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The total monophonic number of a graph

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ABSTRACT

For a connected graph G of order at least two, a total monophonic set of a graph G is a monophonic set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total monophonic set of G is the total monophonic number of G and is denoted by $m_t(G)$. We determine bounds for it and characterize graphs which realize the lower bound. Also, some general properties satisfied by this concept are studied. It is shown that for positive integers a, b such that $3 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $m(G) = a$ and $m_t(G) = b$. Further, if p, a, b are positive integers such that $4 \leq a \leq b \leq p$, then there exists a connected graph G of order p with $m_t(G) = a$ and $m_c(G) = b$, where $m_c(G)$ is the connected monophonic number of G.

Keywords: monophonic set, monophonic number, total monophonic set, total monophonic number

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic* [1]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v of G is called an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v of G is called a *support vertex* of G if it is adjacent to an end-vertex of G.

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A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on a $x-y$ monophonic path for some elements x and y in S. The *monophonic number* of G is the minimum cardinality of its monophonic sets and is denoted by $m(G)$. The monophonic number of a graph and its related concepts were studied by several authors in [2, 3, 4, 6, 9]. A *connected monophonic set* of G is a monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected monophonic set of G is the *connected monophonic number* of G and is denoted by $m_c(G)$. The connected monophonic number of a graph was introduced and studied in [10]. There are useful applications of these concepts to protected facility location problems in real life situations, and also to security based communication network designs [10].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\ \{d_m(v,u) : u \in V\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min\ \{e_m(v) : v \in V\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max\ \{e_m(v) : v \in V\}$. The monophonic distance was introduced in [7] and further studied in [8]. This paper explores total monophonic sets as a structural extension of monophonic sets, examining properties and comparisons with other graph parameters. Future work could explore specific applications.

The following theorems will be used in the sequel.

Theorem 1.1. [9] *Every extreme vertex of a connected graph G belongs to every monophonic set of G. In particular, if the set S of all extreme vertices of G is a monophonic set, then S is the unique minimum monophonic set of G.*

Theorem 1.2. [9] *Let G be a connected graph with cut-vertices and S a monophonic set of G. If v is a cut-vertex of G, then every component of $G-v$ contains an element of S.*

Theorem 1.3. [9] *For the complete graph K_p $(p \geq 2)$, $m(K_p) = p$.*

Theorem 1.4. [10] *Every cut-vertex of a connected graph G belongs to every connected monophonic set of G.*

Theorem 1.5. [10] *For the complete graph K_p $(p \geq 2)$, $m_c(K_p) = p$.*

Throughout this paper, G denotes a connected graph with at least two vertices.

2. Total monophonic number

Definition 2.1. A *total monophonic set* of a graph G is a monophonic set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total monophonic set of G is the *total monophonic number* of G and is denoted by $m_t(G)$.

Example 2.2. For the graph G given in Figure 1, it is clear that $S = \{u, z\}$ is the unique minimum monophonic set of G and so $m(G) = 2$. Since the subgraph induced by S has the isolated vertices u and z, S is not a total monophonic set of G. Also for any $x \in V - S$, $S \cup \{x\}$ is not a total monophonic set of G. Clearly, $S_1 = S \cup \{v, y\}$ is a total monophonic set of G and so $m_t(G) = 4$. Since the subgraph induced by S_1 is not connected, S_1 is not a connected monophonic set of G. Note that $S_1 \cup \{w\}$ is a minimum connected monophonic set of G and so $m_c(G) = 5$. Thus the monophonic number, the total monophonic number and the connected monophonic number of a graph G are different.

