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On the determinant of Kőnig–egerváry graphs

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ABSTRACT

Several graph decompositions that factorize the determinant of the adjacency matrix isolate a Kőnig–Egerváry part, such as the SD–KE decomposition and the critical independence decomposition of Larson. This suggests that the study of graph unimodularity can be approached, to a large extent, through the structure of Kőnig–Egerváry graphs. In this paper, we advance this point of view by introducing a new determinant factorization within the class of Kőnig–Egerváry graphs. More precisely, given a Kőnig–Egerváry graph G, we consider the partition of $V(G)$ into its perfect-flower part $PF(G)$ and its perfect-flower-free part $PFF(G)$, and prove that

$$\det(G) = \det(G[PF(G)])\det(G[PFF(G)]).$$

We also obtain the analogous factorization for the permanent. This decomposition provides a new tool for the study of unimodularity, reducing the problem to two induced subgraphs of very different nature: the graph $G[PF(G)]$, whose structure is closely related to Sterboul–Deming configurations with perfect matchings, and the graph $G[PFF(G)]$, which is governed by the theory of critical independent sets. In this way, the paper provides a new structural framework for the study of unimodular graphs through Kőnig–Egerváry theory.

Keywords: Sachs subgraph, Kőnig–Egerváry graphs, perfect matching, graph decomposition

2020 Mathematics Subject Classification: 05C70, 05C75.

1. Introduction

Let $\alpha(G)$ denote the cardinality of a maximum independent set, and let $\mu(G)$ denote the size of a maximum matching in $G = (V, E)$. It is known that $\alpha(G) + \mu(G)$ may

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equal the order of G; in this case, G is called a Kőnig–Egerváry graph [3, 7, 28]. Kőnig–Egerváry graphs have been extensively studied [2, 9, 17, 18, 10]. It is also known that every bipartite graph is a Kőnig–Egerváry graph [6]. These graphs were independently introduced by Deming [3], Sterboul [28], and Gavril [7].

In this paper, the term subgraph is understood as a subgraph defined by a graph together with a given matching in that graph. In [5], Edmonds introduced the following concepts relative to a matching M of a graph G and its subgraphs. An M-blossom of G is an odd cycle of length $2k+1$ with k edges in M. The vertex not saturated by M in the cycle is called the *base* of the blossom. An M-stem is an M-alternating path of even length, possibly zero, connecting the base of the blossom with a vertex not saturated by M in G. The base is the only common vertex between the blossom and the stem. An M-flower is a blossom joined with a stem. The vertex not saturated by M in the stem is called the *root* of the flower.

In [28], Sterboul introduced the concept of a *posy* for the first time. An M-posy consists of two, not necessarily disjoint, M-blossoms joined by an M-alternating path that starts and ends with edges in M. The endpoints of the path are the bases of the two blossoms. There are no internal vertices of the path in the blossoms.

Sterboul [28] was the first to characterize Kőnig–Egerváry graphs by forbidden configurations relative to a maximum matching. Subsequently, Korach, Nguyen, and Peis [15] reformulated this characterization in terms of simpler configurations, unifying the structures of flowers and posies. Later, Bonomo et al. [1] obtained a purely structural characterization based on forbidden subgraphs. More recently, in [11, 13, 14], results were obtained that simplify working with flower and posy structures.

Theorem 1.1 ([28]). *For a graph G, the following properties are equivalent:*

- *G is a non-Kőnig–Egerváry graph.*
- *For every maximum matching M, there exists an M-flower or an M-posy in G.*
- *For some maximum matching M, there exists an M-flower or an M-posy in G.*

The set of vertices of G lying in a flower or posy, for any maximum matching, is denoted by $SD(G)$, and we write

$$KE(G) = V(G) - SD(G).$$

The sets $SD(G)$ and $KE(G)$ constitute the SD–KE decomposition of the graph. A graph G is called a Sterboul–Deming graph if $KE(G) = \emptyset$. Essentially, such a graph may be regarded as the structural counterpart of a Kőnig–Egerváry graph. Characterizations of Sterboul–Deming graphs can be found in [23].

The SD–KE decomposition admits a factorization for the determinant of the adjacency matrix of a graph [14, 22], namely

$$\det(G) = \det(G[KE(G)]) \det(G[SD(G)]).$$

We say that a graph is *unimodular* if the determinant of its adjacency matrix is ±1. The preceding factorization naturally reduces the problem of studying the unimodularity of graphs to the study of the unimodularity of Kőnig–Egerváry graphs or Sterboul–Deming graphs. Unimodular Sterboul–Deming graphs have been studied in [12, 20]. Another graph decomposition that factorizes the determinant is the critical independence decomposition of Larson [16], which partitions a graph into a Kőnig–Egerváry graph and a 2-bicritical graph [26].

In this work, we advance the study of the unimodularity of Kőnig–Egerváry graphs through decompositions.

Given a matching M of G, an *M-perfect flower* is a pair (C, P) with the following properties. The subgraph C is an odd cycle of length $2k+1$ containing exactly k edges of M. The path P is a non-trivial M-alternating path, say

$$P = p_1, p_2, \dots, p_t,$$

whose first and last edges belong to M. Moreover,

$$V(C) \cap V(P) = \{p_1\}.$$

Thus, paths of length one are allowed in the definition of an M-perfect flower; in that case, the unique edge of the path is both the first and the last edge, and hence it must belong to M. Paths of length zero are not allowed in this definition.

We define the *perfect-flower part* of G as the set of vertices of G that belong to some M-perfect flower, for some matching M, and we denote it by $PF(G)$. That is,

$$PF(G) := \{v \in V(G) : v \text{ belongs to an } M\text{-perfect flower for some maximum matching } M\}.$$

We define the *perfect-flower-free part* of G as

$$PFF(G) := V(G) - PF(G).$$

This decomposition was first defined in [21] in order to study the structure of critical independent sets in Kőnig–Egerváry graphs.

