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Trees with the connected domination number twice the distance-2 domination number

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ABSTRACT

In a graph, the distance between two vertices is the length of a shortest path connecting them. The distance-2 domination number $\gamma_2(G)$ of a graph G is the minimum size of a vertex subset such that every vertex outside it is within distance two of some subset vertex. In a connected graph, a connected dominating set is a subset S whose induced subgraph is connected and in which every vertex not in S is adjacent to some vertex in S; the connected domination number $\gamma_c(G)$ is the size of a smallest such set. For $k \geq 1$, let $\mathscr{T}_k$ be the set of trees T satisfying $\gamma_c(T) = 2\gamma_2(T) = 2k$. The collection $\mathscr{T}_1$ is the set of all double stars. In this paper, we provide a constructive characterization of $\mathscr{T}_k$ for all $k \geq 2$.

Keywords: connected domination number, distance-2 domination

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1. Introduction

Domination and related subset problems constitute a rapidly expanding area of graph theory. Extensive studies on domination parameters can be found in the two books by Haynes et al. ([7, 8]). Determining the domination number of a graph is NP-complete, even for bipartite graphs ([8]).

Let G be a connected graph. A vertex subset S is a *connected dominating set* (CDS) if every vertex not in S is adjacent to at least one vertex in S, and the subgraph $\prec S \succ_G$ induced by S is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a CDS. A connected dominating set (CDS) S is called a γ_c*-set* of G if $|S| = \gamma_c(G)$. Desormeaux et al. ([6]) established several upper bounds for $\gamma_c(G)$

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and characterized the graphs achieving equality. Connected domination has important applications in routing problems and in the design of virtual backbone-based routing for wireless networks ([5, 9, 12]).

The concept of distance domination was introduced by Slater ([10]) and leads to NP-complete problems for general graphs ([3]). In this paper, we consider distance-2 domination. A set S is a *distance-2 dominating set* (D2DS) if every vertex not in S is within distance two from some vertex in S. The *distance-2 domination number* of G, denoted by $\gamma_2(G)$, is the minimum cardinality of a D2DS. A distance-2 dominating set (D2DS) S is called a γ_2*-set* of G if $|S|= \gamma_2(G)$. Sridharan et al. ([11]) established upper bounds for $\gamma_2(G)$ and characterized the corresponding extremal graph classes. Subsequently, Bibi et al. ([1]) proposed algorithms for identifying minimal and minimum distance-2 dominating sets and explored related network applications ([2]).

In this paper, we investigate the particular domination parameters: distance-2 domination and connected domination. For $k \geq 1$, let $\mathscr{T}_k$ be the set of trees T satisfying $\gamma_c(T) = 2\gamma_2(T) = 2k$. The collection $\mathscr{T}_1$ is the set of all double stars. In this paper, we provide a constructive characterization of $\mathscr{T}_k$ for all $k \geq 2$.

2. Notations and preliminary results

All graphs considered in this paper are finite and simple. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For $v \in V(G)$, the *open* and *closed neighborhoods* of v are

$$N_G(v) = \{u \in V(G) : uv \in E(G)\}, \qquad N_G[v] = N_G(v) \cup \{v\}.$$

For $A \subseteq V(G)$, define

$$N_G(A) = \bigcup_{v \in A} N_G(v), \qquad N_G[A] = \bigcup_{v \in A} N_G[v].$$

The *degree* of v is $\deg_G(v) = |N_G(v)|$. A vertex of degree one is a *leaf*, and a vertex adjacent to a leaf is a *support vertex*. Let $L(G)$ and $U(G)$ denote the sets of leaves and support vertices of G, respectively. For sets A and B, the *set difference* $A - B$ consists of all elements of A not in B. For $A \subseteq V(G)$, the *vertex-deletion* $G - A$ is obtained by removing all vertices in A and their incident edges; for $B \subseteq E(G)$, the *edge-deletion* $G - B$ is obtained by removing all edges in B. Let P_n denote a path on n vertices, which has length $n - 1$. For two distinct vertices $u, v \in V(G)$, a u–v *path* is a sequence $u = v_1, \ldots, v_k = v$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i < k$. The *distance* between u and v, denoted by $\text{dist}_G(u, v)$, is the minimum length of a u–v path. The *diameter* of a nontrivial connected graph G, denoted by $diam(G)$, is the maximum distance between any two vertices of G. The *distance-2 closed neighborhood* of v is

