vertex may change $h(G)$ by $\Theta(|V(G)|)$.

A natural problem is to characterize graph classes for which $h(G) = \mathrm{ch}(G)$. We resolve this question for three fundamental families: complete multipartite graphs, 2-trees, and wheels. In these settings the equality is driven by distinct structural features: inheritance from multipartite cores, minimal induced Hall-tight obstructions together with bounded degeneracy in 2-trees, and parity constraints in wheels. Together these results clarify when Hall's condition captures the full content of list colorability.

2. Complete multipartite graphs and threshold behavior

Let $K(n_1, \ldots, n_k)$ denote the complete k-partite graph with parts of sizes $n_1, \ldots, n_k$. When $n_i \geq 2$ for all i, we have $\chi(G) = k$. The choice number of such graphs has been studied extensively, beginning with Erdős, Rubin, and Taylor and refined by Gravier–Maffray and Enomoto–Ohba–Ota–Sakamoto [3, 4, 6].

It was shown in [1] that $h(K(m,2,\dots,2)) = \mathrm{ch}(K(m,2,\dots,2))$ for all $m \geq 2$, and it was conjectured that the same equality holds whenever no part is a singleton. We confirm this and then determine when the equality persists once singleton parts are introduced.

Lemma 2.1. *Let H be an induced subgraph of G. If $h(H) = \mathrm{ch}(G) = k$, then $h(G) = k$.*

Proof. By (H2), $h(G) \geq h(H) = k$, while (H1) gives $h(G) \leq \mathrm{ch}(G) = k$. □

Theorem 2.2. *If $G = K(n_1,\dots,n_k)$ with $n_i \geq 2$ for all i, then $h(G) = \mathrm{ch}(G)$.*

Proof. If $\mathrm{ch}(G) > \chi(G) = k$, then $h(G) = \mathrm{ch}(G)$ by (H3). Otherwise $\mathrm{ch}(G) = k$, and G contains an induced copy of the complete k-partite graph $K(2,\dots,2)$. By the Erdős–Rubin–Taylor theorem [4], $\mathrm{ch}(K(2,\dots,2)) = k$. Since [1] establishes $h(K(2,\dots,2)) = \mathrm{ch}(K(2,\dots,2))$, we have $h(K(2,\dots,2)) = k$. Lemma 2.1 yields $h(G) = k$. □

Thus in the absence of singleton parts, equality is inherited from $K(2,\dots,2)$ and propagated by induced-subgraph monotonicity.

Lemma 2.3. *If G contains an induced subgraph H with $\mathrm{ch}(H) \geq \chi(G)+1$, then $h(G) = \mathrm{ch}(G)$.*

Proof. Since $\mathrm{ch}(G) \geq \mathrm{ch}(H) \geq \chi(G) + 1$, we have $\mathrm{ch}(G) > \chi(G)$, and the conclusion follows from (H3). □

Consider $G = K(m,n,1,\dots,1)$, a complete k-partite graph with two nontrivial parts of sizes $m \geq n \geq 2$ and $k-2$ singleton parts. Then $\chi(G) = k$, and $K_{m,n}$ appears as an induced subgraph.

Lemma 2.4. *Fix $m \geq 2$ and set $\lambda_m := \log\left(\frac{m}{m-1}\right)$. For every $\varepsilon \in (0,1)$ there exists $n_0 = n_0(m,\varepsilon)$ such that $\mathrm{ch}(K_{m,n}) \geq (1-\varepsilon)\frac{\log n}{\lambda_m}$ for all $n \geq n_0$.*

Proof. For fixed $m \geq 2$, Gazit and Krivelevich [5] establish

$$\mathrm{ch}(K_{m,n}) = (1+\eta(n))\frac{\log n}{\lambda_m}, \qquad \eta(n) \to 0 \text{ as } n \to \infty,$$

where all logarithms are natural. (The matching upper bound uses a probabilistic argument due to Alon [2].) Choosing n_0 so that $\eta(n) \geq -\varepsilon$ for $n \geq n_0$ gives the stated lower bound. □

