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The existence of C_4-saturated graphs having sizes close to the lower bound

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ABSTRACT

A graph G is *H-saturated* if G does not have H as a subgraph but $G + uv$ has at least one copy of H for any edge $uv \notin E(G)$. The smallest number of edges of all H-saturated graphs of order n is called H-saturation number and is denoted by $sat(n; H)$. In this paper, we establish the existence of C_4-saturated graphs with prescribed the number of edges in some length that is close to $sat(n; C_4)$.

Keywords: extremal number, even cycle, saturation number, saturation spectrum

2020 Mathematics Subject Classification: 05C38, 05C35.

1. Introduction and motivation

We let $G = (V(G), E(G))$ be a simple graph. The *order* of G is $n = |V(G)|$ and the *size* of G is $m = |E(G)|$. The *complement* of G is denoted by $\overline{G}$. A *cycle* of n vertices is denoted by C_n. A *path* of n vertices is denoted by P_n. A *complete k-partite* graph with the partite sets of $n_1, n_2, n_3, ..., n_k$ vertices is denoted by $K_{n_1,n_2,n_3,...,n_k}$. When $k = 2$, the graph K_{n_1,n_2} is called *complete bipartite*. A *star* is a complete bipartite K_{n_1,n_2} when $1 \in \{n_1, n_2\}$. For graphs G and H, we say that G is *H-saturated* if G does not have H as a subgraph but $G + uv$ has a copy of H as a subgraph for any $uv \notin E(G)$. The smallest size (the number of edges) of all H-saturated graphs of n vertices is called the *H-saturation number* and is denoted by $sat(n; H)$. On the other hand, the largest size of all H-saturated graphs of n vertices is called the *H-extremal number* and is denoted by $ex(n; H)$.

In 1907, Mantel [12] proved that graphs with maximum size that do not have a copy of K_3 as a subgraph is the complete bipartite whose the two partite sets have equally likely the number of vertices. That is:

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Theorem 1.1. [12] *For a natural number $n \geq 3$, we have that*

$$ex(n;K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Further, a K_3-saturated graph G of order n satisfies $|E(G)|= ex(n;K_3)$ if and only if G is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

In 1941, Turán [14] extended Theorem 1.1 by establishing K_p-saturated graphs, for any integer $p \geq 3$, with the largest size.

Theorem 1.2. [14] *For natural numbers $n \geq p \geq 3$, we have that*

$$|E(G)| \leq ex(n;K_p) = \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2} + o(n^2).$$

Further, a K_p-saturated graph G of order n satisfies $|E(G)|= ex(n;K_p)$ if and only if G is $K_{n_1,n_2,n_3,\ldots,n_{p-1}}$ where $n_1 + n_2 + n_3 + \cdots + n_{p-1} = n$ and $|n_i - n_j| \leq 1$ for all $i, j \in \{1, 2, 3, \ldots, p-1\}$.

These has motivated many studies in $ex(n;H)$ for arbitrary graphs H. For instant, see Erdős and Gallai [5] for the study of $ex(n;P_k)$, Dogan [4] for the study of $ex(n;K_{1,k})$ and Győri et al. [10] for the study of $ex(n;C_{2k})$.

It can be asked, on the other hand, how the lower bound, $sat(n;H)$, grows. In 1964, Erdős et al. [6] found the minimum size of K_p-saturated graphs. Interestingly, the growth of $sat(n;K_p)$ is just linear in terms of n.

Theorem 1.3. [6] *For integers n and p such that $n \geq p-2 \geq 1$, we have that*

$$sat(n;K_p) = \binom{p-2}{2} + (n-p+2)(p-2).$$

A K_p-saturated graph G satisfying $|E(G)|= sat(n;K_p)$ if and only if G is obtained by joining every vertex of $(p-2)$-clique to every vertex of $\overline{K}_{n-p+2}$.