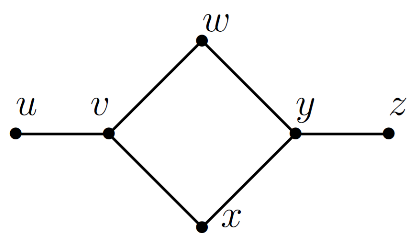

Fig. 1. G

Theorem 2.3. *All the extreme vertices and support vertices of a connected graph G belong to every total monophonic set of G. If the set S of all extreme vertices and support vertices of G is a total monophonic set, then it is the unique minimum total monophonic set of G.*

Proof. Since every total monophonic set of G is a monophonic set of G, by Theorem 1.1, every extreme vertex belongs to every total monophonic set of G. Since a total monophonic set of G contains no isolated vertices, it follows that every support vertex of G also belongs to every total monophonic set of G. Thus, if S is the set of all extreme vertices and support vertices of G, then $m_t(G) \geq |S|$. On the other hand, if S is a total monophonic set of G, then $m_t(G) \leq |S|$. Therefore $m_t(G) = |S|$ and S is the unique minimum total monophonic set of G. □

Corollary 2.4. *For the complete graph $K_p (p \geq 2)$, $m_t(G) = p$.*

Remark 2.5. The converse of Corollary 2.4 need not be true. For the path P_4, every vertex of P_4 is either an extreme vertex or a support vertex. By Theorem 2.3, $V(P_4)$ is the unique minimum total monophonic set of P_4 and so $m_t(P_4) = 4 = p$.

Theorem 2.6. *Let G be a connected graph with cut-vertices and let S be a total monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Since every total monophonic set of G is a monophonic set of G, the result follows from Theorem 1.2. □

Theorem 2.7. *For a connected graph G of order p, $2 \leq m(G) \leq m_t(G) \leq p$.*

Proof. Any monophonic set of G needs at least two vertices and so $m(G) \geq 2$. Every total monophonic set of G is also a monophonic set of G so that $m(G) \leq m_t(G)$. Since $V(G)$ induces a total monophonic set of G, it is clear that $m_t(G) \leq p$. Hence $2 \leq m(G) \leq m_t(G) \leq p$. □

Corollary 2.8. *Let G be a connected graph. If $m_t(G) = 2$, then $m(G) = 2$.*

The converse of Corollary 2.8 need not be true. For the cycle C_5, $m(C_5) = 2$ and $m_t(C_5) = 3$.

Remark 2.9. The bounds in Theorem 2.7 are sharp. For the complete graph $G = K_2$, $m(G) = 2$ and for the path P_4, $m_t(P_4) = 4 = p$. For the graph G given in Figure 2 of order $p = 7$, it is clear that no 2-element subset of $V(G)$ is a monophonic set of G. Note that the set $S = \{v_2, v_6, v_7\}$ is a minimum monophonic set of G and so $m(G) = 3$. Since the subgraph induced by S has an isolated vertex v_2, S is not a total monophonic set of G. Clearly, $S_1 = S \cup \{v_1\}$ is a minimum total monophonic set of G and so $m_t(G) = 4$. Thus, we have $2 < m(G) < m_t(G) < p$. Hence all the inequalities in Theorem 2.7 are strict.

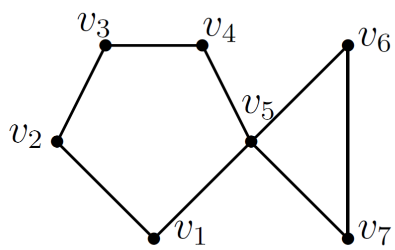

Fig. 2. G

Theorem 2.10. *For any non-trivial tree T, the set of all end-vertices and support vertices of T is the unique minimum total monophonic set of T.*

Proof. Since the set of all end-vertices and support vertices of T form a total monophonic set, the result follows from Theorem 2.3. □

The next theorem gives a characterization of graphs G for which $m_t(G) = 2$.