Theorem 2.5. *Let $G = K(m,n,1,\dots,1)$ be a complete k-partite graph with $m \geq n \geq 2$ and $k-2$ singleton parts. There exists an integer $N = N(m,k)$ such that $h(G) = \mathrm{ch}(G)$ for all $n \geq N$.*

Proof. By definition, $\chi(G) = k$. Applying Lemma 2.4 with $\varepsilon = \dfrac{1}{2}$, we obtain $\operatorname{ch}(K_{m,n}) \geq \frac{\log n}{2\lambda_m}$ for all $n \geq n_0(m, \dfrac{1}{2})$. Set

$$N \;:=\; \max\left(n_0\left(m, \frac{1}{2}\right),\, \left\lceil e^{2k\lambda_m}\right\rceil + 1\right).$$

For any $n \geq N$, this choice ensures $\frac{\log n}{2\lambda_m} > k$, and thus $\operatorname{ch}(K_{m,n}) \geq k + 1$. Since $K_{m,n}$ is an induced subgraph of G, monotonicity of the choice number yields

$$\operatorname{ch}(G) \;\geq\; \operatorname{ch}(K_{m,n}) \;\geq\; k+1 \;>\; k = \chi(G).$$

Lemma 2.3 then yields $h(G) = \operatorname{ch}(G)$. ☐

Corollary 2.6. *For $G = K_{n,\ldots,n}$ with $k \geq 2$ parts, $h(G) = \operatorname{ch}(G)$ and, as $n \to \infty$, $\operatorname{ch}(G) = (1 + o(1))\, \dfrac{\log n}{\log(k/(k-1))}$.*

Proof. Equality $h(G) = \operatorname{ch}(G)$ follows from Theorem 2.2, and the asymptotic formula is due to Gazit and Krivelevich [5]. ☐

3. 2-Trees

A 2-tree is obtained from K_3 by repeatedly adjoining a new vertex adjacent to the endpoints of an existing edge. Equivalently, 2-trees are precisely the edge-maximal chordal graphs of clique number three, or, in structural terms, the chordal graphs of treewidth two. In particular, every 2-tree is 2-degenerate; hence

$$\operatorname{ch}(G) \leq d(G) + 1 \leq 3.$$

We determine $h(G)$ exactly for this class by identifying the minimal induced obstructions to Hall–colorability. The key observation is that every 2-tree on at least five vertices contains, as an induced subgraph, one of the two nonisomorphic 2-trees on five vertices: the fan F_4 or the book B_3. Each of these graphs admits a 2-list assignment that satisfies Hall's condition on every induced subgraph but admits no proper coloring. Consequently,

$$h(F_4) = h(B_3) = 3,$$

and F_4 and B_3 serve as minimal Hall-tight obstructions within the class of 2-trees.

Throughout this section we use the vertex labels $\{a, b, c, d, e\}$ and list assignments shown in Figure 1. The corresponding vertex-deleted colorings, which verify Hall's condition for all proper induced subgraphs, are recorded in Tables 1 and 2 in the Appendix.

The smallest cases are immediate. The unique 2-tree on three vertices is K_3, and $h(K_3) = 1$ by (H4). The unique 2-tree on four vertices is the diamond $K_4 - e$; since it contains an induced C_4, monotonicity (H2) gives $h(K_4 - e) \geq h(C_4) = 2$, and the equality $h(K_n - e) = n - 2$ (see [9]) yields $h(K_4 - e) = 2$.

