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Hamiltonicity of Bell and Stirling colour graphs

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ABSTRACT

For a graph G and a positive integer k, the k-Bell colour graph of G is the graph whose vertices are the partitions of V into at most k independent sets, with two of these being adjacent if there exists a vertex x such that the partitions are identical when restricted to $V - \{x\}$. The k-Stirling colour graph of G is defined similarly, but for partitions into exactly k independent sets. Building on the existing result that for each $k \geq 3$, the k-Bell colour graph of a tree with at least 4 vertices is Hamiltonian, we show that every graph on n vertices, except K_n and $K_n - e$, has a Hamiltonian n-Bell colour graph, and this result is best possible. It is also shown that, for $k \geq 4$, the k-Stirling colour graph of a tree with at least $k+1$ vertices is Hamiltonian.

Keywords: Hamilton cycle, reconfiguration problem, combinatorial Gray code, colour graph

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1. Introduction

For a set S of combinatorial objects, the *reconfiguration graph for S* has the elements of S as its vertices, with two of these being adjacent if they differ in some small, specified way. When this graph is connected, any element of S can be reconfigured into any other via a sequence of small changes of the specified type. When it is Hamiltonian, there is a cyclic list that contains all elements of S, and consecutive elements of the list differ by a small change of the specified type (a cyclic *combinatorial Gray code* for the objects in question). The paper by Ito [11] and the survey by van den Heuvel [14] give a sense of the literature and wide variety of reconfiguration problems that have been considered. The survey by Savage [12] gives pointers to the vast literature on combinatorial Gray codes.

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In this paper, we are interested in combinatorial Gray codes for a type of graph colouring.

A considerable amount is known about the *k-colour graph* of a graph G, denoted by $\mathcal{C}_k(G)$ (e.g., see [14]). It has the k-colourings of G enumerated by the chromatic polynomial as vertices, with two of these being adjacent when they differ in the colour of exactly one vertex. This graph is connected whenever k is at least one more than the colouring number of G [7], and is Hamiltonian when k is at least two more than the colouring number of G [5]. Necessary, and usually sufficient conditions, on k for the existence of a Hamilton cycle in a k-colour graph of a tree, cycle, complete graph, complete bipartite graph, 2-tree, or complete multipartite graph have been found [1, 2, 4, 5].

Much less is known about the *canonical k-colouring of a graph G with respect to a vertex ordering $\pi = x_1, x_2, \ldots, x_n$*. A k-colouring of G is *canonical* if it has the property that, for every $c \geq 1$, if x_j is assigned colour c then each of the colours $1, 2, \ldots, c-1$ has been assigned some vertex that precedes x_j in π. The *canonical k-colour graph of G with respect to π*, denoted $\mathrm{Can}_k^\pi(G)$, has as vertices the canonical k-colourings of G with respect to π, with two canonical k-colourings c_1 and c_2 being adjacent when they agree on all but one vertex of G. Haas proved that for any tree T with at least 4 vertices there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(G)$ has a Hamilton cycle for all $k \geq 3$ [9]. By contrast, a canonical k-colouring graph of a complete multipartite graph almost never has a Hamilton path or cycle, but for all $k \geq 3$ there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(K_{m,n})$ has a Hamilton path for $m, n \geq 2$ [10].

For a given graph G and positive integer k, the *k-Bell colour graph of G*, denoted $\mathcal{B}_k(G)$, has as vertices the set of partitions of $V(G)$ into k or fewer independent sets, with different partitions p_1 and p_2 being adjacent if there is a vertex $x \in V(G)$ such that the restrictions of p_1 and p_2 to $V(G) - \{x\}$ are equal. The *k-Stirling colour graph*, $\mathcal{S}_k(G)$, is defined similarly, but for partitions of $V(G)$ into exactly k independent sets. Observe that $\mathcal{S}_k(G)$ is the subgraph of $\mathcal{B}_k(G)$ induced by $V(\mathcal{B}_k(G)) - V(\mathcal{B}_{k-1}(G))$.

The name k-Bell colour graph of the graph G comes from the k-Bell number of G, which is the number of partitions of $V(G)$ into at most k independent sets. Similarly, the k-Stirling number of G is the number of partitions of $V(G)$ into exactly k independent sets. Notice that the Bell number, $B(n, k)$, is the number of partitions of the vertices of $\overline{K}_n$, into at most k independent sets, and the Stirling number of the second kind, $S(n, k)$, is the number of partitions of the vertices of $\overline{K}_n$ into exactly k independent sets. Refer to [6, 8] for pointers to the literature on Bell and Stirling numbers of graphs.

The k-Bell colour graph has been considered by Haas under the name the *isomorphic colour graph* [9]. This name arises from defining two vertices of $C_k(G)$ to be *equivalent* (or, saying that these colourings are isomorphic) if they give rise to the same partition of the vertex set into independent sets. The vertices of $\mathcal{B}_k(G)$ correspond to these equivalence classes, with $[x]$ and $[y]$ being adjacent if and only if some member of $[x]$ is adjacent to a member of $[y]$ in $\mathcal{C}_k(G)$. The graph $\mathrm{Can}_k^\pi(G)$ is the subgraph of $C_k(G)$ induced by the set of lexicographically least representatives with respect to π from the equivalence classes. It follows that, for any vertex ordering π, the graph $\mathrm{Can}_k^\pi(G)$ is a spanning subgraph of $\mathcal{B}_k(G)$. Hence, if there exists a vertex ordering π such that $\mathrm{Can}_k^\pi(G)$ is connected, or Hamiltonian, then the same is true of $\mathcal{B}_k(G)$.

In Section 2 we give some of the basic properties of k-Bell colour graphs and in Section 3 we show that, for every graph which is neither a complete graph nor a complete graph less an edge, there is a threshold k_0, so that for every $k \geq k_0$, the k-Bell colour graph of G is Hamiltonian. In Section 4, we show that for $k \geq 4$, the k-Stirling colour graph of any tree with at least $k+1$ vertices is Hamiltonian, but the 3-Stirling colour graph of a star with an odd number of vertices is not Hamiltonian. Since we are viewing colourings as partitions of the vertex set into independent sets, through the remaining sections we will use the terms "colouring" and "partition" interchangeably in cases where each colour class is non-empty.

2. Known results and basic properties

In this section we explore properties of the k-Bell colour graph of G. While this graph is defined for every non-negative integer k, for every $k \geq |V(G)|$ we have $\mathcal{B}_k(G) \cong \mathcal{B}_{|V(G)|}(G)$. By contrast, the k-colour graph of G is different for every positive integer $k \geq \chi(G)$.

Haas proved that, for any integer $k \geq 3$ and any tree T with at least 4 vertices, there is an enumeration π of $V(T)$ such that $\mathrm{Can}_k^\pi(T)$ is Hamiltonian. Hence we have the following.

Theorem 2.1. [9] *For any tree with at least four vertices, $\mathcal{B}_k(T)$ is Hamiltonian for every $k \geq 3$.*

Corollary 2.2. *For any tree with at least three vertices, $\mathcal{S}_3(T)$ has a Hamilton path.*

Proof. The only tree with 3 vertices is P_3, and $\mathcal{S}_3(P_3) \cong K_1$. Suppose T is a tree with at least 4 vertices. By Theorem 2.1, $\mathcal{B}_3(T)$ is Hamiltonian. The graph $\mathcal{S}_3(T)$ is obtained by deleting the unique 2-colouring of T from $\mathcal{B}_3(T)$. Deleting a vertex from a graph with a Hamilton cycle gives a graph with a Hamilton path. □

The *colouring number of G*, denoted *col*(G), is the smallest integer c for which there exists an ordering of the vertices such that for all i, the degree of the ith vertex in the subgraph induced by the first i vertices in the ordering is less than c.

Proposition 2.3. [9] *If $\mathcal{C}_k(G)$ is connected, then $\mathcal{B}_k(G)$ is connected.*

Since $\mathcal{C}_k$ is connected when $k \geq \mathrm{col}(G)+1$ [7] (also see [3]), the following is now an immediate consequence of Proposition 2.3.

Corollary 2.4. *If $k \geq \mathrm{col}(G)+1$, then $\mathcal{B}_k(G)$ is connected.*

The graph $L_{n,n} = K_{n,n} - M$, where M is a perfect matching, shows that this bound is best possible. It is easy to see that $col(L_{n,n}) = n$. On the other hand, the n-colouring (partition) where each pair of end vertices of the edges in M are assigned the same colour has no neighbours in $\mathcal{B}_n(L_{n,n})$.