$$N_G^2[v] = \{u \in V(G) : \text{dist}_G(u, v) \leq 2\},$$

and for $A \subseteq V(G)$, let

$$N_G^2[A] = \bigcup_{v \in A} N_G^2[v].$$

A *forest* is an acyclic graph, and a *tree* is a connected forest. For $A \subseteq V(G)$, the *induced subgraph* $\prec A \succ_G$ has vertex set A and edge set $\{uv \in E(G) : u, v \in A\}$. The *union* $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *star* S_n is the graph $K_{1,n-1}$, and the *double star* $S_{n,m}$ is obtained by joining the centers of two stars S_n and S_m. For undefined terminology and notation, we refer the reader to [4].

We begin with the following observations.

Observation 2.1. *If S is a D2DS of a graph G, then $N_G^2[S] = V(G)$.*

Observation 2.2. *If S is a γ_c-set of a tree T, where $|V(T)| \geq 3$, then $S \cap L(T) = \emptyset$ and $U(T) \subseteq S$.*

Suppose, by contradiction, there exists a γ_c-set S of T such that $S \cap L(T) \neq \emptyset$. Let $x \in S$ and $x \in L(T)$. Suppose y is the support vertex adjacent to x in T. Since $|V(T)| \geq 3$, it follows that $y \notin L(T)$. Since $\prec S \succ_T$ is connected and $x \in S$, we obtain that $y \in S$. Let $S' = S - \{x\}$. Then S' is a CDS of T with cardinality $|S'| = |S| - 1 = \gamma_c(T) - 1$. This is a contradiction. So every γ_c-set of T contains no leaf, we obtain Observation 2.2.

The following lemmas are quite useful.

Lemma 2.3. *Suppose T is a tree of order $n \geq 3$, then there exists a γ_2-set S of T satisfying $S \cap L(T) = \emptyset$.*

Proof. Let S be a γ_2-set of T. If $S \cap L(T) = \emptyset$, then we are done. So we assume that $S \cap L(T) = \{x_1, \ldots, x_k\}$, where $k \geq 1$. Let $y_i \in N_T(x_i)$, where $i = 1, \ldots, k$. By the minimality, it follows that $y_i \notin S$ for all i. Let $S^* = (S - \{x_1, \ldots, x_k\}) \cup \{y_1, \ldots, y_k\}$. Then S^* is a D2DS of T with cardinality $|S^*| = |S|$. Thus we have that $\gamma_2(T) \leq |S^*| = |S| = \gamma_2(T)$, so $|S^*| = \gamma_2(T)$. Hence S^* is a γ_2-set of T satisfying $S^* \cap L(T) = \emptyset$. □

The following lemma identifies the unique γ_c-set in a tree T.

Lemma 2.4. *Suppose T is a tree of order $n \geq 3$, then $W(T) = V(T) - L(T)$ is the unique γ_c-set of T and $\gamma_c(T) = |W(T)|$.*

Proof. Let S be a γ_c-set of T. By Observation 2.2, we obtain that $L(T) \cap S = \emptyset$ and $U(T) \subseteq S$. Suppose, by contradiction, $S \neq W(T)$. Then there exists a vertex v, where $v \notin L(T)$, such that $v \notin S$. By Observation 2.2, it follows that $v \notin U(T)$. Thus the vertex-deletion subgraph $T - \{v\}$ contains at least two components. Let C_1 and C_2 be two distinct components of $T - \{v\}$. Since $v \notin L(T)$ and $v \notin U(T)$, we obtain that $|V(C_i)| \geq 2$ and $|V(C_i) \cap S| \geq |V(C_i) \cap U(T)| \geq 1$ for $i = 1$ and 2. Since T is a tree and $v \notin S$, the induced subgraph $\prec S \succ_T$ is disconnected. This is a contradiction, so $S = W(T)$ is the unique γ_c-set of T and $\gamma_c(T) = |W(T)|$. □

The inductive method employed in constructing $\mathscr{T}_k$ relies on Lemma 2.5

Lemma 2.5. *Suppose T is a tree and $P : x, y, z, w, t, p, \ldots$ is a longest path of T, where*

$|V(P)| \geq 6$. *Let* $e = zw$ *and the edge-deletion* $T - \{e\} = T' \cup T''$, *where* $w \in V(T'')$. *Then* $\gamma_2(T'') = \gamma_2(T) - 1$.