Theorem 2.11. *For any connected graph G, $m_t(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $m_t(G) = 2$. Conversely, let $m_t(G) = 2$. Let $S = \{u, v\}$ be a minimum total monophonic set of G. Then uv is an edge. A vertex different from u and v cannot lie on a $u - v$ monophonic path and so $G = K_2$. □

Theorem 2.12. *Let G be a connected graph with at least 2 vertices. Then $m_t(G) \leq 2\,m(G)$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a minimum monophonic set of G. Let $u_i \in N(v_i)$ for $i = 1, 2, \ldots, k$ and let $T = \{u_1, u_2, \ldots, u_k\}$. Then $S \cup T$ is a monophonic set because $S \subseteq S \cup T$, and every vertex already lies on a monophonic path between two vertices of S; also the subgraph induced by $S \cup T$ has no isolated vertices because every v_i is adjacent to $u_i (1 \leq i \leq k)$. Hence $S \cup T$ is a total monophonic set of G so that $m_t(G) \leq |S \cup T| \leq 2k = 2\,m(G)$. □

Theorem 2.13. *For a connected graph G of order p, $2 \leq m_t(G) \leq m_c(G) \leq p$.*

Proof. Any total monophonic set of G needs at least two vertices and so $m_t(G) \geq 2$. Since every connected monophonic set of G is also a total monophonic set of G, it follows that $m_t(G) \leq m_c(G)$. Since $V(G)$ induces a connected monophonic set of G, it is clear that $m_c(G) \leq p$. Hence $2 \leq m_t(G) \leq m_c(G) \leq p$. □

Remark 2.14. The bounds in Theorem 2.13 are sharp. For the complete graph $G = K_2$, $m_t(G) = 2$ and for any path $P_n (n \geq 2)$, $m_c(P_n) = n$. For the graph G given in Figure 2 of order $p = 7$, Remark 2.9 shows that S_1 is a minimum total monophonic set of G and so $m_t(G) = 4$. Since the subgraph induced by S_1 is not connected, S_1 is not a connected monophonic set of G. It is clear that, $S_1 \cup \{v_5\}$ is a minimum connected monophonic set of G and so $m_c(G) = 5$. Thus, we have $2 < m_t(G) < m_c(G) < p$. Hence all the inequalities in Theorem 2.13 are strict.

Theorem 2.15. *For any connected graph G, the following are equivalent*

(i) $G = K_2$

(ii) $m_c(G) = 2$

(iii) $m_t(G) = 2$.

Proof. The result follows from Theorems 2.11 and 2.13. □

Theorem 2.16. *For any connected graph G, $m_t(G) = 3$ if and only if $m_c(G) = 3$.*

Proof. If $m_c(G) = 3$, then $m_t(G) = 2$ or 3. By Theorem 2.15, if $m_t(G) = 2$ then $m_c(G) = 2$. Hence, we have $m_t(G) = 3$. Conversely, suppose that $m_t(G) = 3$. Let $S = \{x, y, z\}$ be a minimum total monophonic set of G. Since S is a total monophonic set of G, the subgraph induced by S has no isolated vertices and so $\delta(G[S]) \geq 1$. Also, hence $|S|= 3$, S is a connected monophonic set of G. Hence $m_c(G) = 3$. □

Next, we determine the total monophonic number for some standard graphs.

Theorem 2.17. *For the cycle $C_n(n \geq 4)$, $m_t(C_n) = 3$.*

Proof. Let $C_n : v_1, v_2, v_3, ..., v_n, v_1$ be a cycle of length n. Any 2-element subset of $V(C_n)$ is not a total monophonic set of C_n. Any set of three consecutive vertices of C_n is a minimum total monophonic set of C_n so that $m_t(C_n) = 3$. □

Theorem 2.18. *For the wheel $W_n = K_1 + C_{n-1}(n \geq 4)$,*

$$m_t(W_n) = \begin{cases} 4 & if\ n = 4 \\ 3 & if\ n \geq 5. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}(n \geq 4)$ be the wheel with $V(C_{n-1}) = \{v_1, v_2, v_3, \cdots, v_{n-1}, v_1\}$ and $V(K_1) = \{x\}$. If $n = 4$, then W_4 is the complete graph K_4, and so by Corollary 2.4, $m_t(W_4) = 4$. For $n \geq 5$, any set S of two non-adjacent vertices of W_n is a minimum monophonic set of W_n. Since the subgraph induced by S has an isolated vertex, S is not a total monophonic set of W_n. Any set of three consecutive vertices of C_{n-1} in W_n is a minimum total monophonic set of W_n and so $m_t(W_n) = 3$. □