Proposition 2.5. *Let k be a positive integer, and H be a uniquely k-colourable graph. If G and H are disjoint, then $\mathcal{B}_k(H \cup G) \cong \mathcal{C}_k(G)$.*

Proof. Suppose that H is a uniquely k-colourable graph. Then H has chromatic number k. Let $\{X_1, X_2, \dots, X_k\}$ be the unique partition of $V(H)$ into independent sets.

Consider the following mapping f from $V(\mathcal{C}_k(G))$ to $V(\mathcal{B}_k(H \cup G))$. Let c be a vertex of $\mathcal{C}_k(G)$. Then c is function from $V(G)$ to $\{1, 2, \dots, k\}$ such that if $xy \in E(G)$ then $c(x) \neq c(y)$. Define $f(c)$ to be the partition $\{c^{-1}(i) \cup X_i : 1 \leq i \leq k\}$ of $V(G \cup H)$. In the mapping f, the cells in the unique partition of $V(H)$ into k independent sets distinguish the colour classes with respect to c. This mapping is clearly a isomorphism. □

Let $G \vee H$ denote the *join* of the disjoint graphs G and H and $G \square H$ denote their *Cartesian product.*

Proposition 2.6. *Let G and H be disjoint graphs and let $k = |V(G \vee H)|$. Then $\mathcal{B}_k(G \vee H) \cong \mathcal{B}_k(G) \square \mathcal{B}_k(H)$.*

Proof. In the graph $G \vee H$, every vertex in the copy of G is adjacent to every vertex in the copy of H. Thus, there is a natural 1-1 correspondence between the set of ordered pairs (P_1, P_2), where P_1 is a partition of $V(G)$ into independent sets and P_2 is a partition of $V(H)$ into independent sets, and the set of partitions of $V(G \vee H)$ into independent sets: the ordered pair (P_1, P_2) corresponds to the partition $P_1 \cup P_2$. It is clear that (P_1, P_2) and (Q_1, Q_2) are adjacent in $\mathcal{B}_k(G \vee H)$ if and only if $P_1 = Q_1$ and $P_2 Q_2 \in E(\mathcal{B}_k(H))$, or $P_2 = Q_2$ and $P_1 Q_1 \in E(\mathcal{B}_k(G))$. The result follows. □

We will require a small variation of the Cartesian product in our subsequent work. Let $K_r \square^+ K_s$ be the graph obtained from $K_r \square K_s$ as follows. Let (a, b) be a vertex of $K_r \square K_s$. Add a new vertex v and edges joining v to (a, b) and all neighbours of (a, b). Clearly any two graphs constructed in this way are isomorphic. In the lemma below, we will refer to (a, b) as the *cloned vertex.*

Lemma 2.7. *Let r and s be positive integers. Then, in the graph $K_r \square^+ K_s$ there is a Hamilton path from the cloned vertex to every other vertex.*

Proof. If $r = 1$, then $K_r \square^+ K_s \cong K_{s+1}$ and the statement follows. The case where $s = 1$ is similar. For $r > 1$ and $s > 1$, the graph $K_r \square K_s$ is Hamilton connected unless $r = s = 2$. Thus, if $r \neq 2$ or $s \neq 2$, the statement follows immediately. It is easy to directly check that the statement holds when $r = s = 2$. □

3. Every graph has a Hamiltonian Bell colour graph

We note that, for any $k \geq n$, the graph $B_k(K_n) \cong K_1$, and hence is not Hamiltonian. Similarly, for any $k \geq n - 1$, the graph $B_k(K_n - e)$ is isomorphic to K_1 or K_2, and hence is not Hamiltonian.

Theorem 3.1. *For any graph G on n vertices which is neither K_n nor $K_n - e$, $\mathcal{B}_n(G)$ is Hamiltonian.*

Proof. Let G be a smallest counterexample. Let $|V(G)|= n$. Since G is not complete, there exist two non-adjacent vertices x and y. Let $G' = G - x - y$.

Every colouring of G can be regarded as an extension of a colouring of G'. Any fixed colouring of G' can be extended to a colouring of G in five possible ways:

(1) both x and y are each assigned the same colour as a vertex of G';
(2) x is assigned a new colour, and y is assigned the same colour as a vertex of G';
(3) x is assigned the same colour as a vertex of G', and y is assigned a new colour;
(4) x and y are assigned different new colours;
(5) x and y are both assigned the same new colour.

Let c be a fixed colouring of G'. Suppose that, in any extension of c to a colouring of G, there are $l_1 - 1$ existing colours available for x, and $l_2 - 1$ existing colours available for y. Then, since colourings are partitions of $V(G)$ into independent sets, there are $l_1 l_2$ extensions of c corresponding to possibilities (1) through (4), and one more corresponding to possibility (5). The subgraph of $\mathcal{B}_n(G)$ induced by any set of extensions that agree on x is a complete graph. Similarly, the subgraph induced by any set of extensions that agree on y is a complete graph. It follows that the subgraph of $\mathcal{B}_n(G)$ induced by the extensions of c corresponding to possibilities (1) through (4) has $K_{l_1} \square K_{l_2}$ as a spanning subgraph. Since the extension corresponding to possibility (5) has the same neighbours in $\mathcal{B}_n(G)$ as the one corresponding to possibility (4), we have shown that the subgraph of $\mathcal{B}_n(G)$ induced by the extensions of c has $K_{l_1} \square^+ K_{l_2}$ as a spanning subgraph.

For a colouring c of G', we will denote the extension of c to G in which x and y are assigned the same new colour by $c(xy)$, and the extension of c to G in which x and y are assigned different new colours by $c(x|y)$. Clearly, if c_1 is adjacent to c_2 in $\mathcal{B}_n(G')$, then $c_1(xy)$ is adjacent to $c_2(xy)$ and $c_1(x|y)$ is adjacent to $c_2(x|y)$ in $\mathcal{B}_n(G)$.

Since G is not a complete graph minus an edge, the graph G' has a nonempty vertex set.

Suppose G' is complete. Then it has only one colouring, c, and every colouring of G is an extension of c. Therefore there are positive integers r and s such that $\mathcal{B}_n(G)$ is a spanning supergraph of $K_s \square^+ K_r$. Since G is not a complete graph minus an edge, either x or y must be non-adjacent to at least one vertex of G'. Hence at least one of r and s is at least 2. It now follows from Lemma 2.7 that $\mathcal{B}_n(G)$ is Hamiltonian.

Now suppose G' is a complete graph minus an edge. Then G' has exactly two colourings. Let c_1 be the colouring of G' where the two non-adjacent vertices are assigned the same colour, and let c_2 be the colouring where the two non-adjacent vertices in G' are assigned different colours. By Lemma 2.7, there is a Hamilton path P_1, between $c_1(xy)$ and $c_1(x|y)$ in the subgraph of $\mathcal{B}_n(G)$ induced by the extensions of the colouring c_1. Similarly, there is a Hamilton path between $c_2(x|y)$ and $c_2(xy)$ in the subgraph of $\mathcal{B}_n(G)$ induced by the extensions of the colouring c_2. The paths P_1 and P_2 and the edges $c_1(xy)c_2(xy)$ and $c_1(x|y)c_2(x|y)$ are a Hamilton cycle in $\mathcal{B}_n(G)$.

Finally, suppose G' is neither a complete graph nor a complete graph minus an edge.

Since G is a smallest counterexample, $\mathcal{B}_n(G') = \mathcal{B}_{n-2}(G')$ has a Hamilton cycle, $c_1, c_2 \ldots c_{m'}$, where $|V(\mathcal{B}_n(G'))| = m'$. By Lemma 2.7, the subgraph of $\mathcal{B}_n(G)$ induced by the extensions of colouring c_i to G has a Hamilton path, P_i, from $c_i(xy)$ to $c_i(x|y)$.

If m' is even then the paths $P_1, P_2, \ldots, P_{m'}$, together with the edges $c_i(xy)c_{i+1}(xy)$ for even $i < m'$, the edges $c_i(x|y)c_{i+1}(x|y)$ for odd i and the edge $c_{m'}(xy)c_1(xy)$ are a Hamilton cycle.

We now consider the situation when m' is odd. First, observe that if $deg_G(x) = deg_G(y) = n-2$, then for each i, c_i has exactly two extensions, $c_i(xy)$ and $c_i(x|y)$. In this case, the edges $c_i(xy)c_{i+1}(xy)$ and $c_i(x|y)c_{i+1}(x|y)$ for $i = 1, 2, \ldots m'-1$, in conjunction with $c_1(xy)c_1(x|y)$ and $c_{m'}(xy)c_{m'}(x|y)$ are a Hamilton cycle. Hence assume that at least one of x and y has degree at most $n-3$.