Proof. Let $\gamma_2(T) = k$. Since $|V(P)| \geq 6$, this implies that $\gamma_2(T) = k \geq 2$. We can see that $dist_T(v, x) \geq 3$ for every vertex $v \in V(T'')$, so $\gamma_2(T) \leq \gamma_2(T) - 1 = k - 1$. Suppose, by contradiction, there exists a γ_2-set S'' of T'' with cardinality $|S''| \leq k - 2$. Then $S = S'' \cup \{z\}$ is a D2DS of T with cardinality $|S| = |S''| + 1 \leq k - 1$. This is a contradiction, so $\gamma_2(T'') = \gamma_2(T) - 1$. □

The following lemma establishes a lower bound on the connected domination number, which is useful for characterizing the trees in $\mathscr{T}_k$.

Lemma 2.6. *Let* T *be a tree and* $T \neq S_n$. *Suppose* $\gamma_2(T) = k$, *then* $\gamma_c(T) \geq 2k$.

Proof. We prove it by induction on k. If $\gamma_2(T) = 1$, then T is a star or a double star. Note that $T \neq S_n$, so T is a double star and $\gamma_c(T) = 2$. Thus it is true for $k = 1$. Assume that it's true for $k-1$, where $k \geq 2$. Let T be a tree and $\gamma_2(T) = k$, where $k \geq 2$. Suppose $P : x, y, z, w, t, p, \ldots$ is a longest path of T and let $e = zw$, where $|V(P)| \geq 6$.. Let $T - \{e\} = T' \cup T''$, where $z \in V(T')$ and $w \in V(T'')$. By Lemma 2.5, $\gamma_2(T'') = k-1$. By the induction hypothesis, $\gamma_c(T'') \geq 2(k-1) = 2k-2$. Since $\{y, z\} \cap L(T) = \emptyset$, by Lemma 2.4, $\gamma_c(T) = |W(T)| = |W(T) \cap V(T')| + |W(T) \cap V(T'')| \geq |\{y, z\}| + \gamma_c(T'') \geq 2 + (2k-2) = 2k$. Thus it is true for k, we complete the proof. □

3. Characterization

In this section, we characterize the set $\mathscr{T}_k$ for all $k \geq 1$, where $\mathscr{T}_k$ is the collection of the trees T satisfying $\gamma_c(T) = 2\gamma_2(T) = 2k$. First, we characterize the set $\mathscr{T}_1$.

Lemma 3.1. $\mathscr{T}_1$ *is the collection of all double stars.*

Proof. Let $T \in \mathscr{T}_1$. Since $\gamma_c(T) = 2$, by Lemma 2.4, $|W(T)| = 2$. So T is a double star. Hence $\mathscr{T}_1$ is the collection of all double stars. □

To provide a constructive characterization of $\mathscr{T}_k$ for $k \geq 2$, we introduce the family $\mathscr{F}_k$. A tree $T \in \mathscr{F}_k$, where $k \geq 2$, is defined to satisfy the following properties (I)∼(VI) (see Figure 1).

(I) $V(T) = A(T) \cup B(T) \cup L_0 \cup (\bigcup_{i=1}^{k} L_i)$.

(II) Let $W(T) = A(T) \cup B(T)$. The induced subgraph $\prec W(T) \succ_T$ is a tree of order $2k$.

(III) $A(T) = \{z_1, z_2, \ldots, z_k\}$ and $\prec A(T) \succ_T$ is a tree.

(IV) $B(T) = \{y_1, y_2, \ldots, y_k\} = L(\prec W(T) \succ_T)$; that is, $B(T)$ is the set of leaves of $\prec W(T) \succ_T$.