In this work, we prove that, for every Kőnig–Egerváry graph G, the perfect-flower decomposition of the vertex set is compatible with both the determinant and the permanent of the adjacency matrix. More precisely, if

$$V(G) = PF(G) \cup PFF(G),$$

then

$$\det(G) = \det(G[PF(G)]) \det(G[PFF(G)]),$$

and

$$\operatorname{perm}(G) = \operatorname{perm}(G[PF(G)]) \operatorname{perm}(G[PFF(G)]).$$

The determinant factorization is the one relevant to unimodularity. Indeed, within the class of Kőnig–Egerváry graphs, it reduces the problem of studying whether $\det(G) = \pm 1$ to the corresponding induced subgraphs $G[PF(G)]$ and $G[PFF(G)]$. In particular, this reduction separates two induced subgraphs with different structural features: the structure of $G[PF(G)]$ is closely related to that of Sterboul–Deming graphs with a perfect matching [12], whereas the structure of $G[PFF(G)]$ can be controlled through the theory of critical independent sets [21].

The permanent factorization has a complementary enumerative meaning. By the Sachs expansion recalled in Theorem 3.1, $\operatorname{perm}(G)$ is the positive weighted sum of the Sachs subgraphs of G, where a Sachs subgraph with m cycle components contributes 2^m. Hence, the permanent identity says that this weighted enumeration splits independently over the two parts $PF(G)$ and $PFF(G)$. In this sense, the permanent factorization is not merely an algebraic analogue of the determinant factorization; it records the decomposition of the Sachs structure itself.

Consequently, the present theorem should be interpreted as an internal reduction for the class of Kőnig–Egerváry graphs. For arbitrary graphs, its use should be understood in combination with previous decompositions that isolate a Kőnig–Egerváry part, such as the SD–KE decomposition discussed above or Larson's critical independence decomposition [16]. Thus, the broader application to general graphs is indirect: the factorization proved here applies to the Kőnig–Egerváry part produced by such decompositions.

The paper is organized as follows. Section 2 contains the terminology and auxiliary results used throughout the paper. Section 3 contains the main results. Section 4 is devoted to open problems.

2. Preliminaries

All graphs considered in this paper are finite, undirected, and simple. For undefined terminology or notation, we refer the reader to Lovász and Plummer [25] or Diestel [4].

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ is the finite vertex set and $E = E(G)$ is the edge set, with

$$E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}.$$

We denote the edge $e = \{u, v\}$ by uv. A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called *spanning* if $V(H) = V(G)$. For two vertex sets $X, Y \subseteq V(G)$, we denote by $E(X, Y)$ the set of edges $uv \in E(G)$ such that $u \in X$ and $v \in Y$.

Let $e \in E(G)$ and $v \in V(G)$. We define

$$G - e := (V, E - \{e\})$$

and

$$G - v := (V - \{v\}, \{uw \in E : u, w \neq v\}).$$

If $X \subseteq V(G)$, the *induced* subgraph of G on X is the subgraph

$$G[X] = (X, F),$$

where

$$F := \{uv \in E(G) : u, v \in X\}.$$

The union of two graphs G and H is the graph $G \cup H$ with

$$V(G \cup H) = V(G) \cup V(H) \quad \text{and} \quad E(G \cup H) = E(G) \cup E(H).$$

If M is a set of edges of G, we denote by $G[M]$ the subgraph of G spanned by the edges of M, that is,

$$V(G[M]) = \{v \in V(G) : v \text{ is an endpoint of some edge in } M\}$$

and

$$E(G[M]) = M.$$

The number of vertices in a graph G is called the *order* of the graph and is denoted by $|G|$. A *cycle* in G is called *odd*, respectively *even*, if it has an odd, respectively even, number of edges.

A *matching* M in a graph G is a set of pairwise non-adjacent edges. The *matching number* of G, denoted by $\mu(G)$, is the maximum cardinality of a matching in G. Matchings induce an involution on the vertex set of the graph:

$$M : V(G) \to V(G),$$

where $M(v) = u$ if $uv \in M$, and $M(v) = v$ otherwise. If $S, U \subseteq V(G)$ with $S \cap U = \emptyset$, we say that M is a matching from S to U if $M(S) \subseteq U$. A matching M is *perfect* if $M(v) \neq v$ for every vertex of the graph. A vertex set $S \subseteq V$ is *independent* if, for every pair of vertices $u, v \in S$, we have $uv \notin E$. The number of vertices in a maximum independent set is denoted by $\alpha(G)$. It is well known that the symmetric difference of two matchings has a very simple structure.

Lemma 2.1. *Let M_1 and M_2 be two maximum matchings of a graph G, and let*

$$H = (V(M_1 \triangle M_2),\ M_1 \triangle M_2).$$

Then every connected component of H is one of the following:

1. *an even cycle whose edges alternate between M_1 and M_2;*
2. *a path of even length whose edges alternate between M_1 and M_2; in this case, one end-vertex is saturated by M_1 and unsaturated by M_2, while the other is saturated by M_2 and unsaturated by M_1.*

3. Main results

In this section, we establish the main results of the paper. Before starting, we introduce the necessary tools and notation.

A spanning subgraph of a graph G is called a *Sachs subgraph* if each of its components is a regular graph of degree one or two. Such subgraphs arise naturally in the study of determinants and permanents of adjacency matrices [8, 27]. They are also known as $\{1, 2\}$-factors and are closely related to perfect 2-matchings. Note that every perfect matching is a Sachs subgraph, since all its components are copies of K_2. The set of all Sachs subgraphs of G is denoted by $\mathrm{Sachs}(G)$.

Let $\mathrm{prk}(G)$ be the maximum order of a subgraph H of G such that $\mathrm{Sachs}(H) \neq \emptyset$. We define

$$\mathrm{Sachs}^{\mathrm{prk}}(G) := \bigcup_{H} \mathrm{Sachs}(H),$$

where the union is taken over all subgraphs of G of order $\mathrm{prk}(G)$. Note that when $\mathrm{Sachs}(G) \neq \emptyset$, we have

$$\mathrm{Sachs}^{\mathrm{prk}}(G) = \mathrm{Sachs}(G).$$

The following theorem shows that there is a natural relationship between the Sachs subgraphs of a graph and the computation of its determinant or permanent.

