UTILITAS MATHEMATICA 126 (2026) 223–234

Utilitas Mathematica
www.combinatorialpress.com/um

Combinatorial Press

An updated table of two–associate–class triangular designs

Shyam Saurabh*

ABSTRACT

The solutions of some new triangular designs not tabulated in Clatworthy (1973) are presented here. These designs are known in the literature but re–presented here with more explicit block lists. As a by–product, resolvable solutions of some designs are also obtained. The work addresses a recognized gap in combinatorial design theory and appears to extend classical catalogs.

Keywords: triangular design, balanced incomplete block design, resolvability, symmetric group, group action, permutation circulant matrix

2020 Mathematics Subject Classification: 05B05, 05B30, 62K10.

1. Introduction

Raghavarao [9] reported eighteen unsettled triangular designs in the range $2 \leq r, k \leq 10$. Subsequently, one of these designs was obtained by Aggarwal [1], three were listed by Clatworthy [3], and Greig et al. [7] settled ten further cases, of which four are non–existent (Table 1) and six are in existence. The remaining four unsettled designs are listed in Table 2, where E denotes the overall efficiency. Aggarwal [1] used balanced incomplete block (BIB) designs, whereas Greig et al. [7] employed group actions on the edge set of complete graphs.

Clatworthy [3] tabulated 100 triangular designs in the range $2 \leq r, k \leq 10$ and discussed their resolvability and duality. The aim of the present paper is to provide a uniform explicit presentation of the seven designs that are not tabulated there but whose existence is known from later work. For each design, the block system is written as a union of orbits of specified base blocks, the corresponding triangular scheme is displayed, and the efficiency

* Corresponding author.

Received 23 Dec 2025; Revised 25 Jan 2026; Accepted 06 Feb 2026; Published Online 30 Mar 2026.

DOI: 10.61091/um126-11

is recorded. In addition, resolvability is made explicit for the cases where it is naturally visible from the construction. Thus the paper serves as a clarified supplement to the classical catalog and as a convenient source of verified data for further theoretical work on triangular designs.

Table 1. Non–existent Triangular designs

No.	v	r	K	b	λ_1	λ_2
1	21	10	5	42	1	3
2	21	10	7	30	5	1
3	55	9	5	99	0	1
4	66	9	6	99	0	1

Table 2. Unsettled Triangular designs

No.	v	r	K	b	λ_1	λ_2	E
1	21	10	6	35	3	2	0.87
2	21	10	7	30	4	2	0.89
3	21	10	10	21	4	5	0.94
4	55	10	10	55	3	1	0.91

1.1. Triangular design

A triangular graph has as its vertices the $v = n(n-1)/2$ unordered pairs of an n-element set (equivalently, the edges of the complete graph K_n on n vertices). Two such pairs are first associates if they have an element in common; otherwise they are second associates. A triangular design is a partially balanced incomplete block (PBIB) design whose strongly regular graph is a triangular graph. Clearly, a triangular design has $v = n(n-1)/2$ elements. A relationship between strongly regular graphs and two–associate–class PBIB designs may be found in Dey [4]. An alternate definition of triangular design is given below.

A *triangular association scheme* is an arrangement of $v = n(n-1)/2$ $(n \geq 5)$ elements in an $n \times n$ array such that the positions on the principal diagonal are left blank, the $n(n-1)/2$ positions above and below the principal diagonal are filled with the v elements in such a way that the resultant arrangement is symmetric about the principal diagonal. Then any two elements which occur in the same row or same column are first associates; otherwise they are second associates.

A PBIB design based on triangular association scheme is called a triangular design. The integers $v = n(n-1)/2, b, r, k, \lambda_1$ and λ_2 are known as parameters of the triangular design and they satisfy the relations: $bk = vr; 2\,(n-2)\,\lambda_1 + \frac{(n-2)(n-3)}{2}\lambda_2 = r\,(k-1)$.

1.2. $(\mu_1, \mu_2, \ldots, \mu_t)$ −resolvable design

If the incidence matrix N of a block design $D(v,\ r,\ k,\ b)$ is decomposed into submatrices as $N = (N_1 \mid N_2 \mid \cdots \mid N_t)$ such that each row sum of $N_i\,(1 \leq i \leq t)$ is μ_i, then the design is $(\mu_1, \mu_2, \ldots, \mu_t)-$resolvable [see Kageyama [8]]. Such design is also denoted as $A-$resolvable designs in design theory [see Ge and Miao [6], p. 261]. Further if $\mu_1 = \mu_2 = \cdots = \mu_t = \mu$, then the design is $\mu-$resolvable.

A practical application of these designs is given in Kageyama [8] and some recent constructions of such designs are described in Saurabh [11] and Saurabh et al. [13].