For more example of the studies of $sat(n;H)$, see Füredi and Kim [8] when H is a cycle and Kászonyi and Tuza [11] when H is a path, a star and a union of edges.

One other direction of the studies of H-saturated graphs is to establish the existence of these graphs having size between the upper and lower bounds. The first result was obtained in 1995 when Barefoot et al. [2] proved the existence of K_3-saturated graphs having sizes between the bounds. The result of [2] was generalized to K_p-saturated graphs when $p \geq 3$ by Amin et al. [1], and was extended to C_{2k+1}-saturated by Gould et al. [9].

When the length of cycle is even, Ollermann [13] and Tuza [15] independently established $sat(n;C_4)$ and Erdős [7] established $ex(n;C_4)$.

Theorem 1.4. *([7],[13] and [15]) Let G be a C_4-saturated graph containing n vertices. Then*

$$\left\lfloor \frac{3n-5}{2} \right\rfloor \leq |E(G)| \leq \frac{n}{4}\left(1 + \sqrt{4n-3}\right).$$

To the best of our knowledge, there is no work that provided exact description of the edge spectrum for C_4-saturated graphs. Although uniquely C_4-saturated graphs (the graphs which the addition of any missing edge creates exactly one copy of C_4) have been investigated in [3], this work focused on structural characterizations rather than on attainable edge counts. As we mentioned earlier, the edge spectrum of odd cycle saturated graphs has been studied by Gould et al. [9] by determining the saturation spectrum of obtained by bipartite-based construction. However, we can not apply this construction to our C_4-saturated because a subgraph whose vertices in different partite sets always have C_4. In this paper, we establish that, for infinitely many n, there exist C_4-saturated graphs of order n and size m for every integer m ranging from $3n/2$ to $2n$.

2. The existence of C_4-saturated graphs with prescribed sizes

In this section, we show our main results concerning the existence of C_4-saturated graphs. In the first subsection, we display C_4-saturated graphs of small orders while a construction for arbitrary large C_4-saturated graphs is provided in the next subsection.

2.1. Collections of C_4-saturated graphs of small orders

In this section, we list C_4-saturated graphs of order at most 12, some of which may be the same as a particular case of our constructions in the next subsection. The bounds are obtained from Theorem 1.4.

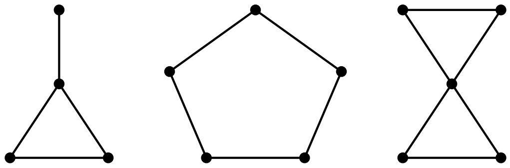

Fig. 1. Graphs F_1, F_2 and F_3 (from left to right)

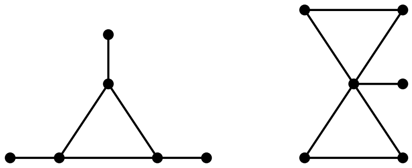

Fig. 2. Graphs F_4 (left) and F_5 (right)

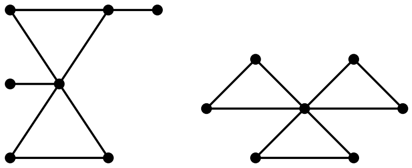

Fig. 3. Graphs F_6 (left) and F_7 (right)

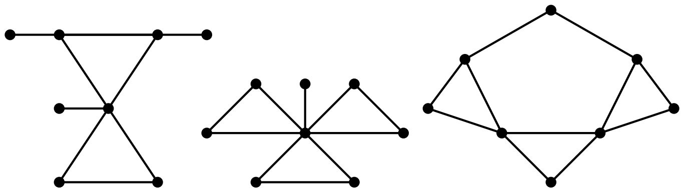

Fig. 4. Graphs F_8, F_9 and F_{10} (from left to right)