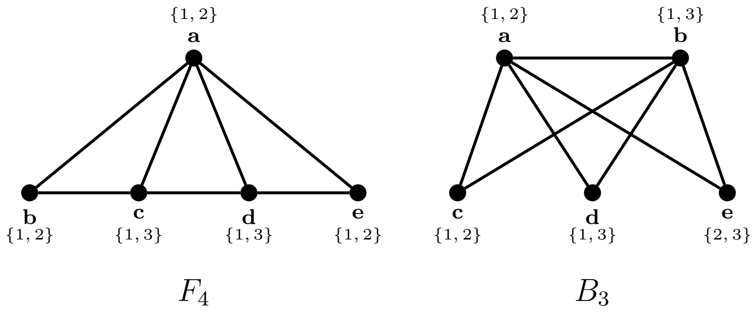

Fig. 1. F_4 and B_3 with lists of size 2.

Lemma 3.1. $h(F_4) = 3$.

Proof. Let a be the hub adjacent to the path b–c–d–e, and assign

$$L(a) = L(b) = L(e) = \{1, 2\}, \qquad L(c) = L(d) = \{1, 3\}.$$

Since F_4 is 2-degenerate, $\mathrm{ch}(F_4) \le 3$, and (H1) gives $h(F_4) \le 3$. The following two claims establish $h(F_4) \ge 3$.

Claim 1. F_4 admits no proper L-coloring.

Proof. If $\varphi(a) = 1$, every neighbor of a avoids 1, so $\varphi(c) = \varphi(d) = 3$, contradicting the edge c–d. If $\varphi(a) = 2$, then $\varphi(b) = \varphi(e) = 1$, again forcing $\varphi(c) = \varphi(d) = 3$, a contradiction. □

Claim 2. L satisfies Hall's condition on every induced subgraph of F_4.

Proof. For the full graph, $(F_4)_1 = F_4$, $(F_4)_2 = F_4[\{a, b, e\}]$, and $(F_4)_3 = F_4[\{c, d\}]$. Since b and e are nonadjacent, $\alpha((F_4)_1) = 2$, $\alpha((F_4)_2) = 2$, $\alpha((F_4)_3) = 1$, giving $\sum_{\sigma} \alpha((F_4)_\sigma) = 5 = |V(F_4)|$. Every proper induced subgraph of F_4 is contained in $F_4 - v$ for some $v \in V(F_4)$. Since a proper L-coloring of $F_4 - v$ satisfies Hall's condition for all induced subgraphs thereof, it suffices to show one such coloring for each v; these are recorded in Table 1. □

Claims 1 and 2 show that L satisfies Hall's condition on every induced subgraph yet admits no proper coloring, so $h(F_4) \ge 3$, and hence $h(F_4) = 3$. □

Lemma 3.2. $h(B_3) = 3$.

Proof. Let B_3 have spine edge ab and leaves c, d, e, each adjacent to both a and b, and assign

$$L(a) = \{1, 2\}, \quad L(b) = \{1, 3\}, \quad L(c) = \{1, 2\}, \quad L(d) = \{1, 3\}, \quad L(e) = \{2, 3\}.$$

Since B_3 is 2-degenerate, $\mathrm{ch}(B_3) \le 3$, and (H1) gives $h(B_3) \le 3$. The following two claims establish $h(B_3) \ge 3$.

Claim 3. B_3 admits no proper L-coloring.

Proof. Since $L(a) \cap L(b) = \{1\}$, the spine assignments are in $\{(1,3),(2,1),(2,3)\}$. Assignment $(1,3)$ exhausts $L(d)$; $(2,1)$ exhausts $L(c)$; $(2,3)$ exhausts $L(e)$. □

Claim 4. L satisfies Hall's condition on every induced subgraph of B_3.