Theorem 2.19. *For the complete bipartite graph $K_{m,n}(2 \leq m \leq n)$,*

$$m_t(K_{m,n}) = \begin{cases} 3 & if\ 2 = m \leq n \\ 4 & if\ 3 \leq m \leq n. \end{cases}$$

Proof. Let $V_1 = \{x_1, x_2, ..., x_m\}$ and $V_2 = \{y_1, y_2, ..., y_n\}$ be the partite sets of $K_{m,n} = G$. When $m = 2$, $S = V_1$ is the unique minimum monophonic set of G. Since the subgraph induced by S has the isolated vertices x_1 and x_2, S is not a total monophonic set of G. It is clear that for any vertex $y_i \in V_2$, $S \cup \{y_i\}$ for some i $(i \leq i \leq n)$, is a minimum total monophonic set of G and so $m_t(G) = 3$.

Now, let $m \geq 3$. Then $S' = \{x_1, x_2, y_1, y_2\}$ is clearly a total monophonic set of G, and it follows that $m_t(G) \leq 4$. It is enough to show that no 3-element subset of $V(G)$ forms a total monophonic set of G. Let X be a 3-element subset of $V(G)$. If $m = 3$ and $X \subseteq V_1$, then it is clear that X is a monophonic set of G and the subgraph induced by X has isolated vertices. Hence X is not a total monophonic set of G. If $m \geq 4$ and $X \subseteq V_1$. Since X contains three elements from V_1, there exists an element $u \in V_1$ and $u \notin X$. Since the vertex u is not an internal vertex of any $v - w$ monophonic path, for some elements v and w in X, X is not a monophonic set of G. Therefore, we may take that $X \cap V_1 = \{x_i, x_j\}$ and $X \cap V_2 = \{y_k\}$. It is clear that X is not a total monophonic set of G.

If $n = 3$ and $X \subseteq V_2$, then it is clear that X is a monophonic set of G and the subgraph induced by X has isolated vertices. Hence X is not a total monophonic set of G. If $n \geq 4$ and $X \subseteq V_2$. Since X contains three elements from V_2, there exists an element $v \in V_2$ and $v \notin X$. Since the vertex v is not an internal vertex of any $r - s$ monophonic path, for some elements r and s in X, X is not a monophonic set of G. Therefore, we may take that $X \cap V_1 = \{x_l\}$ and $X \cap V_2 = \{y_i, y_j\}$. It is clear that X is not a total monophonic set of G. Thus X is not a total monophonic set of G. □

Theorem 2.20. *Let G be a connected graph of order $p \geq 3$. If $G = \overline{K_2} + H$ where H is a graph of order $p - 2$ and $\overline{K_2}$ is the complement of K_2, then $m_t(G) = 3$.*

Proof. If $G = \overline{K_2} + H$, where H is a graph of order $p - 2$. Let $S = V(\overline{K_2}) = \{u, v\}$. It is clear that S is a monophonic set of G. Since the subgraph induced by S has isolated vertices u and v, S is not a total monophonic set of G. Also, it can be easily verified that any 2-element subset of $V(G)$ is not a total monophonic set of G. Then, for any $x \in V(H)$, $S \cup \{x\}$ is a total monophonic set of G and so $m_t(G) = 3$. □

3. Realisation Results

Theorem 3.1. *For positive integers a, b such that $3 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $m(G) = a$ and $m_t(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case 1. $3 \leq a = b$. By Theorem 1.3 and Corollary 2.4, the complete graph K_a has the desired properties.