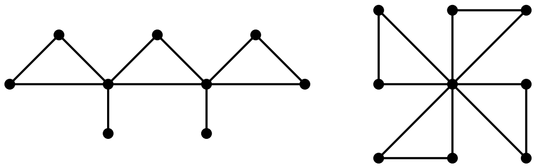

Fig. 5. Graphs F_{11} (left), F_{12} (right)

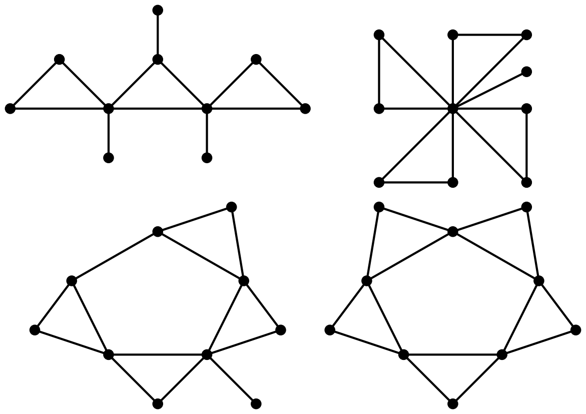

Fig. 6. Graphs F_{13} (top left),F_{14} (top right), F_{15} (bottom left) and F_{16} (bottom right)

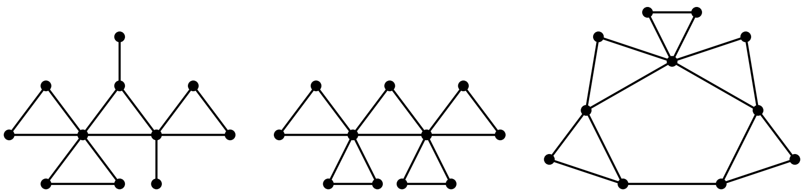

Fig. 7. Graphs F_{17} (left), F_{18} (middle) and F_{19} (right)

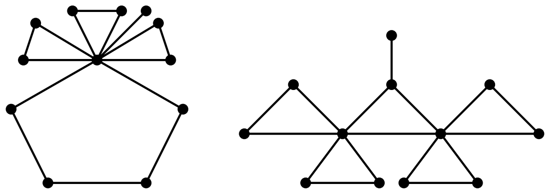

Fig. 8. Graphs F_{20} (left), F_{21} (right)

We summarize the existences of C_4-saturated graphs from the list in the following table. The symbol "N/A" (not investigated) shows that we are not able to find a C_4-satirated graph having the order and size of that entry in which the graph might or might not exists.

Table 1. The existence of C_4-saturated graphs of orders at most 12

n \m	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
4	-	F_1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	-	-	F_2	F_3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
6	-	-	-	F_4	F_5	N/A	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	F_6	F_7	N/A	-	-	-	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	F_8	F_9	F_{10}	N/A	-	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	F_{11}	F_{12}	$G(3)$	N/A	N/A	-	-	-	-	-	-	-	-
10	-	-	-	-	-	-	-	-	-	F_{13}	F_{14}	F_{15}	F_{16}	-	-	-	-	-	-	-	-
11	-	-	-	-	-	-	-	-	-	-	-	F_{17}	F_{18}	F_{19}	$G(4)$	N/A	N/A	N/A	-	-	-
12	-	-	-	-	-	-	-	-	-	-	-	-	F_{20}	F_{21}	N/A	N/A	N/A	N/A	N/A	N/A	N/A

2.2. *Constructions of C_4-saturated graphs*

We first consider the case when the order n of C_4-saturated graphs is odd.