(V) For each $i = 1, \ldots, k$, $L_i = L(T) \cap N_T(y_i) \neq \emptyset$.

(VI) $L_0 = L(T) \cap N_T(A(T))$, which may be empty.

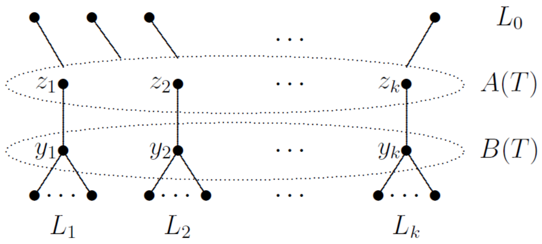

Fig. 1. The figure of $T \in \mathscr{F}_k$, $k \geq 2$

The following lemma establishes that every tree in $\mathscr{F}_k$ belongs to $\mathscr{T}_k$.

Lemma 3.2. *For $k \geq 2$, $\mathscr{F}_k \subseteq \mathscr{T}_k$.*

Proof. Suppose $T \in \mathscr{F}_k$, then $\gamma_c(T) = |W(T)| = 2k$. Since $A(T)$ is a D2DS of T, it follows that $\gamma_2(T) \leq |A(T)| = k$. Suppose, by contradiction, there exists a γ_2-set S of T with cardinality $|S| \leq k-1$. There exists an index i, where $1 \leq i \leq k$, such that $S \cap L_i = \emptyset$ and $S \cap \{y_i, z_i\} = \emptyset$. Then $N_T^2[S] \neq V(T)$, so S is not a D2DS of T. This is a contradiction, so $\gamma_2(T) = k$. Then $T \in \mathscr{T}_k$. We complete the proof. □

Lemmas 3.3 and 3.4 prove the converse, showing that the reverse inclusion also holds.

Lemma 3.3. *Suppose $T \in \mathscr{T}_2$, then $T \in \mathscr{F}_2$.*

Proof. Let $T \in \mathscr{T}_2$. Then $\gamma_2(T) = 2$ and $|W(T)| = \gamma_c(T) = 4$. Suppose $T' = \prec W(T) \succ_T$. Then T' is a tree of order 4, so $T' = S_4$ or P_4. Suppose, by contradiction, $T' = S_4$. Then $\gamma_2(T) = 1$, this is a contradiction. Thus $T' = P_4$, therefor $T \in \mathscr{F}_2$. □

Lemma 3.4. *For $k \geq 2$, $\mathscr{T}_k \subseteq \mathscr{F}_k$.*

Proof. We prove it by induction on k. By Lemma 3.3, it is true for $k = 2$. Assume that it is true for $k-1$, where $k \geq 3$. Let $T \in \mathscr{T}_k$. Then $\gamma_2(T) = k$ and $\gamma_c(T) = 2k$. Since $\gamma_2(T) = k \geq 3$, this follows that $diam(T) \geq 5$. Let $P : v_0, v_1, v_2, v_3, v_4, v_5, \ldots$ be a longest path of T and let $e = v_2v_3$. Suppose the edge-deletion $T - \{e\} = T^* \cup T''$, where $v_2 \in V(T^*)$ and $v_3 \in V(T'')$. Note that T^* and T'' are trees.

Claim 1. $T'' \in \mathscr{T}_{k-1}$.

By Lemma 2.5 and Lemma 2.6, $\gamma_2(T'') = k-1$ and $\gamma_c(T'') \geq 2(k-1) = 2k-2$. Then $2k = \gamma_c(T) = |W(T)| = |W(T) \cap V(T^*)| + |W(T) \cap V(T'')| \geq |\{v_1, v_2\}| + |W(T'')| \geq 2 + (2k-2) = 2k$. The equalities all hold, thus $\gamma_c(T'') = 2(k-1) = 2\gamma_2(T)$. Therefore, $T'' \in \mathscr{T}_{k-1}$.

By Claim 1 and the induction hypothesis, $T'' \in \mathscr{F}_{k-1}$. Then $W(T'') = A(T'') \cup B(T'')$.

Claim 2. $v_3 \in A(T'')$.