Without loss of generality, c_1 is the colouring in which every vertex of G' is assigned a different colour, so that $n-2$ colours are used and every colour class is a singleton. Then in the colouring c_2, there is exactly one colour class of size 2, say $\{z_1, z_2\}$.

We claim that x and y are adjacent to every vertex in $V(G')-\{z_1, z_2\}$. Suppose, by way of contradiction, x is not adjacent to $v \in V(G')-\{z_1, z_2\}$. Let w_1 and w_2 be, respectively, be the extensions of c_1 and c_2 in which x is assigned the same colour as v, and y is assigned a new colour. Note that w_1 and w_2 are adjacent. By Lemma 2.7, there is a Hamilton path, Q_1, from $c_1(xy)$ to w_1 in the subgraph induced by the extensions of c_1 to colourings of G, and a Hamilton path Q_2 from $c_2(x|y)$ to w_2 in the subgraph induced by the extensions of c_2 to colourings of G. The paths $Q_1, Q_2, P_3, P_4, \ldots, P_{m'}$, together with the edges w_1w_2, $c_{m'}(xy)c_1(xy)$, edges of the form $c_i(xy)c_{i+1}(xy)$ when i is odd $(3 \le i < m')$, and of the form $c_i(x|y)c_{i+1}(x|y)$ when i is even, are a Hamilton cycle in $\mathcal{B}_n(G)$. A contradiction, as G is a counterexample, which proves the claim.

Next, we claim that x is adjacent, in G, to either z_1 or z_2 and that y is adjacent, in G, to either z_1 or z_2 . Suppose, by way of contradiction, x is adjacent, in G, to neither z_1 nor z_2. Let w_1 be the extension of c_1 in which x is assigned the same colour as z_1 and y is assigned a new colour. Let w_2 be the extension of c_2 in which x, z_1 and z_2 are all assigned the same colour, and y is assigned a new colour. The same technique as in the first claim yields a Hamilton cycle in $\mathcal{B}_n(G)$, contradicting G is a counter example, which proves the claim.

Recall G is a minimal counterexample. By the first claim x is adjacent to every vertex of $V(G') \setminus \{z_1, z_2\}$. By the second claim x is adjacent to either z_1 or z_2 or both. We conclude that $deg_G(x) \ge n-3$ and x is adjacent to every vertex of $V(G') \setminus \{z_1, z_2\}$. The same argument implies $deg_G(y) \ge n-3$ and y is adjacent to every vertex of $V(G') \setminus \{z_1, z_2\}$. Furthermore, as x and y were chosen to be arbitrary non-adjacent vertices in G, it follows that $\delta(G) \ge n-3$.

By our previous argument, one of x or y has degree at most $n-3$, so that one of them has degree $n-3$, say x. Without loss of generality, x is not adjacent to z_1, and is adjacent to every vertex in $V(G') \setminus \{z_1\}$.

By choice of c_1, in the colouring $c_{m'}$ there is exactly one colour class of size two, say $\{a, b\}$, and all other colour classes have size 1. It follows from symmetry and the first claim that $z_1 \in \{a, b\}$; without loss of generality, $a = z_1$. Since $c_2 \ne c_{m'}$, $b \ne z_2$. Therefore,

z_1 is not adjacent to any of the vertices x, z_2 and b, so that $deg_G(z_1) \leq n-4 < \delta(G)$, a contradiction. It follows that G does not exist. □

It follows from Theorem 3.1 that, for every other graph G, there is a least integer n_0 such that $B_k(G)$ is Hamiltonian whenever $k \geq n_0$. We now show that there exist graphs for which the k-Bell colour graph is Hamiltonian if and only if $k = |V(G)|$, that is, for which $n_0 = |V(G)|$.

For $t \geq 1$, let $G_t = K_{2t} - M$, where $M = \{x_1y_1, x_2y_2, \dots, x_ty_t\}$ is a perfect matching. Then $\chi(G_t) = t$. For each ℓ with $0 \leq \ell \leq t$, there is a 1-1 correspondence between the colourings of G_t with $t+\ell$ colours and the ℓ-element subsets of M consisting of the edges of M whose ends are assigned different colours. Thus there is a 1-1 correspondence between the vertices of $\mathcal{B}_{t+\ell}(G_t)$ and the binary sequences of length t with at most ℓ ones: the i-th term of the sequence equals 1 if x_i and y_i are assigned different colours, and equals 0 otherwise. Two vertices of $\mathcal{B}_{t+\ell}(G_t)$ are adjacent if and only if the corresponding sequences differ in exactly one place. Thus, $\mathcal{B}_{t+\ell}(G_t)$ is the subgraph of the t-dimensional hypercube induced by the binary sequences with at most ℓ ones. This graph is bipartite, and has bipartition (A, B), where A is the set of binary sequences with an even number of ones (and at most ℓ ones), and B is the set of binary sequences with an odd number of ones (and at most ℓ ones). We claim that if $\ell < t$, then $|A| \neq |B|$, hence $\mathcal{B}_{t+\ell}(G_t)$ is not Hamiltonian. Now,

$$|A| - |B| = \binom{t}{0} - \binom{t}{1} + \binom{t}{2} - \cdots + (-1)^t \binom{t}{\ell} = \pm \binom{t-1}{\ell} \neq 0,$$

where we have used the result of [13], page 128, question 44(c). Therefore, $\mathcal{B}_{t+\ell}(G_t)$ is not Hamiltonian for all $\ell < t$, that is, for all $\ell + t < 2t = |V(G_t)|$.

4. The Hamiltonicity of Stirling colour graphs of trees

In this Section, we explore the Hamiltonicity of the k-Stirling colour graph of trees. The $|V(G)|$-Stirling colour graph is a singleton and not Hamiltonian, therefore we consider first the case where $|V(G)|-1$ colours are allowed.

Lemma 4.1. *Let G be a graph on n vertices which is not complete. Then*

(i) *there is a 1-1 correspondence between the set of colourings of G that use exactly $n-1$ colours and the set of edges of $\overline{G}$;*

(ii) *the graph $S_{n-1}(G)$ is isomorphic to the line graph of $\overline{G}$.*

(iii) *the graph $S_{n-1}(G)$ is connected if and only if $\overline{G}$ is connected;*

(iv) *the graph $S_{n-1}(G)$ has a Hamilton cycle if and only if $\overline{G}$ has a circuit that contains an endpoint of every edge of $\overline{G}$.*

Proof. In a colouring of G that uses exactly $n-1$ colours there are exactly two vertices that are assigned the same colour, say x and y. Then $xy \notin E(G)$, so $xy \in E(\overline{G})$. This proves (i).

Let c_1 and c_2 be two colourings of G that use exactly $n-1$ colours. These are adjacent in $S_{n-1}(G)$ if and only if there is a vertex x which belongs to the unique cell of size two in each one. That is, if and only if c_1 and c_2 correspond to adjacent edges of $\overline{G}$. This proves (ii). Statement (iii) now follows from properties of the line graph, as does statement (iv). □

Lemma 4.2. *Let T be a tree on $n \geq 5$ vertices. Then $S_{n-1}(T)$ is Hamiltonian.*

Proof. The statement follows from part *(iv)* of Lemma 4.1 on noting that if T is not a star, then $\overline{T}$ is Hamiltonian, and if T is a star, then $\overline{T}$ has a cycle of length $n-1$. □

We now consider the k-Stirling colour graph of trees using fewer colours. We use Q_n to denote the n-dimensional hypercube. The vertices of Q_n are the binary sequences of length n, and the partition of $V(Q_n)$ into the set of binary sequences with an even number of ones, and the set of sequences with an odd number of ones, is a bipartition. We will use the following well known result, which can be proved by induction.

Lemma 4.3. *Suppose $x, y \in V(Q_n)$ be vertices belonging to different cells of the bipartition. Then, there is a Hamilton path from x to y.*

Any non-trivial tree, T, is uniquely 2-colourable; hence $\mathcal{B}_2(T) = \mathcal{S}_2(T) \cong K_1$. The graph $\mathcal{S}_3(T)$ can have a much richer structure. The star is a special case, similarly to the situation for k-colour graphs [5]. We now characterize the situations where $\mathcal{S}_3(K_{1,n})$ is, and is not, Hamiltonian. By contrast, Theorem 2.1 states that for any tree with at least four vertices, the graph $\mathcal{B}_3(T)$ is Hamiltonian.