Theorem 3.1 ([8, 19]). *Let G be a graph. Then*

$$\begin{aligned} \det(G) &= \sum_{H \in \mathrm{Sachs}(G)} (-1)^{k(H)} 2^{m(H)}, \\ \mathrm{perm}(G) &= \sum_{H \in \mathrm{Sachs}(G)} 2^{m(H)}, \end{aligned}$$

where $k(H)$ is the number of even components of H, and $m(H)$ is the number of cycles of H.

Denote by $\det(G)$ the determinant $\det(A(G))$, and define $\det(G[\emptyset]) = 1$.

For the reader's convenience, we include a short proof of Lemma 3.2 for completeness.

Lemma 3.2 ([24]). *Let G be a Kőnig–Egerváry graph and $H \in \mathrm{Sachs}^{\mathrm{prk}}(G)$. Then H has no odd cycles.*

Proof. Let I be a maximum independent set of G, and set $X := V(G) - I$. Since G is a Kőnig–Egerváry graph, we have

$$|X| = |G| - \alpha(G) = \mu(G).$$

Moreover, X is a vertex cover of G, and therefore $X \cap V(H)$ is a vertex cover of H.

Since every maximum matching of G is a Sachs subgraph of order $2\mu(G)$, it follows that

$$\mathrm{prk}(G) \geq 2\mu(G).$$

As $H \in \mathrm{Sachs}^{\mathrm{prk}}(G)$, we have $|H| = \mathrm{prk}(G)$, and hence

$$|H| \geq 2\mu(G).$$

Assume that H has an odd cycle component. Since each component of H is either a copy of K_2 or a cycle, every vertex cover of H has size at least $|H|/2$, and in fact strictly greater than $|H|/2$ whenever H has an odd cycle component. Therefore,

$$|X \cap V(H)| > \frac{|H|}{2}.$$

On the other hand,

$$|X \cap V(H)| \leq |X| = \mu(G) \leq \frac{|H|}{2},$$

which is a contradiction. Therefore, H has no odd cycles. □

Corollary 3.3. *Let G be a Kőnig–Egerváry graph. Then*

$$\operatorname{prk}(G) = 2\mu(G).$$

In general, the decomposition $PFF(G), PF(G)$ does not respect the Sachs structure of every Kőnig–Egerváry graph in the following sense: for the graph G in Figure 1, there exists $H \in \mathrm{Sachs}^{\mathrm{prk}}(G)$ such that

$$E(PFF(G), PF(G)) \cap E(H) \neq \emptyset.$$

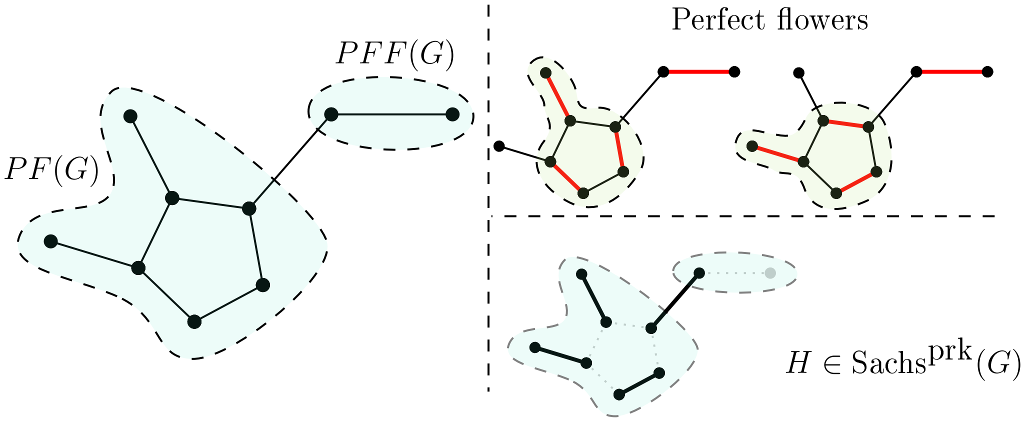

Fig. 1. The partition into $PF(G)$ and $PFF(G)$, and a Sachs subgraph crossing the partition.

Moreover, note that G satisfies

$$\mu(G) + \alpha(G) = 4 + 5 = 9 = |G|,$$

that is, G is a Kőnig–Egerváry graph. However, for Kőnig–Egerváry graphs, the graph has a perfect matching if and only if $\mathrm{Sachs}(G) = \mathrm{Sachs}^{\mathrm{prk}}(G)$. This reduces the study of the determinant and permanent of the graph via Theorem 3.1 to the study of Kőnig–Egerváry graphs with a perfect matching. Therefore, from this point of view, we are only interested in Kőnig–Egerváry graphs with a perfect matching. We will show in Theorem 3.4 that the decomposition $PFF(G), PF(G)$ respects the Sachs structure in any Kőnig–Egerváry graph with a perfect matching.

Theorem 3.4. *Let G be a Kőnig–Egerváry graph with a perfect matching. Then, for every $H \in \mathrm{Sachs}(G)$, it holds that*

$$E(PFF(G), PF(G)) \cap E(H) = \emptyset.$$

Proof. Assume, for contradiction, that there exists $H \in \mathrm{Sachs}(G)$ such that

$$E(PFF(G), PF(G)) \cap E(H) \neq \emptyset.$$

Let

$$e \in E(PFF(G), PF(G)) \cap E(H).$$

Since H is a Sachs subgraph, each connected component of H is either a copy of K_2 or a cycle. By Lemma 3.2, H has no odd cycles, and therefore every cycle component of H is even. Hence, each cycle component has a perfect matching, and if e lies on such a cycle, we may choose a perfect matching of that cycle containing e. If e belongs to a K_2-component, then it is forced to belong to the unique perfect matching of that component. Choosing arbitrarily a perfect matching on every other component of H, we obtain a perfect matching M_e of H such that $e \in M_e$. Since every Sachs subgraph is spanning by definition, it follows that M_e is also a perfect matching of G.