Case 2. $3 \leq a < b$. Let $b = a + k$, where $1 \leq k \leq a$. For $k = 1$, the star $K_{1,a}$ has the desired properties. Now, let $k \geq 2$. Let $C_i : v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,1}\,(1 \leq i \leq k - 1)$ be $(k - 1)$ copies of C_4. Let H be the graph formed by identifying the vertices $v_{i,4}$ of $C_i\,(1 \leq i \leq k - 1)$, say x the identified vertex and also joining the vertices $v_{i,1}$ and $v_{i,3}\,(1 \leq i \leq k - 1)$. Let G be the graph obtained from H by adding the $a - k + 1$ new vertices $u_1, u_2, \ldots, u_{a-k+1}$ and joining each $u_i (1 \leq i \leq a - k + 1)$ with x. The graph G is shown in Figure 3. Let $S = \{u_1, u_2, \ldots, u_{a-k+1}, v_{1,2}, v_{2,2}, \cdots, v_{k-1,2}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every monophonic set of G contains S. It is clear that S is the unique minimum monophonic set of G and $m(G) = a$. Since $S_1 = S \cup \{x\}$ is the set of all extreme vertices and support vertex of G, by Theorem 2.3, every total monophonic set of G contains S_1. Since S_1 is a monophonic set of G and the subgraph induced by S_1 has the isolated vertices $v_{1,2}, v_{2,2}, \cdots, v_{k-1,2}$, S_1 is not a total monophonic set of G. Note that any set S' of G with cardinality atmost $b - 1$ is not a total monophonic set of G. Clearly, $S_1 \cup \{v_{1,1}, v_{2,1}, \cdots, v_{k-1,1}\}$ is a minimum total monophonic set of G and so $m_t(G) = a + k = b$. □

In view of Theorem 2.13, we have the following realisation result.

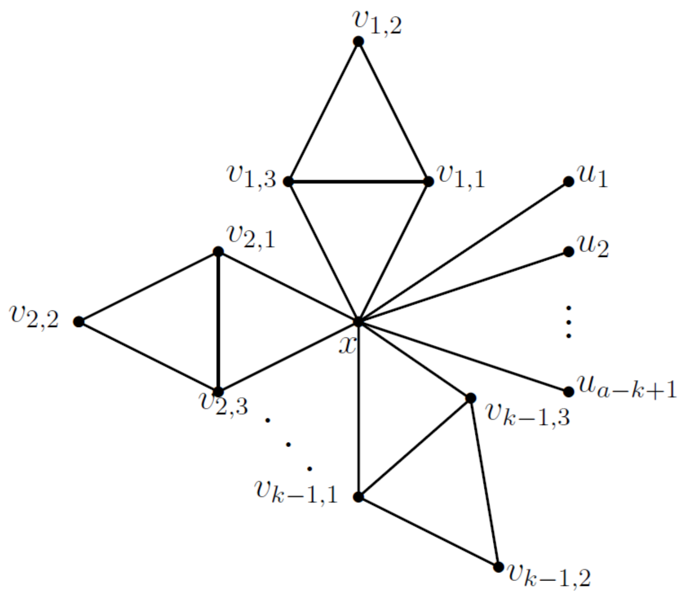

Fig. 3. G

Theorem 3.2. *If p, a, b are positive integers such that $4 \leq a \leq b \leq p$, then there exists a connected graph G of order p with $m_t(G) = a$ and $m_c(G) = b$.*

Proof. We prove this theorem by considering four cases.
Case 1. $4 \leq a = b = p$. Let $G = K_p$. Then by Corollary 2.4 and Theorem 1.5, we have $m_t(G) = m_c(G) = p$.
Case 2. $4 \leq a < b = p$. Let $P_{b-a+3} : u_1, u_2, \ldots, u_{b-a+3}$ be a path of order $b-a+3$. Add $a-3$ new vertices $v_1, v_2, \ldots, v_{a-3}$ to P_{b-a+3} and join each $v_i (1 \leq i \leq a-3)$ with u_{b-a+3}, thereby producing the graph G in Figure 4 of order $b = p$. Let $S = \{u_1, v_1, v_2, \ldots, v_{a-3}, u_2, u_{b-a+3}\}$ be the set of all extreme vertices and support vertices of G. By Theorem 2.3, every total monophonic set of G contains S. It is clear that S is the unique minimum total monophonic set of G and so $m_t(G) = a$.