Proof. For the full graph, $(B_3)_1 = B_3[\{a,b,c,d\}]$, $(B_3)_2 = B_3[\{a,c,e\}]$, $(B_3)_3 = B_3[\{b,d,e\}]$. In each case the two leaves present are nonadjacent, so $\alpha((B_3)_\sigma) = 2$ for each σ, giving $\sum_\sigma \alpha((B_3)_\sigma) = 6 \geq 5 = |V(B_3)|$. Every proper induced subgraph of B_3 is contained in $B_3 - v$ for some $v \in V(B_3)$. Since a proper L-coloring of $B_3 - v$ satisfies Hall's condition for all induced subgraphs thereof, it suffices to show one such coloring for each v; these are recorded in Table 2. □

Claims 3 and 4 show that L satisfies Hall's condition on every induced subgraph yet admits no proper coloring, so $h(B_3) \geq 3$, and hence $h(B_3) = 3$. □

Lemma 3.3. *Every 2-tree on $n \geq 5$ vertices contains an induced copy of F_4 or B_3.*

Proof. Induct on n. For $n = 5$, start from the unique 2-tree on four vertices, the diamond on $\{a,b,c,d\}$ with triangles abc and acd (so the shared edge is ac). A 5-vertex 2-tree is obtained by adjoining a new vertex e adjacent to the ends of some edge of this diamond. There are only two cases, up to symmetry.

(i) e attached to the shared edge ac. Then e is adjacent to a and c, and the three vertices b, d, e are pairwise nonadjacent, each adjacent to both a and c. Hence the resulting graph is the book B_3 (as in Figure 1).

(ii) e attached to a boundary edge (say ab). Then a is adjacent to b, c, d, e, and the remaining edges among $\{b,c,d,e\}$ are exactly bc, cd, de, so $\{b,c,d,e\}$ induces the path b–c–d–e. Hence the resulting graph is the fan F_4 (as in Figure 1). Thus the only 2-trees on five vertices are F_4 and B_3.

For $n > 5$, write G as $G' + v$ where G' is a 2-tree on $n-1$ vertices and v is adjacent precisely to the ends of some edge of G'. By induction, G' contains an induced copy of F_4 or B_3, which remains induced in G since $v \notin V(G')$. □

Theorem 3.4. *Let G be a 2-tree on n vertices. Then*

$$h(G) = \begin{cases} 1, & n = 3, \\ 2, & n = 4, \\ 3, & n \geq 5. \end{cases}$$

Proof. The cases $n = 3$ and $n = 4$ were established above. For $n \geq 5$, Lemma 3.3 provides an induced copy of F_4 or B_3 in G, each with $h = 3$ by Lemmas 3.1 and 3.2. Property (H2) gives $h(G) \geq 3$, while 2-degeneracy and (H1) yield $h(G) \leq \mathrm{ch}(G) \leq 3$. □

Corollary 3.5. *Every 2-tree on at least five vertices satisfies* $h(G) = \mathrm{ch}(G) = 3$.

Proof. Since G contains a triangle, $\chi(G) \geq 3$. The 2-degeneracy bound gives $\mathrm{ch}(G) \leq 3$, so $\mathrm{ch}(G) = 3$, and Theorem 3.4 gives $h(G) = 3$. □

4. Hall numbers of wheel graphs

Let W_n denote the wheel on $n \geq 4$ vertices, obtained from the cycle C_{n-1} by adjoining a universal vertex x. Note that n odd corresponds to an even rim C_{n-1}, while n even corresponds to an odd rim C_{n-1}; this parity distinction governs all three cases below.

Theorem 4.1. *For* $n \geq 4$,

$$h(W_n) = \begin{cases} 1, & n = 4, \\ 3, & n \geq 5 \text{ odd}, \\ 4, & n \geq 6 \text{ even}. \end{cases}$$

Proof. Since $W_4 \cong K_4$, property (H4) gives $h(W_4) = 1$. Label the rim vertices $v_1, v_2, \ldots, v_{n-1}$ consecutively.

Case 1. $n = 2t + 1$ *odd,* $t \geq 2$.

Since the rim C_{2t} is an even cycle, $\chi(W_n) = 3$. To see that $\mathrm{ch}(W_n) = 3$, consider any 3-assignment of the vertices of W_n. Color the hub x with any color $c \in L(x)$ and remove c from each adjacent rim list; the residual lists have size at least 2 on C_{2t}, and even cycles are 2-choosable [4], so a proper coloring exists. Hence $\mathrm{ch}(W_n) \leq 3$, and equality $\mathrm{ch}(W_n) = 3$ follows from $\chi(W_n) = 3$. By (H1), $h(W_n) \leq 3$.