By the choice of e, there exists $v \in PF(G) \cap e$. Note that $M_e(v) \in PFF(G)$ and $e = vM_e(v) \in M_e$. On the other hand, there exists a perfect matching M_v of G such that v lies in an M_v-perfect flower (P, C) of G, as shown in Figure 2.

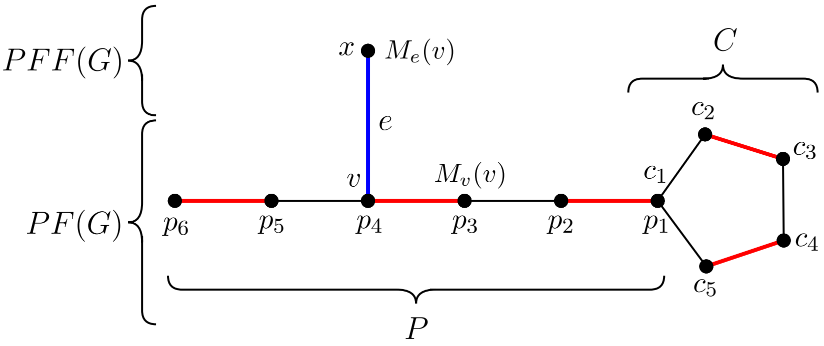

Fig. 2. An edge between $PF(G)$ and $PFF(G)$, and an M_v-perfect flower containing v

Write

$$C = c_1, c_2, \ldots, c_{2r+1}, c_1, \qquad P = p_1, p_2, \ldots, p_t,$$

where $p_1 = c_1$. The common vertex $p_1 = c_1$ is the base of the M_v-blossom C; otherwise p_1 would be incident with an edge of M_v in C and also with the first edge of P, which belongs to M_v. We orient P from p_1 to its other endpoint. Thus

$$p_\ell p_{\ell+1} \in M_v \iff \ell \text{ is odd}, \qquad 1 \leq \ell < t.$$

In particular, since the first and last edges of P belong to M_v, the number t is even. We also choose the cyclic orientation of C so that

$$c_\ell c_{\ell+1} \in M_v \iff \ell \text{ is even}, \qquad 1 \leq \ell \leq 2r,$$

and

$$c_{2r+1}c_1 \notin M_v.$$

Therefore C is an M_v-blossom based at $p_1 = c_1$, and

$$|E(C) \cap M_v| = r = \frac{|V(C)|-1}{2}.$$

Define

$$x := M_e(v).$$

Then note that the edges $vM_v(v)$ and vx are distinct. Set

$$S := V(P) \cup V(C).$$

Since (P, C) is an M_v-perfect flower, every vertex of S is matched by M_v to another vertex of S. In particular, no edge of M_v has exactly one endpoint in S. By Lemma 2.1, the connected component K of

$$(V(M_v \triangle M_e),\ M_v \triangle M_e),$$

containing v is an even cycle whose edges alternate between M_v and M_e. Since $x = M_e(v) \notin S$, if we traverse K starting at v along the edge $vx \in M_e$, we eventually meet a vertex of S again, because K is a cycle and $v \in S$. Let u be the first such vertex after v, and let

$$Q = q_1, \ldots, q_w,$$

be the corresponding subpath of K, where $q_1 = v$ and $q_w = u$. By construction,

$$V(Q) \cap S = \{q_1, q_w\} = \{v, u\}.$$

Moreover, Q alternates between edges of M_e and edges of M_v, and its first edge is

$$q_1 q_2 = vx \in M_e.$$

Since the internal vertices of Q lie outside S and no edge of M_v joins S to $V(G) \setminus S$, the last edge $q_{w-1}q_w$ cannot belong to M_v. Hence

$$q_{w-1}q_w \in M_e.$$

Therefore,

$$q_\ell q_{\ell+1} \in M_e \quad \Longleftrightarrow \quad \ell \text{ is odd},$$

and

$$q_\ell q_{\ell+1} \in M_v \quad \Longleftrightarrow \quad \ell \text{ is even}, \qquad 1 \leq \ell < w.$$

In particular, w is even. Since $q_2 = x \notin V(P) \cup V(C)$ and $q_w = u \in V(P) \cup V(C)$, we have $w \geq 4$. Hence q_3 is defined and

$$q_3 \notin V(P) \cup V(C).$$

We now consider the possible positions of v and u in $V(P) \cup V(C)$. We keep the roles of v and u fixed throughout the argument: the vertex v is the endpoint of e that belongs to

$PF(G)$, whereas $x = q_2 = M_e(v)$ is the vertex that must be forced to belong to $PF(G)$. Thus no case below is obtained by interchanging u and v.

In each case, we explicitly identify an odd cycle D which is an M-blossom for the relevant perfect matching M, and an M-alternating path R whose first and last edges belong to M. We also check that $V(D) \cap V(R)$ consists of exactly one vertex.

Case 1. Assume that $v \in V(P)$, say $v = p_i$ for some odd i. According to the position of the vertex u, we consider the following subcases.

Case 1.1. Assume that $u \in V(P)$, say $u = p_j$ for some even j, see Figure 3. Consider the path

$$R := p_1, p_2, \dots, p_j (= q_w), q_{w-1}, q_{w-2}, \dots, q_2.$$

The cycle C is an M_v-blossom based at $p_1 = c_1$. Let us check that R is a valid attaching path. Its first edge is $p_1p_2 \in M_v$. Since j is even, the edge $p_{j-1}p_j$ also belongs to M_v. The next edge, namely q_wq_{w-1}, does not belong to M_v, because $q_{w-1}q_w \in M_e$. Along the reversed part of Q, the edges alternate with respect to M_v, and the last edge q_3q_2 belongs to M_v. Hence R is M_v-alternating and its first and last edges belong to M_v.