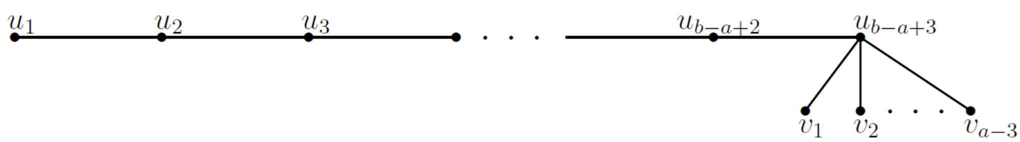

Fig. 4. G

Note that $V(G)$ is the set of all extreme vertices and cut-vertices of G. Then by Theorems 1.1 and 1.4, $V(G)$ is the unique minimum connected monophonic set of G and so $m_c(G) = b = p$.
Case 3. $4 \leq a = b < p$. The graph G is obtained from the cycle $C_{p-a+3} : v_1, v_2, \cdots, v_{p-a+3}, v_1$ of length $p - a + 3$ by adding $a - 3$ new vertices $u_1, u_2, \cdots, u_{a-3}$ and joining each $u_i (1 \leq i \leq a-3)$ to the vertex v_1 of C_{p-a+3}. The graph G of order p is shown in Figure 5. Let $S = \{u_1, u_2, \cdots, u_{a-3}\}$ be the set of all extreme vertices and $S' = S \cup \{v_1\}$

be the set of all extreme vertices and cutvertices of G. Since v_1 is the unique support vertex of any vertex in S, by Theorem 2.3, S' is a subset of any total monophonic set of G and so $m_t(G) \geq |S'| = a-2$. Since no vertex in C_{p-a+3} other than v_1 does not lie on any monophonic path joining any two vertices in S', S' is not a total monophonic set of G. Hence $m_t(G) \geq a-1$. Let $S'' = S' \cup \{x\}$, where $x \in V(C_{p-a+3}) - \{v_1\}$. If $x = v_2$ or $x = v_{p-a+3}$, then the vertex $v_i(3 \leq i \leq p-a+2)$ does not lie any monophonic path joining any two vertices in S''. If $x = v_i(3 \leq i \leq p-a+2)$, then its supporting vertices v_{i-1} and v_{i+1} do not lie on S''. Hence S'' is not a total monophonic set of G and so $m_t(G) \geq |S''| = a-1$. Let $S''' = S' \cup \{v_2, v_{p-a+3}\}$. It is clear that every vertex of G lies on a monophonic path joining any two vertices in S''' and no vertex in S''' is isolated, S''' is a total monophonic set of G. Hence $m_t(G) = a$. In a similar way we can easily verify that S''' is a minimum connected monophonic set of G and so $m_c(G) = a$.

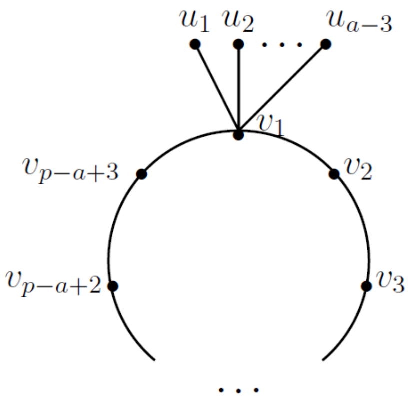

Fig. 5. G

Case 4. $4 \leq a < b < p$. Let H be the graph obtained from the path P_{b-a+3} : $u_1, u_2, \ldots, u_{b-a+3}$ of order $b-a+3$ by adding $a-3$ new vertices $v_1, v_2, \cdots, v_{a-3}$ and joining each $v_i(1 \leq i \leq a-3)$ to the vertex u_1 of P_{b-a+3}. If $p-b=1$, the graph G is obtained from H by adding a new vertex u and joining the vertex u to the both the vertices u_1, u_3 of H. The graph G is of order p. Let $S = \{v_1, v_2, \cdots, v_{a-3}, u_{b-a+3}, u_{b-a+2}, u_1\}$ be the set of all extreme vertices of G and support vertices of G. By Theorem 2.3, every total monophonic set of G contains S. It is clear that S is the unique minimum total monophonic set of G and $m_t(G) = a$. Let $S_1 = \{v_1, v_2, \ldots, v_{a-3}, u_{b-a+3}, u_1, u_3, u_4, \ldots, u_{b-a+2}\}$ be the set of all extreme vertices and cut-vertices of G. By Theorems 1.1 and 1.4, every connected monophonic set of G contains S_1. Since S_1 is a monophonic set of G and the subgraph induced by S_1 is not connected, S_1 is not a connected monophonic set of G. For any $x \in V - S_1$, $S_1 \cup \{x\}$ is a minimum connected monophonic set of G and so $m_c(G) = b$.