It therefore suffices to exhibit a 2-assignment L that satisfies Hall's condition on every induced subgraph of W_n yet admits no proper L-coloring. Set $u := v_{2t-1}$ and $w := v_{2t}$ as the last two rim vertices, so that the rim path is v_1–v_2–$\cdots$–v_{2t-2}–u–w–v_1. Set $L(x) = \{1, 2\}$, $L(u) = \{3, 4\}$, $L(w) = \{1, 4\}$, and for $1 \leq i \leq 2t - 2$,

$$L(v_i) = \begin{cases} \{1, 2\}, & i \text{ odd}, \\ \{1, 3\}, & i \text{ even}. \end{cases}$$

The four colors partition the vertex set as follows: V_σ consists of all rim vertices whose lists contain color σ, together with x if $\sigma \in L(x)$. Explicitly,

$$\begin{aligned} V_1 &:= V(W_n) \setminus \{u\}, & V_2 &:= \{x\} \cup \{v_i : 1 \leq i \leq 2t-3,\ i \text{ odd}\}, \\ V_4 &:= \{u, w\}, & V_3 &:= \{u\} \cup \{v_i : 2 \leq i \leq 2t-2,\ i \text{ even}\}. \end{aligned}$$

Claim 5. W_n admits no proper L-coloring.

Proof. If $\varphi(x) = 1$, every rim vertex avoids 1; for $2 \leq i \leq 2t - 2$ this forces $\varphi(v_i) = 3$ for even i and $\varphi(v_i) = 2$ for odd i, so $\varphi(v_{2t-2}) = 3$. Then $\varphi(u) = 4$ (since $L(u) = \{3, 4\}$ and v_{2t-2} is colored 3), but w must avoid both 1 (hub spoke) and 4 (rim edge uw), exhausting its list $\{1,4\}$, a contradiction. If $\varphi(x)=2$, then $\varphi(v_1)=1$. The same propagation gives $\varphi(v_{2t-2})=3$, hence $\varphi(u)=4$ and $\varphi(w)=1$, which contradicts the rim edge wv_1 since $\varphi(v_1)=1$. $\square$

Claim 6. L satisfies Hall's condition on every induced subgraph of W_n.

Proof. For an induced subgraph $H=W_n[S]$ and $\sigma\in\{1,2,3,4\}$, let $H_\sigma := H[S\cap V_\sigma]$. Since $V_2\setminus\{x\}$ is independent and x is adjacent to all its members, $\alpha(H_2)=\max(\mathbf{1}_{x\in S}, |S\cap(V_2\setminus\{x\})|)$. Since H_4 is an induced subgraph of the edge uw, $\alpha(H_4)=\min(1,|S\cap\{u,w\}|)$. As $V_3\setminus\{u\}$ is independent on the rim and $w\notin V_3$,

$$\alpha(H_3)=|S\cap(V_3\setminus\{u\})|+\mathbf{1}_{[u\in S,\, v_{2t-2}\notin S]}.$$

We verify $\sum_{\sigma=1}^{4}\alpha(H_\sigma)\geq|S|$ in three cases.

Case (i). $x\notin S$.

Then $|S|=|S\cap(V_2\setminus\{x\})|+|S\cap(V_3\setminus\{u\})|+|S\cap\{u,w\}|$. With $\alpha(H_2)=|S\cap(V_2\setminus\{x\})|$ and $\alpha(H_3)\geq|S\cap(V_3\setminus\{u\})|$, it suffices to cover $|S\cap\{u,w\}|$. If $|S\cap\{u,w\}|\leq 1$ then $\alpha(H_4)$ suffices. If $u,w\in S$, then $w\in V_1$ gives $\alpha(H_1)\geq 1$, so $\alpha(H_1)+\alpha(H_4)\geq 2$.