Moreover,

$$V(C) \cap V(R) = \{p_1\},$$

because $V(C) \cap V(P) = \{p_1\}$ and the vertices $q_2, q_3, \dots, q_{w-1}$ lie outside $V(P) \cup V(C)$. Therefore C together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

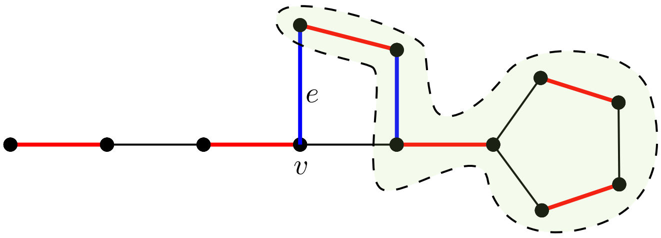

Fig. 3. Case 1.1: both vertices lie on P

Case 1.2. Assume that $u \in V(P)$, say $u = p_j$ for some odd j, see Figure 4 for the case $j > i$. Since $u \neq v$, we have $i \neq j$. Let

$$a := \min\{i, j\}, \qquad b := \max\{i, j\}, \qquad z := p_b.$$

Let D be the cycle obtained by joining Q with the p_j-p_i subpath of P. Equivalently,

$$D = \begin{cases} q_1, q_2, \dots, q_w, p_{j-1}, p_{j-2}, \dots, p_i, & \text{if } i < j, \\ q_1, q_2, \dots, q_w, p_{j+1}, p_{j+2}, \dots, p_i, & \text{if } j < i. \end{cases}$$

Since i and j are odd, the two edges of D incident with $z = p_b$ do not belong to M_v, whereas all the remaining edges of D alternate with respect to M_v. Thus D is an

M_v-blossom based at z. In particular,

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

Since b is odd and t is even, the vertex p_{b+1} exists and

$$zp_{b+1} = p_b p_{b+1} \in M_v.$$

The length-one path

$$R := z, p_{b+1},$$

is therefore M_v-alternating and its unique edge belongs to M_v. Also, $p_{b+1} \notin V(D)$, so

$$V(D) \cap V(R) = \{z\}.$$

Hence D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

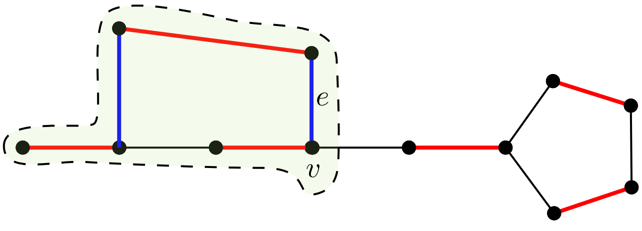

Fig. 4. Case 1.2: both vertices lie on P

Case 1.3. Assume that $u \in V(C) \setminus V(P)$, say $u = c_j$, see Figure 5. Let X be the u-p_1 path in C whose first edge belongs to M_v. Such a path exists because exactly one of the two edges of C incident with u belongs to M_v. Its last edge does not belong to M_v, because both edges of C incident with $p_1 = c_1$ are outside M_v.

Let $P[p_1, p_i]$ denote the p_1-p_i subpath of P, possibly of length zero if $i = 1$. Consider the cycle

$$D := X\, P[p_1, p_i]\, Q,$$

where the paths are concatenated in the natural way. At the vertex $v = p_i$, the two incident edges of D do not belong to M_v: one is the first edge q_1q_2 of Q, and the other is either the last edge of $P[p_1, p_i]$, when $i > 1$, or the last edge of X, when $i = 1$. All other edges of D alternate with respect to M_v. Hence D is an M_v-blossom based at v, and

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

The length-one path

$$R := v, M_v(v),$$

has its unique edge in M_v. Moreover, $M_v(v) \notin V(D)$, and hence

$$V(D) \cap V(R) = \{v\}.$$

Thus D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

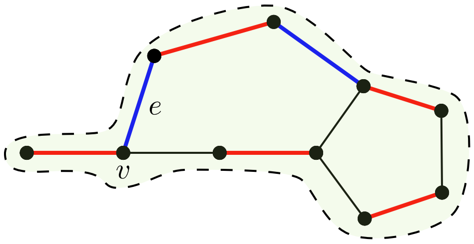

Fig. 5. Case 1.3: one vertex lies on P and the other on C

Case 2. Assume that $v \in V(P)$, say $v = p_i$ for some even i. Figure 6 illustrates the subcase in which u also lies on P.

Since $w \geq 4$, the vertex q_3 is defined and lies outside $V(P) \cup V(C)$. Consider the path

$$R := p_1, p_2, \dots, p_i (= q_1), q_2, q_3.$$

The cycle C is an M_v-blossom based at $p_1 = c_1$. The path R is M_v-alternating: its first edge p_1p_2 belongs to M_v, the edge $p_{i-1}p_i$ belongs to M_v because i is even, the edge q_1q_2 does not belong to M_v, and the edge q_2q_3 belongs to M_v. Hence the first and last edges of R belong to M_v.

Furthermore,

$$V(C) \cap V(R) = \{p_1\},$$

because $V(C) \cap V(P) = \{p_1\}$ and $q_2, q_3 \notin V(P) \cup V(C)$. Therefore C together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

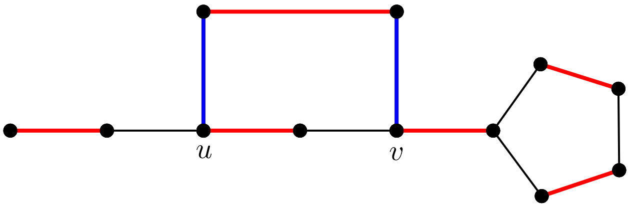

Fig. 6. Case 2: v lies on P at an even position; the drawing shows the subcase in which u also lies on P

Case 3. Assume that $v \in V(C) - V(P)$, say $v = c_i$ for some i. According to the position of the vertex u, we consider the following subcases.