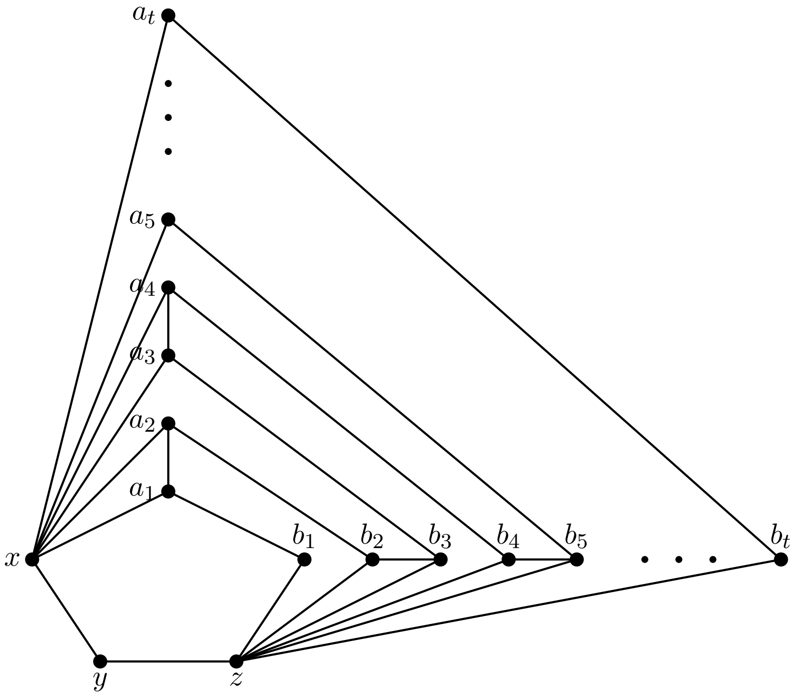

Fig. 9. A graph C_4-saturated graph

Construction. Let $P = xyz$ be a path of length 2 and $P^1 = a_1b_1, P^2 = a_2b_2, ..., P^t = a_tb_t$ be t paths of length 1. The graph $G(t)$ is constructed from $P, P^1, P^2, ..., P^t$ by adding edges from x to all $a_1, a_2, a_3, ..., a_t$, from z to all $b_1, b_2, b_3, ..., b_t$, and adding edges $a_1a_2, a_3a_4, a_5a_6, ..., a_{t-2}a_{t-1}, b_2b_3, b_4b_5, b_6b_7..., b_{t-1}b_t$ when t is odd, and adding edges a_1a_2, $a_3a_4, a_5a_6, ..., a_{t-1}a_t, b_2b_3, b_4b_5, b_6b_7, ..., b_{t-2}b_{t-1}$ when t is even. Observe that $|V(G(t))|= 2t+3$ and $|E(G(t))|= 4t+1$.

Lemma 2.1. *For $t \geq 2$, there exists a C_4-saturated graph with $2t+3$ vertices and $4t+1$ edges.*

Proof. We first show that $G(t)$ does not contain C_4 as a subgraph. We may consider our graph according to the edges from the construction. We see that the smallest cycles that contain the edge a_1b_1 are only $a_1b_1zyxa_1$ and $a_1b_1zb_2a_2a_1$, which are not cycles of length 4. Hence, the edge a_1b_1 is not contained in any C_4. Similarly, the edges $a_2b_2, \ldots, a_tb_t$ are not contained in any C_4. We see that the smallest cycles that contain the edge a_1a_2 must pass through either x or some vertex b_j, and hence have length 3 or at least 5. Therefore, the edge a_1a_2 is not contained in any C_4. Similarly, the edges $a_3a_4, a_5a_6, \ldots$ are not contained in any C_4. Moreover, when t is odd, the edges $b_2b_3, b_4b_5, b_6b_7, \ldots, b_{t-1}b_t$ are added, and when t is even, the edges $b_2b_3, b_4b_5, \ldots, b_{t-2}b_{t-1}$ are added. In both cases, these edges are contained only in cycles of length 3 or at least 5, and hence are not contained in any C_4. Thus, the edges $xy, yz, xa_1, \ldots, xa_t, zb_1, \ldots, zb_t$ are not contained in any C_4. Therefore, $G(t)$ does not contain C_4 as a subgraph.

It remains to show that $G(t) + ab$ has a C_4 for any non-edge $ab \notin E(G(t))$.