Case (ii). $x\in S$, $S\cap(V_2\setminus\{x\})=\emptyset$.

Then $S\setminus\{x\}\subseteq(V_3\setminus\{u\})\cup\{u,w\}$, and $S\cap(V_3\setminus\{u\})$ together with $\{w\}\cap S$ form an independent set in H_1-x, so $\alpha(H_1)\geq|S\cap(V_3\setminus\{u\})|+\mathbf{1}_{[w\in S]}$. Summing with $\alpha(H_2)=1$ and $\alpha(H_4)\geq\mathbf{1}_{[u\in S]}$ gives $\alpha(H_1)+\alpha(H_2)+\alpha(H_4)\geq|S\cap(V_3\setminus\{u\})|+\mathbf{1}_{[w\in S]}+1+\mathbf{1}_{[u\in S]}=|S|$.

Case (iii). $x\in S$, $S\cap(V_2\setminus\{x\})\neq\emptyset$.

Here $\alpha(H_2)=|S\cap(V_2\setminus\{x\})|$ and $\alpha(H_1)\geq 1$. It suffices to show $\alpha(H_1)+\mathbf{1}_{[u\in S,\, v_{2t-2}\notin S]}+\min(1,|S\cap\{u,w\}|)\geq 1+|S\cap\{u,w\}|$. If $|S\cap\{u,w\}|\leq 1$ this follows from $\alpha(H_1)\geq 1$. If $u,w\in S$ and $v_{2t-2}\notin S$, the indicator provides $+1$. If $u,w,v_{2t-2}\in S$, then $u\notin V_1$ makes v_{2t-2} and w nonadjacent in H_1-x, yielding $\alpha(H_1)\geq 2$. $\square$

Claims 5 and 6 give $h(W_n)\geq 3$, completing Case 1.

Case 2. $n=2t$ even, $t\geq 3$.

The rim C_{2t-1} is an odd cycle, so $\chi(W_n)=4$. Every induced subgraph $H\subseteq W_n$ contains a vertex of degree at most 3: if $x\notin V(H)$ then H is a subgraph of C_{2t-1} and every vertex has degree at most 2; if $x\in V(H)$ every rim vertex has at most two rim-neighbors and one spoke. Hence W_n is 3-degenerate, and $\mathrm{ch}(W_n)\leq d(W_n)+1\leq 4$ [4]. As $\chi(W_n)=4$, we have $\mathrm{ch}(W_n)=4$, and (H1) yields $h(W_n)\leq 4$.

Claim 7. W_n is not 3-colorable.

Proof. Immediate from $\chi(W_n)=4>3$. $\square$

Claim 8. The uniform assignment $L(v)=\{1,2,3\}$ for all $v\in V(W_n)$ satisfies Hall's condition on every induced subgraph of W_n.

Proof. Since every vertex carries the same list, $H_\sigma=H$ for each σ, and Hall's condition reduces to $3\alpha(H)\geq|V(H)|$. For a proper induced subgraph $H\subsetneq W_n$, we have $\chi(H)\leq 3$:

if $x \notin V(H)$, then H is an induced subgraph of C_{2t-1}, hence a disjoint union of paths, which is 2-colorable; if $x \in V(H)$, then $H - x$ is an induced subgraph of a path, hence bipartite, so H is 3-colorable. In either case $\alpha(H) \geq |V(H)|/\chi(H) \geq |V(H)|/3$. For $H = W_n$, $\alpha(W_n) = \alpha(C_{2t-1}) = t - 1$, so $3\alpha(W_n) = 3t - 3 \geq 2t = |V(W_n)|$ for $t \geq 3$. ☐

Claims 7 and 8 together show that $L(v) = \{1, 2, 3\}$ satisfies Hall's condition on every induced subgraph of W_n (including W_n itself) yet W_n is not L-colorable, so $h(W_n) \geq 4$, and hence $h(W_n) = 4$. ☐

5. Conclusion

The results obtained here isolate three distinct structural sources governing the identity $h(G) = \mathrm{ch}(G)$.