Case 3.1. Assume that $u \in V(P)$, say $u = p_j$ for some odd j. Let Y be the p_1-v path in C whose last edge belongs to M_v. Such a path exists because exactly one of the two edges of C incident with v belongs to M_v. Its first edge does not belong to M_v, since both edges of C incident with $p_1 = c_1$ are outside M_v.

Let $P[u, p_1]$ denote the u-p_1 subpath of P, possibly of length zero if $u = p_1$. Consider the cycle

$$D := Y\, Q\, P[u, p_1],$$

where the paths are concatenated in the natural way. At the vertex $u = p_j$, the two incident edges of D do not belong to M_v: one is the last edge $q_{w-1}q_w$ of Q, and the other is either the first edge of $P[u, p_1]$, when $j > 1$, or the first edge of Y, when $j = 1$. All other edges of D alternate with respect to M_v. Hence D is an M_v-blossom based at u, and

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

The length-one path

$$R := u, M_v(u),$$

has its unique edge in M_v. Moreover, $M_v(u) \notin V(D)$, and hence

$$V(D) \cap V(R) = \{u\}.$$

Therefore D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

Case 3.2. Assume that $u \in V(P)$, say $u = p_j$ for some even j, see Figure 7. Consider the path

$$R := p_1, p_2, \ldots, p_j(= q_w), q_{w-1}, q_{w-2}, \ldots, q_2.$$

The cycle C is an M_v-blossom based at $p_1 = c_1$. Since j is even, the edge $p_{j-1}p_j$ belongs to M_v. The next edge $q_w q_{w-1}$ does not belong to M_v, and the reversed part of Q alternates with respect to M_v, ending with the edge $q_3q_2 \in M_v$. Therefore R is M_v-alternating and its first and last edges belong to M_v.

Moreover,

$$V(C) \cap V(R) = \{p_1\},$$

because $q_2, q_3, \ldots, q_{w-1}$ lie outside $V(P) \cup V(C)$. Hence C together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

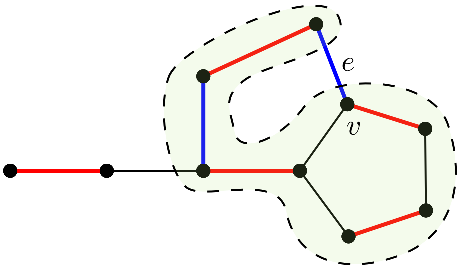

Fig. 7. Case 3.2: one vertex lies on $C - V(P)$ and the other on P.

Case 3.3. Assume that $u \in V(C) \setminus V(P)$. Then both vertices u and v lie in $V(C) \setminus V(P)$. Write

$$v = c_i, \qquad u = c_j.$$

By reversing the cyclic orientation of C, if necessary, we may assume that

$$2 \leq i < j \leq 2r + 1.$$

This does not interchange u and v; it only fixes the notation along the cycle C. Let

$$X := c_i, c_{i+1}, \dots, c_j.$$

Since X does not contain the base c_1, it is an M_v-alternating path. We consider the four possibilities determined by the first and last edges of X.

Case 3.3.1. If the first and last edges of X belong to M_v, then

$$C^* := q_1, q_2, \dots, q_w(= c_j), c_{j-1}, c_{j-2}, \dots, c_i(= q_1),$$

is an even M_v-alternating cycle, see Figure 8. Therefore

$$M := M_v \triangle E(C^*),$$

is again a perfect matching of G.

Now consider the cycle

$$D := c_1, c_2, \dots, c_i(= q_1), q_2, \dots, q_w(= c_j), c_{j+1}, \dots, c_{2r+1}, c_1,$$

where the segment $c_{j+1}, \dots, c_{2r+1}$ is omitted if $j = 2r + 1$. After rotating the matching along C^*, the cycle D is an M-blossom based at $c_1 = p_1$. Indeed, the edges of Q have been switched, the edges of $C \setminus X$ have not been switched, and the only two consecutive non-M edges on D are the two edges incident with c_1. Hence

$$|E(D) \cap M| = \frac{|V(D)| - 1}{2}.$$

The length-one path

$$R := p_1, p_2,$$

has its unique edge in M, and

$$V(D) \cap V(R) = \{p_1\}.$$

Therefore D together with R is an M-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

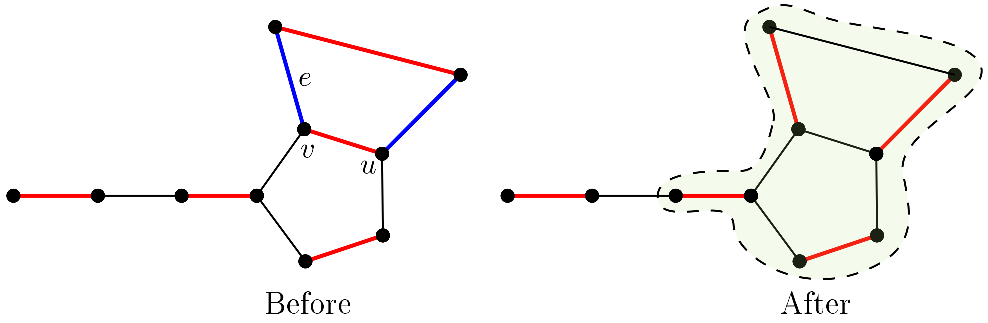

Fig. 8. Case 3.3.1: rotating the matching on the alternating cycle

Case 3.3.2. If neither the first nor the last edge of X belongs to M_v, then consider the cycle

$$D := c_1, c_2, \dots, c_i(= q_1), q_2, \dots, q_w(= c_j), c_{j+1}, \dots, c_{2r+1}, c_1,$$

where the segment $c_{j+1}, \dots, c_{2r+1}$ is omitted if $j = 2r + 1$. The cycle D is an M_v-blossom based at $c_1 = p_1$, see Figure 9. Indeed, the two edges of D incident with c_1 do not belong to M_v, while all remaining edges alternate with respect to M_v. Hence