Case 1. $y \in \{a, b\}$ Without loss of generality $y = a$. Clearly $b \in \{a_1, a_2, a_3, ..., a_t, b_1, b_2, b_3, ..., b_t\}$. If $b \in \{a_1, a_2, a_3, ..., a_t\}$, $b = a_i$ says, then ya_ib_izy is a C_4. If $b \in \{b_1, b_2, b_3, ..., b_t\}$, $b = b_i$ says, then yb_ia_ixy is a C_4. This proves Case 1.

Case 2. $x \in \{a, b\}$ Without loss of generality $x = a$. Thus $b \in \{b_1, b_2, b_3, ..., b_t, z\}$. If $b = z$, then xzb_1a_1x is a C_4. Hence we let $b \in \{b_1, b_2, b_3, ..., b_t\}$, $b = b_i$ says, We have xb_izyx is a C_4. This proves Case 2.

Case 3. $z \in \{a, b\}$ By similar arguments as Case 2, we prove this case.

Case 4. $a_i \in \{a, b\}$ for some $i \in \{1, 2, 3, ..., t\}$ Without loss of generality $a_i = a$. Thus $b \in \{y, z, b_1, b_2, b_3, ..., b_t, a_1, a_2, a_3, ..., a_t\} \backslash \{b_i, a_j\}$, when $a_ia_j \in E(G(t))$. It is possible that a_j does not exist when t is odd and $i = t$. If $b = y$, then $a_iyzb_ia_i$ is a C_4. If $b = z$, then a_izyxa_i is a C_4. If $b = b_{i'}$, for some $i' \in \{1, 2, 3, ..., t\} \backslash \{i\}$, then $a_ib_{i'}a_{i'}xa_i$ is a C_4. Thus we consider when $b = a_{i'}$ for some $i' \in \{1, 2, 3, ..., t\} \backslash \{i\}$. If i is even, then $a_ia_{i'}xa_{i-1}a_i$ is a C_4. If i is odd but $i < t$, then $a_ia_{i'}xa_{i+1}a_i$ is a C_4. If i is odd but $i = t$, then $a_ia_{i'}a_{i'-1}xa_i$ is a C_4 when i' is even and $a_ia_{i'}a_{i'+1}xa_i$ is a C_4 when i' is odd. This proves Case 4.

Case 5. $b_i \in \{a, b\}$ By similar arguments as Case 4, we prove this case. and this completes the proof. □

By Lemma 2.1 and the construction of $G(t)$, we see that the graph $G(3)$ has 9 vertices and 13 edges while the graph $G(4)$ has 11 vertices and 17 edges. These graphs are counted in the list of Table 1.

A *windmill graph* $W(t, k)$ is obtained from t cycles of k vertices by identifying one vertex of each cycle together and we call this the vertex c. When $t \geq 2$, it can be observed that $W(t, 3)$ is a C_4-saturated graph of order $2t+1$ and size $3t$. Further, for $t \geq 1$, we let the graph $W'(t,3)$ be obtained from the graph $W(t,3)$ and two vertex x and y by joining x to a vertex of degree two of one cycle of length 3 and joining y to the other vertex of degree two of the same cycle. It can be observed that $W'(t,3)$ is a C_4-saturated graph of $2t+3$ vertices, containing $3t+2$ edges. Hence, by Lemma 2.1 and the graphs $W(t,3)$ and $W'(t,3)$, we have the existence of C_4-saturated graphs when n is odd.

Corollary 2.2. *For an odd natural number $n \geq 5$, there exists a C_4-saturated graph of n vertices, containing m edges where $m \in \left\{\frac{3(n-1)}{2} - 1, \frac{3(n-1)}{2}, 2n-5\right\}$.*

When n is even, we let $G'(t)$ be the graph obtained from $G(t)$ and a vertex u by joining u to x and y. By similar arguments of the proof of Lemma 2.1, we can show that $G'(t)$ is a C_4-saturated graph of $2t+4$, having $4t+3$ edges. Further, we let $G''(t)$ be the graph obtained from $G'(t)$ and vertices v, w by joining v to y and z and joining w to y. It can be checked that $G''(t)$ is a C_4-saturated graph of $2t+6$ vertices, having $4t+6$ edges.