Theorem 4.4. *For any $n \geq 2$, $\mathcal{S}_3(K_{1,n})$ is Hamiltonian if and only if n is odd.*

Proof. The vertex of degree n in $K_{1,n}$ will always be in its own cell in a partition of the vertices into independent sets. Hence we must consider the partitions of n independent vertices (each of which has degree 1) into exactly two cells. One of the leaves, say x, is used to label the two cells and will be considered to correspond to colour 0. The colourings can now be listed as the binary sequences of length $n-1$ containing at least one 1. Two of these are adjacent in $\mathcal{S}_3(K_{1,n})$ if they either differ in exactly one entry (corresponding to colourings that agree on $K_{1,n} - y$, where $y \neq x$) or they differ in every entry (corresponding to colourings that agree on $K_{1,n} - x$). This graph is obtained from an $n-1$ dimensional hypercube by adding edges joining antipodal vertices, and deleting the vertex corresponding to the sequence of all zeros.

For even n, it is easy to verify that the graph is bipartite (the sequences with an even (odd) number of ones are independent). Hence $\mathcal{S}_3(K_{1,n})$ is a bipartite graph with an odd number of vertices and not Hamiltonian. For $n = 3$, $\mathcal{S}_3(K_{1,3}) = K_3$ and hence is Hamiltonian.

Let $n \geq 5$ be an odd integer. By the above discussion, the subgraph of $\mathcal{S}_3(K_{1,n})$ induced by the binary sequences in which the first two elements are 0 is isomorphic to

$Q_{n-3} - \{000\dots0\}$. As Q_{n-3} is Hamiltonian, by symmetry, there is a Hamilton cycle in Q_{n-3} on which the vertices adjacent to $000\dots000$ are $000\dots001$ and $000\dots010$. It follows that in $Q_{n-3} - \{000\dots0\}$ there is a Hamilton path from $000\dots001$ to $000\dots010$. This corresponds to a path in $\mathcal{S}_3(K_{1,n})$ containing all of the binary sequences in which the first two elements are 0.

The subgraph of $\mathcal{S}_3(K_{1,n})$ induced by the binary sequences where the first two elements are either 10 or 11 is isomorphic to Q_{n-2}. By Lemma 4.3 and since n is odd (and therefore the length of each binary sequence of length $n-1$ is even), there is a Hamilton path, P, in this graph from $100\dots001$ to $111\dots101$ which corresponds to a path in $\mathcal{S}_3(K_{1,n})$ containing all of the binary sequences in which the first two elements are 10 or 11. Further, we note that in the subgraph where the first two elements are 10 or 11, each element of the form $11x$ has exactly one neighbour where the first two elements are 10. It follows that there are two consecutive vertices on P in which the first two entries are 11.

As $100\dots001$ is adjacent to $000\dots001$ and $111\dots101$ is adjacent to $000\dots010$, there is a cycle, C, containing all of the binary sequences which start with 00, 10 or 11 and, by the above, there are two consecutive vertices on C which begin with 11, say $11x$ and $11y$.

The subgraph of $\mathcal{S}_3(K_{1,n})$ induced by the binary sequences where the first two elements are 01 is isomorphic to Q_{n-3}. Let P be the path obtained from C by deleting the edge joining $11x$ and $11y$. By Lemma 4.3, there is a path in $\mathcal{S}_3(K_{1,n})$ from $01x$ to $01y$ containing all of the vertices in which the first two elements are 01. It now follows that there is a Hamilton cycle in $\mathcal{S}_3(K_{1,n})$, when n is odd. □

From Theorem 4.4, $\mathcal{S}_3(T)$ is not necessarily Hamiltonian. Lemma 4.5 is a technical Lemma that can used in the proof that $\mathcal{S}_4(T)$ is Hamiltonian for trees with sufficient vertices.

Lemma 4.5. *For any tree T with at least four vertices, and any vertex $x \in V(T)$, there are distinct vertices $a, b \in V(T)$ and a Hamilton path P in $S_3(T)$ for which the end vertices are $\{A-\{a\}, B-\{a\}, \{a\}\}$ and $\{A-\{b\}, B-\{b\}, \{b\}\}$ where $x \neq a$, $x \neq b$ and $\{A, B\}$ is the 2-colouring of T.*

Proof. Suppose T is the star on $n+1$ vertices. By Theorem 2.1, $\mathcal{B}_3(T)$ is Hamiltonian. The two neighbours of the two colouring of T in every Hamilton cycle of $\mathcal{B}_3(T)$ are of the form $\{\{s\}, X \cup \{b\}, \{a\}\}$ and $\{\{s\}, X \cup \{a\}, \{b\}\}$ where s is the vertex of degree n and a and b are leaves. By the symmetry of stars, this implies the lemma holds for stars.

We need to establish the result for trees which are not stars. The proof is by induction on the number of vertices. If $|V(T)| = 4$, then $T \cong P_4$ and an appropriate Hamilton path is given in Figure 1 for any choice of the vertex x. If $|V(T)| = 5$, then either $T \cong P_5$ (and appropriate Hamilton paths are given in Figure 2 for any choice of the vertex x) or T is isomorphic to the tree shown in Figure 3 (and two appropriate Hamilton paths in $\mathcal{S}_3(T)$ are given depending on the choice of the vertex x). In all three cases the lemma holds.