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

The length-one path

$$R := p_1, p_2,$$

has its unique edge in M_v, and

$$V(D) \cap V(R) = \{p_1\}.$$

Therefore D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

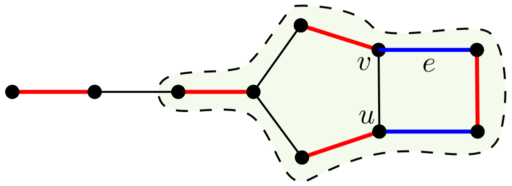

Fig. 9. Case 3.3.2: both end edges of X are outside M_v

Case 3.3.3. Assume that the first edge of X belongs to M_v, and the last edge of X does not belong to M_v. Consider the cycle

$$D := q_1, q_2, \dots, q_w(= c_j), c_{j-1}, c_{j-2}, \dots, c_i(= q_1).$$

Then D is an M_v-blossom based at $u = c_j$, see Figure 10. Indeed, the last edge $q_{w-1}q_w$ of Q and the last edge $c_{j-1}c_j$ of X both lie outside M_v, while all remaining edges of D alternate with respect to M_v. Hence

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

Since the last edge of X does not belong to M_v, the edge $c_jc_{j+1} = uM_v(u)$ belongs to M_v. Thus the length-one path

$$R := c_j, c_{j+1},$$

has its unique edge in M_v, and

$$V(D) \cap V(R) = \{c_j\}.$$

Therefore D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

Case 3.3.4. Assume that the first edge of X does not belong to M_v, and the last edge of X belongs to M_v. Consider again the cycle

$$D := q_1, q_2, \ldots, q_w(= c_j), c_{j-1}, c_{j-2}, \ldots, c_i(= q_1).$$

Then D is an M_v-blossom based at $v = c_i$. Indeed, the first edge q_1q_2 of Q and the first edge c_ic_{i+1} of X both lie outside M_v, while all remaining edges of D alternate with respect to M_v. Hence

$$|E(D) \cap M_v| = \frac{|V(D)| - 1}{2}.$$

Since the first edge of X does not belong to M_v, the edge $c_ic_{i-1} = vM_v(v)$ belongs to M_v. Thus the length-one path

$$R := c_i, c_{i-1},$$

has its unique edge in M_v, and

$$V(D) \cap V(R) = \{c_i\}.$$

Therefore D together with R is an M_v-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$.

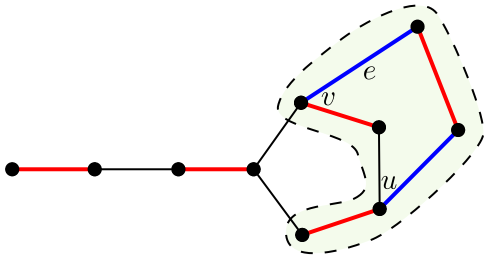

Fig. 10. Case 3.3.3: the first edge of X belongs to M_v, but the last does not

The four subcases in Case 3.3 exhaust all possibilities for the first and last edges of X. Together with Cases 1, 2, 3.1, and 3.2, this covers all possible positions of v and u in $V(P) \cup V(C)$, without interchanging their roles. In every case we have obtained a perfect matching M and an M-perfect flower containing $x = q_2$. This contradicts $x \in PFF(G)$. Therefore no such edge e exists, and

$$E(PFF(G), PF(G)) \cap E(H) = \emptyset.$$

□

Then, from Theorem 3.4 we obtain the main result of this work.

Theorem 3.5. *Let G be a Kőnig-Egerváry graph. Then*

$$\begin{aligned}\det(G) &= \det\left(G[PF(G)]\right)\det\left(G[PFF(G)]\right),\\ \operatorname{perm}(G) &= \operatorname{perm}(G[PF(G)])\operatorname{perm}(G[PFF(G)]).\end{aligned}$$

Proof. Set

$$G_1 := G[PF(G)] \quad \text{and} \quad G_2 := G[PFF(G)].$$

If $\mathrm{Sachs}(G) = \emptyset$, then by Theorem 3.1,

$$\det(G) = 0 \quad \text{and} \quad \mathrm{perm}(G) = 0.$$

Moreover, at least one of the graphs G_1 and G_2 has no Sachs subgraphs. Indeed, if there existed

$$H_1 \in \mathrm{Sachs}(G_1) \quad \text{and} \quad H_2 \in \mathrm{Sachs}(G_2),$$

then $H_1 \cup H_2$ would be a Sachs subgraph of G, a contradiction. Hence at least one of $\det(G_1), \det(G_2)$ is zero, and at least one of $\mathrm{perm}(G_1), \mathrm{perm}(G_2)$ is zero. Therefore,

$$\det(G) = \det(G_1)\det(G_2) \quad \text{and} \quad \mathrm{perm}(G) = \mathrm{perm}(G_1)\mathrm{perm}(G_2).$$

Assume now that $\mathrm{Sachs}(G) \neq \emptyset$. Since every Sachs subgraph is spanning, we have $\mathrm{prk}(G) = |G|$. Thus, by Corollary 3.3,

$$|G| = \mathrm{prk}(G) = 2\mu(G),$$

and therefore G has a perfect matching.

Let $H \in \mathrm{Sachs}(G)$. By Theorem 3.4,

$$E(PF(G), PFF(G)) \cap E(H) = \emptyset.$$

Hence every connected component of H lies entirely in $PF(G)$ or entirely in $PFF(G)$. Therefore

$$H = H_1 \cup H_2,$$

where

$$H_1 := H[PF(G)] \quad \text{and} \quad H_2 := H[PFF(G)].$$

Clearly,

$$H_1 \in \mathrm{Sachs}(G_1) \quad \text{and} \quad H_2 \in \mathrm{Sachs}(G_2).$$

Conversely, if

$$H_1 \in \mathrm{Sachs}(G_1) \quad \text{and} \quad H_2 \in \mathrm{Sachs}(G_2),$$

then $H_1 \cup H_2 \in \mathrm{Sachs}(G)$. Thus

$$H \mapsto (H_1, H_2),$$

is a bijection between $\mathrm{Sachs}(G)$ and $\mathrm{Sachs}(G_1) \times \mathrm{Sachs}(G_2)$.