For the auxiliary of windmill graph, we let $\hat{W}(t,3)$ be obtained from $W(t,3)$ and a vertex z by joining z to the vertex c. Hence, for $t \geq 1$, $\hat{W}(t,3)$ is a C_4-saturated graph of $2t+2$ having $3t+1$ edges. Further, the graph $\tilde{W}(t,3)$ be obtained from $W'(t,3)$ and a vertex z by joining z to c. It can be checked that the graph $\tilde{W}(t,3)$ is a C_4-saturated with $2t+4$ vertices, having $3t+3$ edges. Thus, we have the existence of C_4-saturated graphs when n is even as follows.

Corollary 2.3. *For an even natural number $n \geq 4$, there exists a C_4-saturated graph of n vertices, containing m edges where $m \in \left\{\frac{3n}{2} - 3, \frac{3n}{2} - 2, 2n-6, 2n-5\right\}$.*

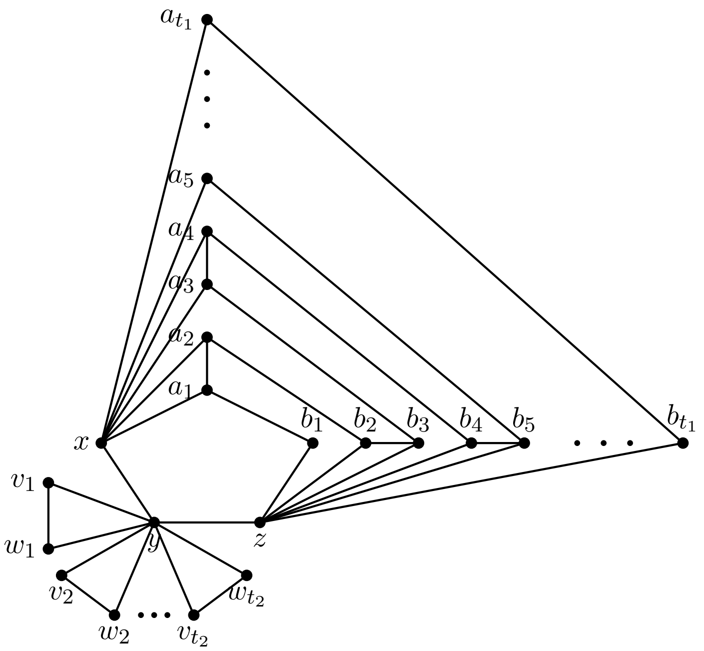

Fig. 10. The graph $H(t_1, t_2)$

Finally, we construct C_4-saturated graphs which can be varied the number of edges while the order is fixed.

Construction. The graph $H(t_1, t_2)$ is constructed from $G(t_1)$ and $W(t_2, 3)$ by identifying the vertex y of $G(t_1)$ with the vertex c of $W(t_2, 3)$. Observe that $|V(H(t_1, t_2))|= 2(t_1 + t_2) + 3$ and $|E(H(t_1, t_2))|= 4t_1 + 3t_2 + 1$. Further, the graph $H'(t_1, t_2)$ is constructed from $G'(t_1)$ and $W(t_2, 3)$ by identifying the vertex y of $G'(t_1)$ with the vertex c of $W(t_2, 3)$. Observe that $|V(H'(t_1, t_2))|= 2(t_1 + t_2) + 4$ and $|E(H'(t_1, t_2))|= 4t_1 + 3t_2 + 3$.