If $p-b \geq 2$, the graph G is obtained from H and the path $P_{p-b} : w_1, w_2, \cdots, w_{p-b}$ by joining the vertex w_1 of P_{p-b} to the vertex u_1 of P_{b-a+3}; and joining the vertex w_{p-b} of P_{p-b} to the vertex u_3 of P_{b-a+3}. The graph G of order p is shown in Figure 6. Let $S = \{v_1, v_2, \cdots, v_{a-3}, u_{b-a+3}\}$ be the set of all extreme vertices and let $S_1 = \{u_1, u_3, \ldots, u_{b-a+2}\}$ be the set of all cut-vertices of G. Since u_1 is the unique support vertex any vertex in S and u_{b-a+2} is the unique support vertex of u_{b-a+3} in G, by Theorem 2.3, $S' = S \cup \{u_1, u_{b-a+2}\}$ is a subset of every total monophonic set of G. It is clear that every vertex of G lies on a monophonic path joining any two vertices in S' and no vertex of S' is an isolated vertex. Hence S' is a minimum total monophonic set of G and so $m_t(G) = |S'| = a$.

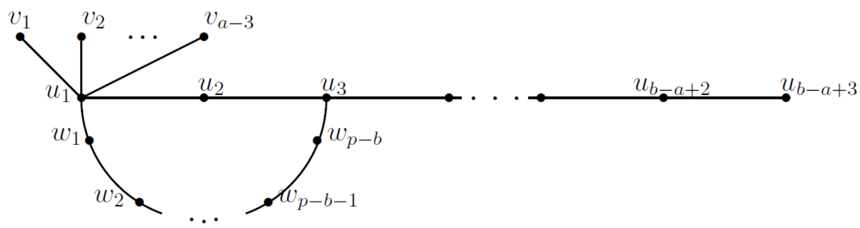

Fig. 6. G

Also, by Theorems 1.1 and 1.4, every connected monophonic set of G contain $S \cup S_1$. It is clear that every vertex of G lies on a monophonic path joining any two vertices in $S \cup S_1$. Since the subgraph induced by $S \cup S_1$ is not connected, $S \cup S_1$ is not a connected monophonic set of G. Hence $m_c(G) > |S \cup S_1| = b - 1$. Let $S_2 = S \cup S_1 \cup \{u_2\}$. Then S_2 is a monophonic set of G and the subgraph induced by S_2 is connected. Hence S_2 is a minimum connected monophonic set of G and $m_c(G) = b$. □

Theorem 3.3. *If p, d and k are positive integers such that $2 \leq d \leq p - 2$, $k \geq 4$ and $p - d - k + 2 \geq 0$, then there exists a connected graph G of order p with monophonic diameter d and $m_t(G) = k$.*

Proof. We prove this theorem by considering two cases.
Case 1. $d = 2$. Let G be the graph obtained from the path $P_3 : x, y, z$ of order 3 by adding $p - 3$ new vertices $v_1, v_2, \ldots, v_{p-k}, w_1, w_2, \ldots, w_{k-3}$ and joining each $w_i (1 \leq i \leq k - 3)$ to y; joining each $v_i (1 \leq i \leq p - k)$ with x, y and z; and joining each $v_i (1 \leq i \leq p - k - 1)$ with $v_j (i + 1 \leq j \leq p - k)$. The graph G of order p is shown in Figure 7.