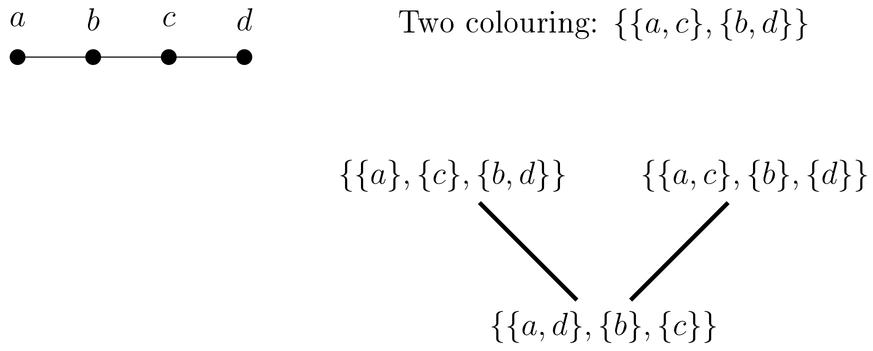

Fig. 1. The 3-Stirling colour graph of P_4

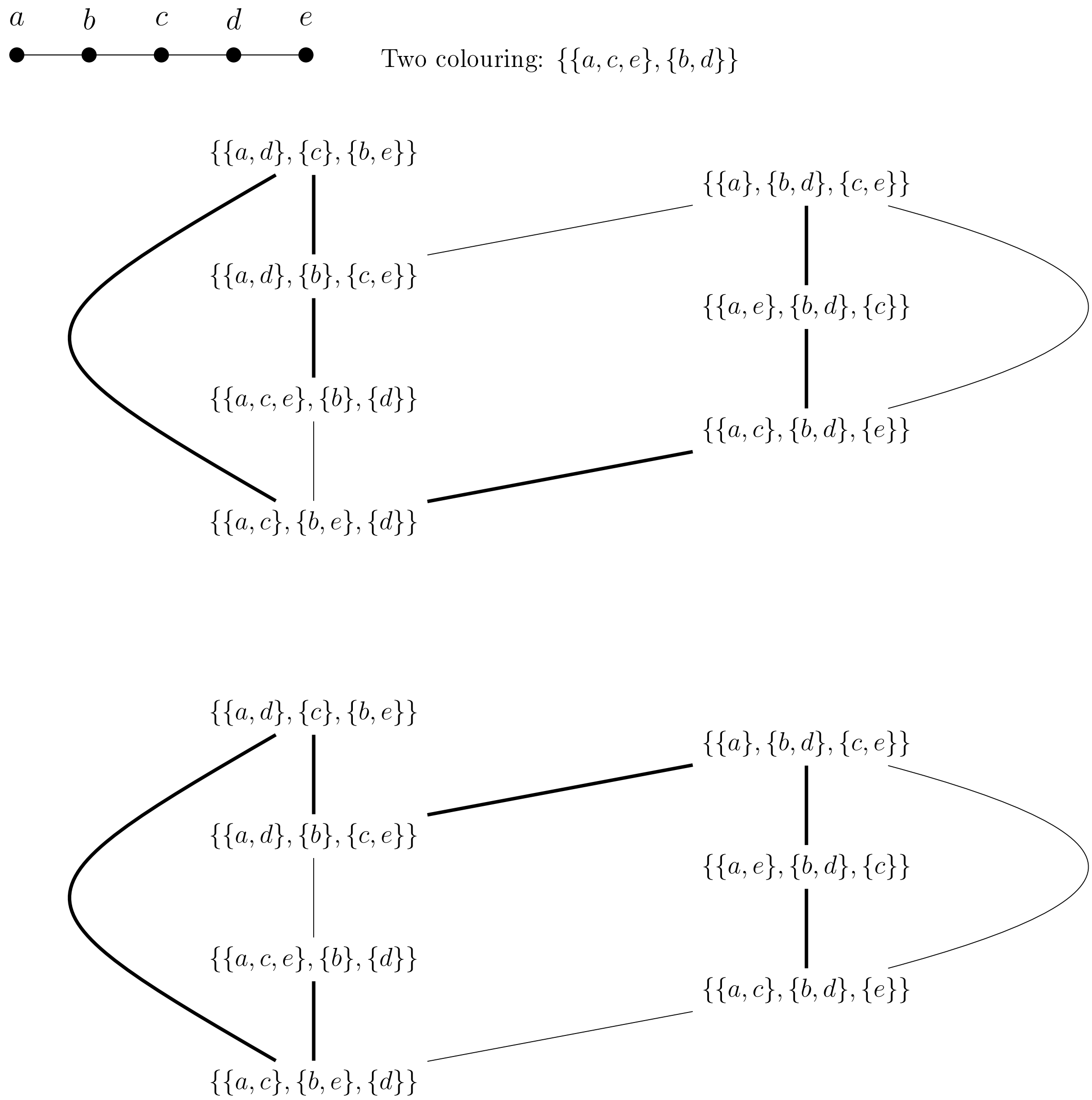

Fig. 2. The 3-Stirling colour graph of P_5 with two of its Hamilton paths highlighted

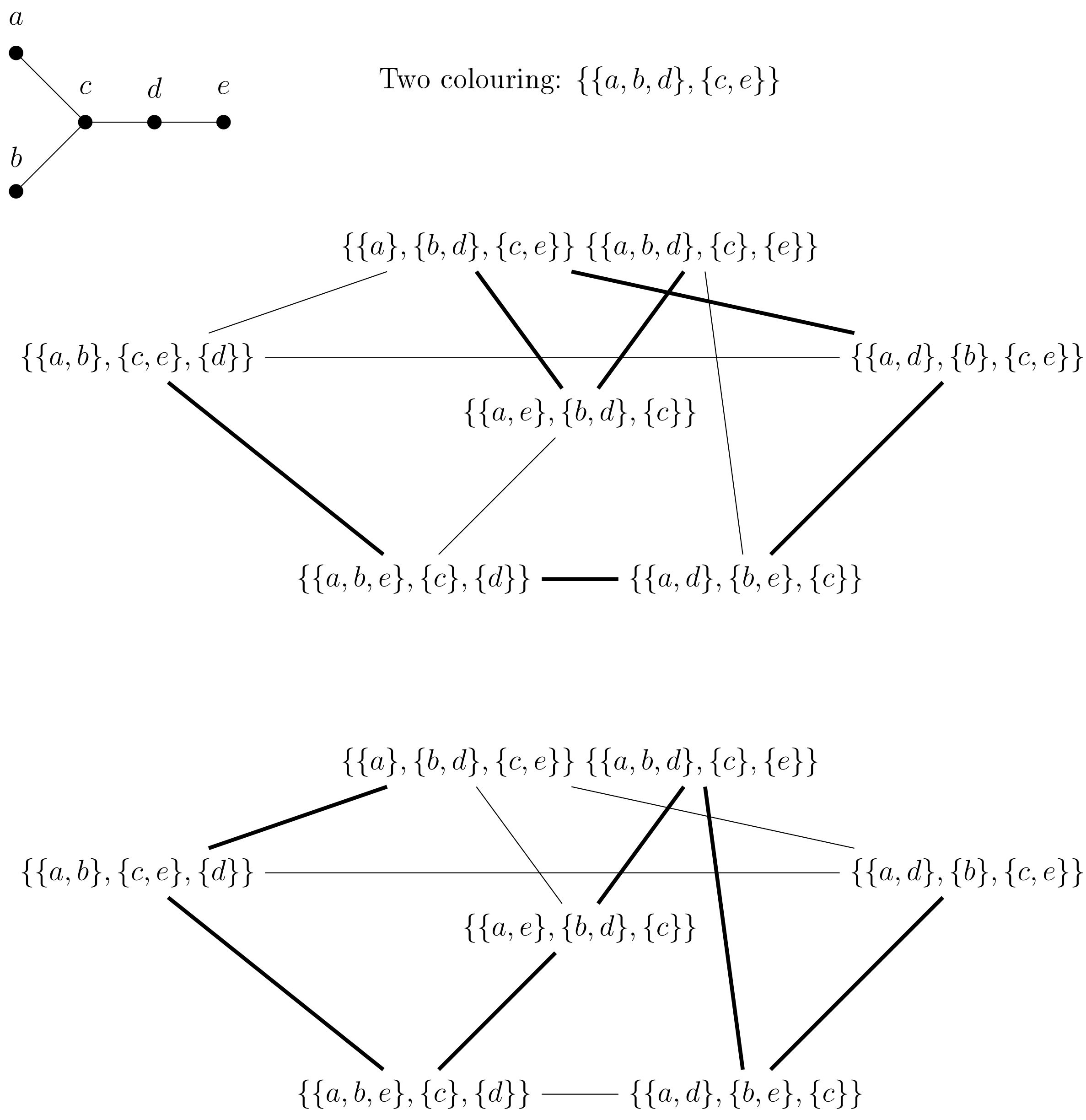

Fig. 3. A tree on five vertices and two Hamilton paths in the 3-Stirling colour graph of the tree

Suppose the result is true for any tree with at least 4 vertices, but fewer than $n \geq 6$ vertices. Let T be a tree with n vertices which is not a star, and let x be an arbitrary vertex of T. If x is a leaf of T, let $x = l_1$ and l_2 be leaves of T with corresponding distinct neighbours p_1 and p_2 (that is $p_1 \neq p_2$). Set $z = p_1$. Otherwise, x is not a leaf and it is possible select leaves l_1 and l_2 of T so that $x \notin \{l_1, l_2\}$ and l_1 and l_2 have corresponding distinct neighbours p_1 and p_2. Set $z = x$. Let $T' = T - \{l_1, l_2\}$.

By the induction hypothesis, $\mathcal{S}_3(T')$ contains a Hamilton path, P, for which the end vertices are $\{A' - \{y_1\}, B' - \{y_1\}, \{y_1\}\}$ and $\{A' - \{y_2\}, B' - \{y_2\}, \{y_2\}\}$, where $y_1 \neq z$, $y_2 \neq z$ and the two colouring of T' is $\{A', B'\}$. Define the cycle C in $\mathcal{S}_3(T')$ to be P together with edges joining its end vertices to $\{A', B'\}$.

Every colouring of T is an extension of exactly one colouring of T'. A given 3-colouring of T' may be extended by choosing to add l_1 to exactly one of the two cells of the colouring not containing p_1, and add l_2 to exactly one of the two cells of the colouring not containing p_2. We note that the four corresponding colourings (or partitions of $V(T)$ into independent sets) induce a 4-cycle in $\mathcal{S}_3(T)$ in which each edge arises from colourings that agree on $T - l_i$ for some i. Note that by similar logic, the extensions of the 2-colouring of T' using two or three colours also form a 4-cycle in $\mathcal{S}_3(T)$ in which each edge arises from colourings that agree on $T - l_i$ for some i. For a colouring u of T', we will call the 4-cycle corresponding to the extensions u, C_u.

For colourings u and v of T', let uv be an edge in the Hamilton path in $\mathcal{S}_3(T')$ corresponding to colourings that agree on $T' - y$ for some y. Now if $y \notin \{p_1, p_2\}$, then every partition in the four extensions of u agrees with one of the four extensions of v on every vertex except y. Hence there is a (perfect) matching between the vertices of C_u and the vertices of C_v. Suppose $y = p_1$. That is, u is a partition $\{\{p_1\} \cup S_1, S_2, S_3\}$ for some independent sets S_1, S_2 and S_3, and v is the partition $\{S_1, \{p_1\} \cup S_2, S_3\}$. Then the corresponding extensions of u and v where l_1 is assigned the same colour as vertices in S_3 (or its own colour if $S_3 = \emptyset$) are adjacent. In particular, the subgraph induced by the four extensions of u and v where l_1 is assigned the same colour as vertices in S_3 contains a four cycle. An identical argument can be made to show if $y = p_2$, the subgraph induced by two of the adjacent extensions of u and two of the extension adjacent of v contains a four cycle. Note that, in the extensions of a colouring v, edges in C_v alternately correspond to colourings that agree on $T - l_1$ and colourings that agree on $T - l_2$. Furthermore, any fixed colouring of $T' - p_1$ (respectively $T' - p_2$) has at most 2 extensions to a colouring of T'. Hence if u, v, w are three consecutive vertices on C, it can not be the case that both uv and vw correspond to colourings that agree on $T' - p_i$ for some $i \in \{1, 2\}$. Therefore, from the above argument, if u, v, w are three consecutive vertices on C, the subgraph of $\mathcal{B}_3(T)$ induced by the vertices belonging to C_u, C_v and C_w is a supergraph of the graph shown in Figure 4.