For complete multipartite graphs $K(n_1, \ldots, n_k)$ with all parts of size at least 2, the equality is inherited from the induced core $K(2, \ldots, 2)$ and lifted to the full graph via induced-subgraph monotonicity (H2). Thus in this setting Hall–colorability is controlled by a rigid multipartite kernel.

For 2-trees, the mechanism is obstruction-based. The five-vertex graphs F_4 and B_3 (Figure 1) constitute the minimal induced Hall-tight configurations. Every 2-tree on at least five vertices contains one of these as an induced subgraph, forcing $h(G) \geq 3$, while 2-degeneracy ensures $\mathrm{ch}(G) \leq 3$. Here equality emerges from the interplay between minimal induced obstructions and bounded degeneracy.

For wheels, parity governs the outcome. If the rim is even (n odd), then W_n is 3-choosable and $h(W_n) = \mathrm{ch}(W_n) = 3$. If the rim is odd ($n$ even), the uniform 3-assignment satisfies Hall's condition on every induced subgraph, including W_n itself, yet W_n is not 3-colorable; hence $h(W_n) = 4 = \mathrm{ch}(W_n)$. In this case Hall's condition detects precisely the chromatic obstruction created by the odd rim.

To summarize, we established exact Hall numbers for all 2-trees and all wheel graphs, proves $h(G) = \mathrm{ch}(G)$ for every complete multipartite graph without singleton parts, and proves the eventual equality $h(G) = \mathrm{ch}(G)$ for $K(m, n, 1, \ldots, 1)$ as $n \to \infty$.

These examples suggest several natural directions. For k-trees, does there exist a threshold $n_0(k)$ such that $h(G) = \mathrm{ch}(G) = k + 1$ for every k-tree G with $|V(G)| \geq n_0(k)$? More generally, can a chordal graph satisfy $h(G) < \mathrm{ch}(G)$? Ohba's theorem [10] asserts that $\mathrm{ch}(G) = \chi(G)$ whenever $|V(G)| \leq 2\chi(G) + 1$. Whether an analogous sharp bound forces $h(G) = \chi(G)$ remains an open problem.

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Appendix A: Vertex-deleted colorings for F_4 and B_3

The following tables record the proper L-colorings used in Claims 2 and 4. Each row gives an explicit proper L-coloring of the induced subgraph $G - v$ under the list assignment defined in the corresponding lemma; a dash indicates the deleted vertex.

Table 1. Proper L-colorings of the vertex-deleted subgraphs of F_4, with $L(a) = L(b) = L(e) = \{1, 2\}$ and $L(c) = L(d) = \{1, 3\}$.

Subgraph	$\varphi(a)$	$\varphi(b)$	$\varphi(c)$	$\varphi(d)$	$\varphi(e)$
$F_4 - a$	—	2	1	3	2
$F_4 - b$	2	—	1	3	1
$F_4 - c$	1	2	—	3	2
$F_4 - d$	1	2	3	—	2
$F_4 - e$	2	1	3	1	—

Table 2. Proper L-colorings of the vertex-deleted subgraphs of B_3, with $L(a) = \{1, 2\}$, $L(b) = \{1, 3\}$, $L(c) = \{1, 2\}$, $L(d) = \{1, 3\}$, $L(e) = \{2, 3\}$.

Subgraph	$\varphi(a)$	$\varphi(b)$	$\varphi(c)$	$\varphi(d)$	$\varphi(e)$
$B_3 - a$	—	1	2	3	2
$B_3 - b$	1	—	2	3	3
$B_3 - c$	2	1	—	3	3
$B_3 - d$	1	3	2	—	2
$B_3 - e$	2	3	1	1	—