Since the connected components of H are precisely the connected components of H_1 together with those of H_2, we have

$$k(H) = k(H_1) + k(H_2) \quad \text{and} \quad m(H) = m(H_1) + m(H_2).$$

Hence

$$(-1)^{k(H)} 2^{m(H)} = (-1)^{k(H_1)+k(H_2)} 2^{m(H_1)+m(H_2)}$$

$$= (-1)^{k(H_1)}2^{m(H_1)}(-1)^{k(H_2)}2^{m(H_2)},$$

and also

$$2^{m(H)} = 2^{m(H_1)}2^{m(H_2)}.$$

Using Theorem 3.1 and the above bijection, we obtain

$$\begin{aligned}
\det(G) &= \sum_{H\in \mathrm{Sachs}(G)} (-1)^{k(H)}2^{m(H)}\\
&= \sum_{H_1\in \mathrm{Sachs}(G_1)} \sum_{H_2\in \mathrm{Sachs}(G_2)} (-1)^{k(H_1)}2^{m(H_1)}(-1)^{k(H_2)}2^{m(H_2)}\\
&= \left(\sum_{H_1\in \mathrm{Sachs}(G_1)} (-1)^{k(H_1)}2^{m(H_1)}\right)\left(\sum_{H_2\in \mathrm{Sachs}(G_2)} (-1)^{k(H_2)}2^{m(H_2)}\right)\\
&= \det(G_1)\det(G_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{perm}(G) &= \sum_{H\in \mathrm{Sachs}(G)} 2^{m(H)}\\
&= \sum_{H_1\in \mathrm{Sachs}(G_1)} \sum_{H_2\in \mathrm{Sachs}(G_2)} 2^{m(H_1)}2^{m(H_2)}\\
&= \left(\sum_{H_1\in \mathrm{Sachs}(G_1)} 2^{m(H_1)}\right)\left(\sum_{H_2\in \mathrm{Sachs}(G_2)} 2^{m(H_2)}\right)\\
&= \mathrm{perm}(G_1)\mathrm{perm}(G_2).
\end{aligned}$$

□

By Lemma 3.2 and Theorem 3.4, one has the additivity of $\mu(G)$ according to the perfect flower decomposition.

Corollary 3.6. *Let G be a Kőnig-Egerváry graph with a perfect matching. Then*

$$\mu(G) = \mu(G[PF(G)]) + \mu(G[PFF(G)]).$$

Proof. Let M_1 and M_2 be maximum matchings of $G[PF(G)]$ and $G[PFF(G)]$, respectively. Since $PF(G)$ and $PFF(G)$ are disjoint, $M_1 \cup M_2$ is a matching of G. Hence

$$\mu(G) \geq \mu(G[PF(G)]) + \mu(G[PFF(G)]).$$

On the other hand, let M be a perfect matching of G. Since $G[M] \in \mathrm{Sachs}(G)$, Theorem 3.4 yields

$$E(PF(G), PFF(G)) \cap M = \emptyset.$$

Therefore,

$$M = (M \cap E(G[PF(G)])) \cup (M \cap E(G[PFF(G)])),$$

and so

$$\mu(G) = |M| \leq \mu(G[PF(G)]) + \mu(G[PFF(G)]).$$

Thus,

$$\mu(G) = \mu(G[PF(G)]) + \mu(G[PFF(G)]).$$

□

To illustrate Theorem 3.5, consider the graph in Figure 11. Here one has $\det(G[PFF(G)]) = -4$, while $\det(G[PF(G)]) = 9$, therefore, by Theorem 3.5, it follows that

$$\det(G) = -4 \cdot 9 = -36.$$

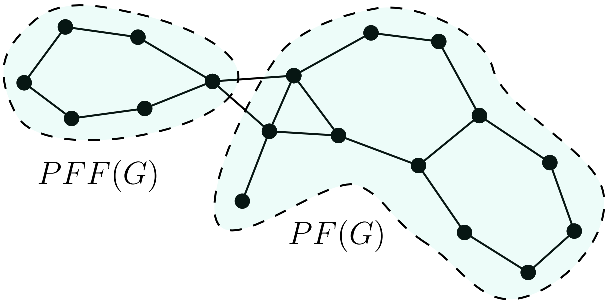

Fig. 11. An example illustrating the determinant factorization via $PF(G)$ and $PFF(G)$

Therefore, by Theorem 3.5, studying when a Kőnig-Egerváry graph is unimodular reduces to studying when the graphs $G[PFF(G)]$ and $G[PF(G)]$ are unimodular. This motivates the following problems.

4. Open Problems

It is known that every Kőnig-Egerváry graph with a unique matching is unimodular; however, the converse is not true. Indeed, unimodularity is only enough to guarantee the existence of a perfect matching. For example, this occurs in the graph obtained by considering two cycles C_4 with one common edge. Consequently, the study of unimodularity in Kőnig–Egerváry graphs can be reduced to the analysis of the following two problems. When G is a Kőnig-Egerváry graph with a perfect matching, then by Theorem 1.1, both graphs

$$G[PF(G)], \qquad G[PFF(G)],$$

also have a perfect matching and are both Kőnig-Egerváry graphs, so the most natural thing in this context is to study the following.

Problem 4.1. *Characterize the unimodular Kőnig-Egerváry graphs G such that $G[PF(G)] = G$.*

Problem 4.2. *Characterize unimodular perfect-flower-free Kőnig-Egerváry graphs, that is, graphs G such that $G[PFF(G)] = G$.*

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT-3.5 in order to improve the grammar of several paragraphs of the text. After using this service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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