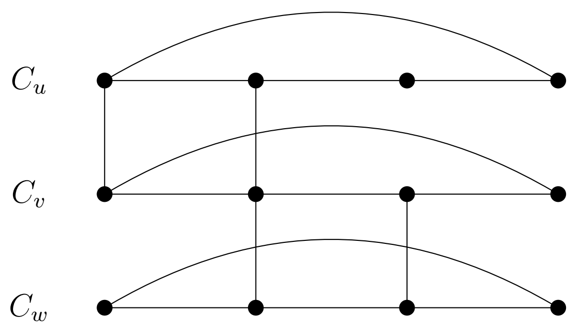

Fig. 4. A subgraph of the extension of three colourings from a path uvw in C where uv and vw correspond to colourings that agree on $T - p_i$ and $T - p_j$ (with $i \neq j$), respectively

The above paragraph justifies the following statements which we will use without reference through the remainder of the proof. Suppose u, v, w are three consecutive vertices on C.

- If the edge uv corresponds to colourings that agree on $T' - q$, where $q \notin \{p_1, p_2\}$, then there is a (perfect) matching from C_u to C_v
- If the edge uv corresponds to colourings that agree on $T' - q$, where $q \in \{p_1, p_2\}$, then the subgraph induced by two of the adjacent extensions of u and two of the adjacent extensions of v contains a four cycle.
- If the edge uv corresponds to colourings that agree on $T' - q_1$, and the edge vw corresponds to colourings that agree on $T' - q_2$, where $\{q_1, q_2\} = \{p_1, p_2\}$, then the subgraph induced by the vertices of C_u, C_v and C_w is a supergraph of the graph shown in Figure 4.

The constructive proof used to show that the 3-colour graph of a tree is Hamiltonian [5] can now be used. We provide a proof here for completeness.

Label the vertices in C in order $v_1, v_2, \ldots, v_k$, where v_1 corresponds to the two colouring, say $\{A, B\}$, of T'.

For $i = 1, 2, \ldots k$, define C_i sequentially as follows.

- Define $C_1 = C_{v_1}$.
- Let s_1 to be the 2-colouring of T. We note s_1 has a neighbour, say s'_2, in $V(C_{v_2})$. Define the two neighbours of s_1 in C_{v_1} as t' and t'', where t' corresponds to the extension of v_1 where $\{l_1\}$ is a cell and t'' corresponds to the extension of v_1 where $\{l_2\}$ is a cell . If t' has a neighbour in $V(C_{v_2})$, define $t_1 = t'$ and define the neighbour of t_1 in $V(C_{v_2})$ as t'_2. Otherwise, define $t_1 = t''$ and there is a neighbour of t_1 in $V(C_{v_2})$ which we define as t'_2.

  Form C_2 as follows. Take C_1 and delete the edge $t_1 s_1$. Add the edges $s_1 s'_2$ and $t_1 t'_2$, together with the path of length three around C_{v_2} which start at s'_2 and ends at t'_2.
- For $1 < i < k$, suppose C_i, s'_i and t'_i have been defined in the previous step. Consider the path of length three around C_{v_i} which start at s'_i and ends at t'_i. We note that by this construction, this path is in C_i. Find an adjacent pair of vertices on this path with neighbours in $V(C_{v_{i+1}})$ and define them to be s_i and t_i in that order. (Note that it is possible to have $s_i = s'_i$ or $t_i = t'_i$). Define the neighbours of s_i and t_i in $V(C_{v_{i+1}})$ to be s'_{i+1} and t'_{i+1}, respectively.

  Form C_{i+1} as follows. Take C_i and delete the edge $t_i s_i$. Add the edges $s_i s'_{i+1}$ and $t_i t'_{i+1}$, together with the path of length three in $C_{v_{i+1}}$ which start at s'_{i+1} and ends at t'_{i+1}.

By this construction, C_k is a Hamilton cycle of $\mathcal{B}_3(T)$. It remains to show that C_k satisfies the conclusion of the lemma. The neighbours of s_1, the two colouring of T, in C_k are the extension of v_1 where $\{l_i\}$ is a cell (for some $i = 1, 2$) and the extension of v_2 where $\{y\}$ is a cell, for some $y \neq z$.

If x is not a leaf, then $x \neq l_1$, $x \neq l_2$ and, as $x = z$, $x \neq y$. In this case, the lemma follows.

Suppose x is a leaf. Then $x = l_1$. As $y \in V(T')$, $x \neq y$. Furthermore $z = p_1$ implies $y \neq p_1$. It follows that the extensions of v_2 with cell $\{x, y\}$ is a neighbour of t' (the extension of v_1 where $\{x\}$ is a cell). It follows that $t' = t_1$ and therefore a neighbour of the two colouring of T in C_k is t'' (the extension of v_1 where $\{l_2\}$ is a cell). As $l_2 \neq x$, this completes the proof. □

Theorem 4.6. *For any tree T with at least 5 vertices, $\mathcal{S}_4(T)$ is Hamiltonian.*

Proof. The proof is by induction on the number of vertices of T. If $|V(T)|= 5$, then Lemma 4.2 implies $S_4(T)$ is Hamiltonian. Suppose T is a tree with $n \geq 6$ vertices and suppose further that for all trees with more than 5 vertices, but fewer than n vertices, the result holds.

Choose a leaf l of T and define $T' = T - l$. Let p be the vertex adjacent to l in T.

Each colouring of the vertices of T that uses exactly 4 colours is an extension of exactly one colouring of T' that uses either 3 or 4 colours. Colourings that extend vertices of $S_3(T')$ correspond to those where l is the only vertex of its colour. For each colouring in $S_4(T')$, l can be added to exactly three sets of the partition. The three corresponding colourings of $V(T)$ form a 3-cycle in $\mathcal{S}_4(T)$ in which each edge arises from colourings that agree on $T - l$. We will call the 3-cycle corresponding to the extensions of the colouring u, C_u.

Let $\{A, B\}$ be the 2-colouring of T'. By Lemma 4.5, there are vertices a and b and a Hamilton path P in $S_3(T')$ for which the end vertices are $\{A - \{a\}, B - \{a\}, \{a\}\}$ and $\{A - \{b\}, B - \{b\}, \{b\}\}$ where $p \neq a$ and $p \neq b$. It follow that there is a path P^* in $S_4(T)$ which starts at $\{A - \{b\}, B - \{b\}, \{b\}, \{l\}\}$ and ends at $\{A - \{a\}, B - \{a\}, \{a\}, \{l\}\}$ that contains all of the colourings in which $\{l\}$ is a cell.

By the induction hypothesis, there is a Hamilton cycle, C, in $S_4(T')$. For each edge uv we consider the edges between C_u and C_v arising from colourings that agree on $T' - y$. If $y \neq p$ then every colouring among the three extensions of u agrees with exactly one of the extensions of v on every vertex except y. Hence there is a (perfect) matching between the vertices of C_u and the vertices of C_v. If $y = p$, then there will be two edges between C_u and C_v. Specifically, if u corresponds to $\{A_1 \cup \{p\}, A_2, A_3, A_4\}$ and v corresponds to $\{A_1, A_2 \cup \{p\}, A_3, A_4\}$, then $\{A_1 \cup \{p\}, A_2, A_3 \cup \{l\}, A_4\}$ is adjacent to $\{A_1, A_2 \cup \{p\}, A_3 \cup \{l\}, A_4\}$ and $\{A_1 \cup \{p\}, A_2, A_3, A_4 \cup \{l\}\}$ is adjacent to $\{A_1, A_2 \cup \{p\}, A_3, A_4 \cup \{l\}\}$. Note that, if there is a path uvw in C where both uv and vw arise from colourings that agree on $T' - p$, then the endpoints of the matching edges are offset in the sense that the extensions of u, v and w have a subgraph of the form shown in Figure 5.

Label the vertices of C in order as $v_1 v_2 \ldots v_k$ with v_1 corresponding to the partition $\{A - \{a, b\}, B - \{a, b\}, \{a\}, \{b\}\}$ of the vertices of T'. Without loss of generality assume $p \in B$. Set $s_1 = \{(A - \{a, b\}) \cup \{l\}, B - \{a, b\}, \{a\}, \{b\}\}$ (in C_{v_1}). This vertex either has a neighbour in C_{v_2} or C_{v_k}. With out loss of generality, s_1 has a neighbour in C_{v_2}. For $i = 2, 3, \ldots, k$, choose s_i, t_i as described below (where indices are taken modulo k).