Suppose, by contradiction, $v_3 \in L(T'')$. Then $v_3 \in W(T)$ and $v_3 \notin W(T'')$. So $2k = \gamma_c(T) = |W(T)| = |W(T) \cap V(T^*)| + |W(T) \cap V(T'')| \geq 2 + (|\{v_3\}| + |W(T'')|) =$

$2+1+2k-2 = 2k+1$, this is a contradiction. Suppose, by contradiction, $v_3 = y_j$ for some $y_j \in B(T'')$. Let $z_m \in A(T'')$ be a neighbor of z_j. Then $N_T[z_j] \subseteq N_T^2[z_m]$ and $L_j \subseteq N_T^2[v_2]$. Let $S = (A(T'') - \{z_j\}) \cup \{v_2\}$. Then S is a D2DS of T with cardinality $|S| = |A(T'')| = k-1$. This is a contradiction again, so $v_3 \in A(T'')$.

Claim 3. T^* is a star or a double star.

Suppose, by contradiction, there exists another support vertex v_1', where $v_1' \neq v_1$, adjacent to v_2 in T^*. Then $2k = \gamma_c(T) = |W(T)| = |W(T) \cap V(T^*)| + |W(T) \cap V(T'')| \geq |\{v_2, v_1, v_1'\}| + |W(T'')| = 3 + 2k - 2 = 2k+1$, this is a contradiction. So $W(T) \cap V(T^*) = \{v_1, v_2\}$, thus T^* is a star or a double star.

By Claim 2 and Claim 3, $T \in \mathscr{F}_k$. Thus it is true for k, we complete the proof. □

As an immediate consequence of Lemmas 3.2 and 3.4, we obtain Theorem 3.5, which establishes that $\mathscr{T}_k = \mathscr{F}_k$ for all $k \geq 2$.

Theorem 3.5. *For $k \geq 2$, $\mathscr{T}_k = \mathscr{F}_k$.*

References

[1] K. A. Bibi, A. Lakshmi, and R. Jothilakshmi. An algorithm for minimal and minimum distance-2 dominating sets of graph. *Global Journal of Pure and Applied Mathematics*, 13(3):1117–1126, 2017.

[2] K. A. Bibi, A. Lakshmi, and R. Jothilakshmi. Applications of distance-2 dominating sets of graph in networks. *Advances in Computational Sciences and Technology*, 10(9):2801–2810, 2017.

[3] J. W. Boland, T. W. Haynes, and L. M. Lawson. Domination from a distance. *Congressus Numerantium*, 103:89–96, 1994.

[4] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. New York, 1976.

[5] B. Das and V. Bhargavan. Routing in ad-hoc networks using minimum connected dominating sets. In *Proceedings of the IEEE International Conference on Communications*, volume 1, pages 376–380, June 1997. https://doi.org/10.1109/ICC.1997.605303.

[6] W. J. Desormeaux, T. W. Haynes, and M. A. Henning. Bounds on the connected domination number of a graph. *Discrete Applied Mathematics*, 161(18):2925–2931, 2013. https://doi.org/10.1016/j.dam.2013.06.023.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Domination in Graphs: Advanced Topics*. Marcel Dekker, New York, 1998.

[8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[9] D. Li, H. Du, P. J. Wan, X. Gao, Z. Zhang, and W. Wu. Construction of strongly connected dominating sets in asymmetric multi-hop wireless networks. *Theoretical Computer Science*, 410(8-10):661–669, 2009. https://doi.org/10.1016/j.tcs.2008.09.058.

[10] P. J. Slater. R-domination in graphs. *Journal of the Association for Computing Machinery*, 23(3):446–450, 1976. https://doi.org/10.1145/321958.321964.

[11] N. Sridharan, V. S. A. Subramanian, and M. D. Elias. Bounds on the distance two-domination number of a graph. *Graphs and Combinatorics*, 18(3):667–675, 2002. https://doi.org/10.1007/s003730200050.

[12] J. Wu and H. Li. On calculating connected dominating set for efficient routing in ad hoc wireless networks. In *Proceedings of the 3rd International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications*, pages 7–14. ACM, 1999. https://doi.org/10.1145/313239.313261.

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