By the above construction, we can establish the following theorem. Let

$$Spec_{C_4}(n) := \{m : \exists \text{ a } C_4 - \text{saturated graph } G \text{ on } n \text{ vertices with } |E(G)|= m\}.$$

Theorem 2.4. *For positive integers n and $t \geq 1$ such that $n = 2t + 3$, we have that*

$$Spec_{C_4}(n) = \{3t + 2, 3t + 3, 3t + 4, ..., 4t + 1\}.$$

Further, if $n = 2t + 4$, we have that

$$Spec_{C_4}(n) = \{3t + 4, 3t + 5, 3t + 6, ..., 4t + 3\}.$$

Proof. Let $t = t_1 + t_2$ where $t_1 \geq 1$ and $t_2 \geq 0$ are integers.

We first consider the case when $n = 2t + 3$.

By construction, the graph $H(t_1, t_2)$ is obtained from $G(t_1)$ and $W(t_2, 3)$ by identifying the vertex y of $G(t_1)$ with the vertex c of $W(t_2, 3)$. Hence,

$$|V(H(t_1, t_2))|= |V(G(t_1))|+|V(W(t_2, 3))|-1 = (2t_1 + 3) + (2t_2 + 1) - 1 = 2t + 3,$$

and

$$|E(H(t_1, t_2))|= |E(G(t_1))|+|E(W(t_2, 3))|= (4t_1 + 1) + 3t_2 = 4t_1 + 3t_2 + 1.$$

Next, we show that $H(t_1, t_2)$ does not contain a 4-cycle. From the construction, by Lemma 2.1, Corollaries 2.2 and 2.3, both $G(t_1)$ and $W(t_2, 3)$ do not contain C_4. Moreover, these two graphs intersect in exactly one vertex. Consequently, any cycle in $H(t_1, t_2)$ must be entirely contained in either $G(t_1)$ or $W(t_2, 3)$. Since neither of them contains a cycle of length four, it follows that $H(t_1, t_2)$ contains no C_4.

We now show that adding any missing edge to $H(t_1, t_2)$ creates a C_4. Let uv be a non-edge of $H(t_1, t_2)$. If both u and v belong to $G(t_1)$, then, since $G(t_1)$ is C_4-saturated, the graph $G(t_1) + uv$ contains a C_4, and hence so does $H(t_1, t_2) + uv$. The same argument applies when both vertices lie in $W(t_2, 3)$. If u lies in $G(t_1)$ and v lies in $W(t_2, 3)$, then together with the common vertex y, the edge uv forms a 4-cycle using edges of the construction. Therefore, every added edge produces a C_4.

Thus, $H(t_1, t_2)$ is C_4-saturated having $n = 2t + 3$ vertices. For each $m \in \{3t + 2, 3t + 3, ..., 4t+1\}$, we can let $m = 3t+1+i$ for some $1 \leq i \leq t$. We let $t_1 = i$ and $t_2 = t-i$. Thus, for a given m, the graph $H(i, t-i)$ has $2t+3$ vertices and $4i+3(t-i)+1 = 3t+1+i = m$ edges as required.

The argument for the case $n = 2t + 4$ is analogous. The graph $H'(t_1, t_2)$ satisfies

$$|V(H'(t_1,t_2))| = 2t + 4 \quad \text{and} \quad |E(H'(t_1,t_2))| = 4t_1 + 3t_2 + 3.$$

As in the previous case, $H'(t_1, t_2)$ contains no C_4, and adding any missing edge creates a C_4. Hence, $H'(t_1, t_2)$ is C_4-saturated.

Therefore, for $n = 2t + 4$,

$$Spec_{C_4}(n) = \{3t + 4, 3t + 5, \ldots, 4t + 3\}.$$

This completes the proof. □

Example 2.5. The following table shows that when $n = 19$. We have

Table 2. The number of edges when $n = 19$

t_1	t_2	m
1	7	26
2	6	27
3	5	28
4	4	29
5	3	30
6	2	31
7	1	32
8	0	33

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