- Suppose s_{i-1} has been previously chosen. Select t_i to be the neighbour of s_{i-1} in C_{v_i} in the matching described above.
- Select s_i to be a vertex in C_{v_i}, different from t_i which has a neighbour in $C_{v_{i+1}}$. If possible (under this constraint), select s_k so that s_k is not adjacent to s_1.

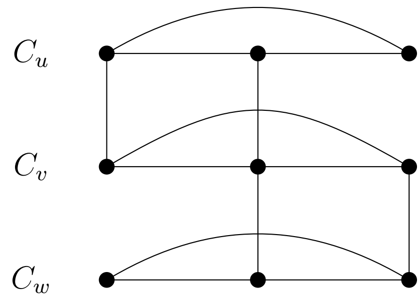

Fig. 5. A subgraph of the extension of three colourings from a path uvw in C where both uv and vw correspond to colourings that agree on $T' - p$

For each i, $2 \leq i \leq k$, there is a path, P_i, of length 2 containing all the vertices in C_{v_i} from s_i to t_i. These paths, together with the edges of the form $s_i t_{i+1}$ for $i = 1, 2, \ldots, k-1$, can be used to form a path P^{**} starting at s_1 and ending at s_k containing all the vertices belonging to $C_{v_2}, C_{v_3}, \ldots, C_{v_{k-1}}$ and C_{v_k}.

If s_k is not adjacent to s_1, then without loss of generality s_k is adjacent to $\{A-\{a,b\}, B-\{a,b\}, \{a,l\}, \{b\}\}$. The sequence $\{A-\{a,b\}, B-\{a,b\}, \{a,l\}, \{b\}\}$ concatenated with P^* concatenated with $\{A-\{a,b\}, B-\{a,b\}, \{a\}, \{b,l\}\}$ concatenated with P^{**}, is a Hamilton cycle of $S_4(T)$.

If s_k is adjacent to s_1, then (by the restriction that we must avoid this case if possible) it must be that there are only two edges of the matching with one end in C_{v_k} and the other end in C_{v_1} and t_k must be incident with one of these edges. Furthermore the edge $v_k v_1$ corresponds to colourings that agree on $T' - p$. By symmetry (as s_k is adjacent to s_1), we may assume the edge $v_1 v_2$ also corresponds to colourings that agree on $T' - p$. Hence v_1, v_2 and v_k correspond to colourings that agree on $T' - p$. Observe that there are at most 3 extensions of any fixed colouring of $T' - p$ to a colouring of T'. Recalling that C is a Hamilton cycle in $S_4(T')$, it therefore can not also be the case that the edge $v_{k-1} v_k$ corresponds to colourings that agree on $T' - p$. Hence there is a matching from $C_{v_{k-1}}$ to C_{v_k}. Set $s'_1 = s_1$. For $2 \leq i \leq k-2$ set $s'_i = s_i$ and $t'_i = t_i$. Choose s'_{k-1} so that s'_{k-1}, s_{k-1} and t_{k-1} are the vertices of $C_{v_{k-1}}$. Choose t'_k to be the neighbour of s'_{k-1} in C_{v_k}. Finally set $s'_k = t_k$. For each i, $2 \leq i \leq k$, there is a path, P_i, of length 2 containing all the edges from s'_i to t'_i. These paths together with the edges of the form $s'_i t'_{i+1}$ (for $i = 1, 2, \ldots, k-1$) can be used to form a path P^{***} starting at s'_1 and ending at s'_k containing all the vertices in $C_{v_2}, C_{v_3}, \ldots, C_{v_{k-1}}$ and C_{v_k}. Suppose, without loss of generality that s'_k is adjacent to $\{A-\{a,b\}, B-\{a,b\}, \{a,l\}, \{b\}\}$. The, the sequence $\{A-\{a,b\}, B-\{a,b\}, \{a,l\}, \{b\}\}$, P^*, $\{A-\{a,b\}, B-\{a,b\}, \{a\}, \{b,l\}\}$, P^{**} is a Hamilton cycle of $S_4(T)$. □

In what follows, we make use of a theorem of Choo and MacGillivray [5] about the existence of Hamilton cycles in graphs with a special structure. A *C-graph with vertex partition* $F_0, F_1, \ldots, F_{N-1}$ is a graph G such that there is a partition $F_0, F_1, \ldots, F_{N-1}$ of $V(G)$ in which, for $i = 0, 1, \ldots, N-1$, the subgraph induced by F_i is a Hamilton connected graph with at least three vertices. If G is a C-graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$, then we define $F_N = F_0$.

If X and Y are disjoint subsets of the vertices of a graph G, then the symbol $[X, Y]$ is used to denote the set of edges of G with one end in X and the other end in Y.

Corollary 4.7. [5] *Let G be a C-graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose that, for $j = 0, 1, \ldots, N-1$, the set $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges. If there exists i, $0 \le i \le N-1$, such that $[F_i, F_{i+1}]$ contains at least three vertex disjoint edges, then G is Hamiltonian. Further, the Hamilton cycle is constructed by finding, for $i = 0, 1, \ldots, N-1$, a Hamilton path through the subgraph induced by $F_0, F_1, \ldots, F_{N-1}$ that begin at an end of a specific edge in $[F_{i-1}, F_i]$ and terminate at an end of a specific edge in $[F_i, F_{i+1}]$.*

Lemma 4.8. *Let T be a tree on at least $k+2 \ge 7$ vertices and let l be a leaf of T. Suppose that $f_0, f_1, \ldots, f_{N-1}$ is a Hamilton cycle in $S_k(T-l)$. For $i = 0, 1, \ldots, N-1$, let F_i be the set of extensions of f_i to a k-colouring of T. Let H be the subgraph of $S_k(T)$ induced by $F_0 \cup F_1 \cup \cdots \cup F_{N-1}$. Then*

(i) *$N \ge k-1 \ge 4$;*

(ii) *for $i = 0, 1, \ldots, N-1$, the subgraph of H induced by F_i is a complete graph on $k-1$ vertices;*

(iii) *for $i = 0, 1, \ldots, N-1$, the set $[F_i, F_{i+1}]$ contains at least three vertex disjoint edges;*

(iv) *H is a C-graph;*

(v) *For any edge xy in the subgraph of H induced by F_i, there is a Hamilton cycle in H that contains the edge xy.*

Proof. Let l' be a leaf of $T-l$ and let $T' = T - l - l'$. It follows that T' has at least k vertices and therefore at least one k-colouring. There are $k-1$ extensions of this colouring to a k-colouring of $T-l$ and therefore $N = |V(S_k(T-l))| \ge k-1$. This proves *(i)*.

For $1 \le i \le N-1$, the colouring f_i has exactly $k-1$ extensions: the vertex l can be inserted into any cell except the one that contains its unique neighbour. Since any two of these extensions of f_i differ only in the cell containing l, they are adjacent in H. This proves *(ii)* and *(iv)*.

Suppose that f_i and f_{i+1} differ in the cell containing the vertex x. If x is not the unique neighbour of l in T, then each colouring $c_i \in F_i$ is adjacent in H to the unique colouring $c_{i+1} \in F_{i+1}$ which is the same when restricted to $V(T) - \{x\}$. Thus, in this case, $[F_i, F_{i+1}]$ contains $k-1 \ge 4$ vertex disjoint edges. If x is adjacent to l in T then, since l and x can not belong to the same cell, there are $k-2$ colourings in $c_i \in F_i$ for which there is a unique colouring $c_{i+1} \in F_{i+1}$ which is the same when restricted to $V(T) - \{x\}$ (the remaining colouring $a \in F_i$ also differs from the remaining colouring $b \in F_{i+1}$ in the cell containing l). Thus, in this case, $[F_i, F_{i+1}]$ contains $k-2 \ge 3$ vertex disjoint edges. This proves *(iii)*.