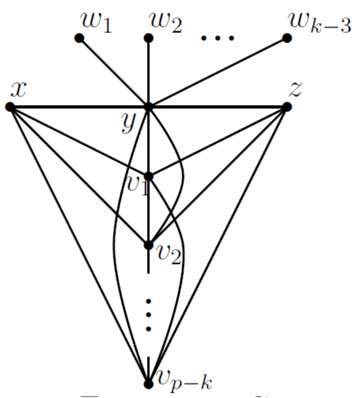

Fig. 7. G

It is clear that for any vertex $u \in V(G)$, $1 \leq e_m(u) \leq 2$, and $e_m(w_i) = 2\,(1 \leq i \leq k-3)$, the monophonic diameter of G is 2. Let $S = \{w_1, w_2, w_3, \ldots, w_{k-3}, x, z, y\}$ be the set of all extreme vertices and support vertex of G. By Theorem 2.3, every total monophonic set of G contains S. It is easily verified that S is the unique minimum total monophonic set of G and so $m_t(G) = k$.

Case 2. $d \geq 3$. Let G be the graph obtained from the cycle $C_{d+1} : v_1, v_2, \cdots, v_{d+1}, v_1$ of order $d+1$ by adding $p-d-1$ new vertices $w_1, w_2, \cdots, w_{p-d-k+2}, u_1, u_2, \cdots, u_{k-3}$ and joining each $w_i (1 \leq i \leq p-d-k+2)$ to the vertices v_1 and v_3; and joining each $u_i (1 \leq i \leq k-3)$ to the vertex v_1. The graph G of order p is shown in Figure 8.

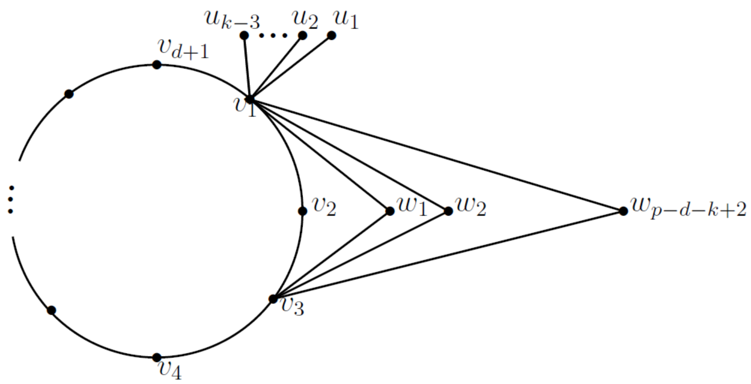

Fig. 8. G

It is clear that $e_m(u_i) = e_m(v_3) = e_m(v_d) = d\,(1 \leq i \leq k-3)$ and for any other vertex u, $e_m(u) = d-1$ so that the monophonic diameter of G is d. Let $S = \{u_1, u_2, \ldots, u_{k-3}\}$ be the set of all extreme vertices of G. Since v_1 is the unique support vertex of every vertex in S, by Theorem 2.3, $S' = S \cup \{v_1\}$ is subset of any total monophonic set of G and so $m_t(G) \geq |S'|= k-2$. Since no vertex in $V(G) - S'$ does not lie on any monophonic path joining any two vertices in S', S' is not a total monophonic set of G. Hence $m_t(G) \geq k-1$. Let $S'' = S' \cup \{x\}$, where $x \in V(G) - S'$. If $x \in \{v_2, v_{d+1}, w_1, w_2, \ldots, w_{p-d-k+2}\}$, then the vertex x does not lie on any monophonic path joining any two vertices in S''. If $x = v_i (3 \leq i \leq d)$ then no one of its supporting vertices does not lie on any monophonic path joining any two vertices in S''. Hence S'' is not a total monophonic set of G and so $m_t(G) > |S''|= k-1$. Let $S''' = S' \cup \{v_2, v_{d+1}\}$. It is clear that every vertex of G lies on a monophonic path joining any two vertices in S''' and no vertex of S''' is an isolated vertex. Hence S''' is a minimum total monophonic set of G and so $m_t(G) = |S'''|= k$. □

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