Finally, let xy be an edge of H in the subgraph induced by F_i. Let H' be the subgraph of H induced by deleting any edge in $[F_{i-1}, F_i] \cup [F_i, F_{i+1}]$ incident with x. The graph H' is a C-graph. Since all the edges which were deleted in the formation of H' were incident with x, by *(iii)* for every $0 \le j \le N-1$, the set $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges. Furthermore, by *(i)*, $N \geq 3$ and therefore by *(iii)* there exists j such that the set $[F_j, F_{j+1}]$ contains at least three vertex disjoint edges. All the premises in Corollary 4.7 are therefore satisfied. By Corollary 4.7 the Hamilton cycle is constructed by finding, for $i = 0, 1, \dots, N-1$, a Hamilton paths through the subgraph induced by $F_0, F_1, \dots, F_{N-1}$ that begin at an end of a specific edge in $[F_{i-1}, F_i]$ and terminate at an end of a specific edge in $[F_i, F_{i+1}]$. By construction of H', the vertex x is neither the start nor end of the Hamilton path in the subgraph induced by F_i. Since the subgraph induced by F_i is a complete graph, the edge xy can be included in the Hamilton path that is used. □

We now outline the main idea in the the proof of the next theorem, and then formalize it in a technical lemma (Lemma 4.9). The argument uses of the same sort of construction as the proof of Theorem 4.6. Let $k \geq 5$ and let T be a tree on at least $k+2$ vertices. Let l be a leaf of T. In a k-colouring of T either the vertex l belongs to a cell of size at least two, or it belongs to a cell of size one. In the former case, deleting l gives a k-colouring of $T - l$ and, for each k-colouring of $T - l$, the vertex l can be inserted into any cell that does not contain its unique neighbour to obtain a k-colourings of T. In the latter case, deleting the cell containing l gives a $(k-1)$-colouring of $T - l$, and each $(k-1)$-colouring of $T - l$ can be uniquely extended to a k-colouring of T by inserting a cell containing only l. A Hamilton cycle in $S_k(T)$ is constructed from a Hamilton cycle C in $S_k(T-l)$ and a Hamilton path in $S_{k-1}(T-l)$ whose ends are joined to consecutive vertices of C.

Lemma 4.9. *Let T be a tree on at least $k+2 \geq 7$ vertices, let l be a leaf of T, and let s be the unique neighbour of l in T. Suppose $S_{k-1}(T-l)$ has a Hamilton path with ends*

$$\alpha = \{X_1 \cup \{w\}, X_2, X_3, \dots, X_{k-2}, \{y\}\}, \quad \text{and} \quad \beta = \{X_1, X_2, X_3, \dots, X_{k-2}, \{w, y\}\},$$

where neither w nor y equals s. Then, if $S_k(T-l)$ has a Hamilton cycle, then so does $S_k(T)$.

Proof. Since there is a 1-1 correspondence between the set of k-colourings of T in which l belongs to a cell of size one and the set of $(k-1)$-colourings of $T-l$, there is a path P in $S_k(T)$ from

$$\alpha' = \{X_1 \cup \{w\}, X_2, X_3, \dots, X_{k-2}, \{y\}, \{l\}\}, \text{ to } \beta' = \{X_1, X_2, X_3, \dots, X_{k-1}, \{w, y\}, \{l\}\},$$

which contains all of the k-colourings where l belongs to a cell of size one.

Let $f_0, f_1, \dots, f_{N-1}$ be a Hamilton cycle in $S_k(T-l)$ where, without loss of generality,

$$f_0 = \{X_1, X_2, X_3, \dots, X_{k-2}, \{y\}, \{w\}\}.$$

For $i = 0, 1, \dots, N-1$, let F_i be the set of extensions of f_i to a k-colouring of T. Then

$$a = \{X_1, X_2, X_3, \dots, X_{k-2}, \{y\}, \{w, l\}\}, \quad \text{and} \quad b = \{X_1, X_2, X_3, \dots, X_{k-2}, \{w\}, \{y, l\}\},$$

both belong to F_0. Furthermore both $a\alpha'$ and $b\beta'$ are edges of $S_k(T)$. The colourings in $F_0 \cup F_1 \cup \cdots \cup F_{N-1}$ are exactly the k-colourings of T in which l belongs to a cell of size at least two. Let H be the subgraph of $S_k(T)$ induced by $F_0 \cup F_1 \cup \cdots \cup F_{N-1}$. By Lemma 4.8, there is a Hamilton cycle C in H that contains the edge ab.

The desired Hamilton cycle in $S_k(T)$ arises from replacing the edge ab of C by the edge $a\alpha'$, followed by the path P and then the edge $\beta' b$. □

Theorem 4.10. *Let $r \geq 4$. If T is a tree on at least $r+1$ vertices, then $S_r(T)$ is Hamiltonian.*

Proof. We prove the stronger statement, that if T is a tree on at least $r+1$ vertices and l is a leaf of T, then $S_r(T)$ has a Hamiltonian cycle constructed as in Lemma 4.9 or Theorem 4.6 from a Hamilton path in $S_{r-1}(T-l)$ and a Hamilton cycle in $S_r(T-l)$.

The statement holds for $r = 4$ by Theorem 4.6. For the induction hypothesis suppose, for some $k \geq 5$ and all r such that $4 \leq r < k$, that if T is a tree with at least $r+1$ vertices, then $S_r(T)$ has a Hamiltonian cycle constructed from a Hamilton path in $S_{r-1}(T-l)$ and a Hamilton cycle in $S_r(T-l)$ as in Lemma 4.9 or Theorem 4.6. We need to show the statement to be proved holds when $r = k$, and do so by induction on the number of vertices in the tree.

The basis, when $|V(T)| = k+1$, was established in Lemma 4.2. We note that in this case, for any leaf l of T, $S_k(T-l) \cong K_1$. Hence any Hamilton cycle in $S_k(T)$ is constructed from the only vertex in $S_k(T-l)$ and a Hamilton path in $S_{k-1}(T-l)$. Suppose that the statement holds for all trees on between $k+1$ and $n-1$ vertices, and let T be a tree on n vertices. Note that $n \geq k+2 \geq 7$.

Let l be an end of a longest path in T. Then l is a leaf. Let s be its unique neighbour. The vertex s either has degree 2, or is adjacent to another leaf $l' \neq l$. We consider these two possibilities.

Case 1. *The vertex s is adjacent to another leaf $l' \neq l$.*

By the induction hypotheses (both), there exists a Hamilton cycle in $S_k(T-l)$ and a Hamilton cycle C in $S_{k-1}(T-l)$.

The Hamilton cycle C can be assumed to be constructed as in Lemma 4.9 or Theorem 4.6 from a Hamilton path A in $S_{k-2}(T-l-l')$ and a Hamilton cycle B in $S_{k-1}(T-l-l')$. Let $\alpha\beta$ be an edge of C that joins a vertex in A to a vertex in B. Then, there is a vertex $w \neq s$ such that

$$\alpha = \{X_1, X_2, \ldots, X_{k-1}, \{l'\}\}, \quad \text{and} \quad \beta = \{X_1 - \{w\}, X_2, \ldots, X_{k-1}, \{l', w\}\}.$$

It follows that $S_{k-1}(T-l)$ has a Hamilton path from α to β.

Since neither w nor l' equals s, the existence of the desired Hamilton cycle in $S_k(T)$ follows from Lemma 4.9.

Case 2. *The vertex s has degree two.*

Again, by the induction hypotheses (both), there exists a Hamilton cycle in $S_k(T-l)$ and a Hamilton cycle C in $S_{k-1}(T-l)$. The Hamilton cycle C can be assumed to be constructed as in Lemma 4.9 from a Hamilton path A in $S_{k-2}(T-l-s)$ and a Hamilton cycle B in $S_{k-1}(T-l-s)$.

Since $k \geq 5$ we have $k-4 \geq 1$. Let $x_1, x_2, \ldots, x_{k-4}$ be vertices of $T-l-s$ such that $T' = T - \{l, s, x_1, x_2, \ldots, x_{k-4}\}$ is a tree. Recall that T has at least $k+1 \geq 6$ vertices, so that T' has at least 3 vertices. Let $\{X, Y\}$ be a 2-colouring of T' such that $|Y| \geq 2$. Let

$$\alpha = \{X, Y, \{x_1\}, \{x_2\}, \ldots, \{x_{k-4}\}, \{s\}\}.$$

Note that α is a colouring of $T-l$. Since $T-l$ and T' are trees, at most one vertex of Y can be adjacent to x_1. Hence, for some $w \in Y$, the vertex α has a neighbour of the form

$$\beta = \{X, Y - \{w\}, \{x_1, w\}, \{x_2\}, \ldots, \{x_{k-4}\}, \{s\}\},$$

in the collection of colourings that extend those on the Hamilton path A to $T-l$. Note that $\{s\}$ is a cell in each of these. It follows that α and β are neighbours on C and therefore, $S_{k-1}(T-l)$ has a Hamilton path from α to β.

Since neither w nor x_1 equals s, the existence of the desired Hamilton cycle in $S_k(T)$ now follows from Lemma 4.9. □

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