1.3. *Tactical decomposable design*

Let

$$N = [N_{ij}]_{\substack{i=1,2,\ldots,s \\ j=1,2,\ldots,t}}$$

be the incidence matrix of a block design $D(v,\ r,\ k,\ b)$, where N_{ij} are submatrices of N of suitable sizes. Then

(i) D is row–wise (column–wise) tactical decomposable design if each row (column) sum of N_{ij} is $r_{ij}(k_{ij}) \forall i, j$ and tactical decomposable design if it is row–wise as well as column–wise tactical.

(ii) D is uniform row (column) tactical decomposable design if $r_{ij} = \alpha\,(k_{ij} = \beta)\,\forall i, j$.

(iii) Further if each N_{ij} is an $m \times m$ matrix and D is a tactical decomposable design, then it is called a square tactical decomposable design, STD (m), see Bekar et al. [2] and Saurabh [12].

1.4. *Efficiency*

The overall efficiency (E) of a partially balanced design $D(v,\ r,\ k,\ b)$ is defined as the ratio of the average variance of an element (treatment) comparison to the variance in a randomized block experiment with the same replication, assuming that the standard errors of individual plots are the same. For a two–associate–class partially balanced design, E is calculated as [see Clatworthy [3], pp. 28–29]:

$$E = \frac{(k-1)(v-1)}{n_1\,(k-c_1) + n_2(k-c_2)},$$

where $n_i(i=1,2)$ is the number of ith associates of any treatment and the computational constants $c_1,\ c_2$ are obtained using the following relations; $a = r\,(k-1)$:

$$k^2\triangle = (a+\lambda_1)\,(a+\lambda_2) + (\lambda_1-\lambda_2)\left\{\left(a\left(p_{12}^1 - p_{12}^2\right) + \lambda_2 p_{12}^1 - \lambda_1 p_{12}^2\right\}\right., \tag{1}$$

$$k\triangle c_1 = \lambda_1\,(a+\lambda_2) + (\lambda_1-\lambda_2)\left(\lambda_2 p_{12}^1 - \lambda_1 p_{12}^2\right), \tag{2}$$

$$k\triangle c_2 = \lambda_2\,(a+\lambda_1) + (\lambda_1-\lambda_2)\left(\lambda_2 p_{12}^1 - \lambda_1 p_{12}^2\right). \tag{3}$$

For a triangular PBIB design, $n_1 = 2\,(n-2)\,, n_2 = \frac{(n-2)(n-3)}{2},\ p_{12}^1 = n-3, p_{12}^2 = 2n-8.$

1.5. *The solutions*

Suppose a design $(X, \mathcal{B})$ is given by a set of base blocks $\{B_1,\ B_2, \ldots,\ B_m\}$ under an automorphism group G, where G is a subgroup of the symmetric group on X. Then the set of blocks is $\mathcal{B} = \{g(B_i) : g \in G,\ i = 1,\ 2,\ 3, \ldots,\ m\}$. The set $\{g(B_i) : g \in G\}$ is called the orbit of the base block B_i under the action of G, and $\mathcal{B}$ is the union of these base–block orbits, i.e. $\mathcal{B} = \bigcup_{i=1}^{m} \{g(B_i) : g \in G\}$.

The design T70a was obtained by Aggarwal [1] using a BIB design, whereas the remaining designs were obtained by Greig et al. [7] through explicit group actions on the unordered pairs forming the edge set of K_n. The present paper does not claim original existence proofs for all of these cases; rather, it supplies a uniform explicit tabulation and structural verification for designs omitted from Clatworthy [3]. In each case, the displayed orbit lists were checked by direct counting: the number of blocks equals b, every treatment occurs r times, and first– and second–associate pairs occur λ_1 and λ_2 times, respectively. Interested readers can find further details on algorithmic constructions based on group action in Kreher and Stinson (1998). The design TX is from Clatworthy [3], whereas TXa lies between TX and T$(X{+}1)$ [see Freeman [5]].

1. T51a: $v = 15,\ r = 9,\ k = 5,\ b = 27,\ n_1 = 8,\ n_2 = 6,\ \lambda_1 = 3,\ \lambda_2 = 2.$

$$\triangle = \frac{288}{5}, \quad c_1 = \frac{3}{8}, \quad c_2 = \frac{1}{4}, \quad p^1_{12} = 3, \quad p^2_{12} = 4, \quad E = 0.85.$$

The group G is:

$$\begin{aligned} G &= \langle (0,1,2),\ (3,4,5) \rangle \\ &= \Big\{ I,\ (0,1,2),\ (0,2,1),\ (3,4,5),\ (3,5,4), \\ &\quad (0,1,2)(3,4,5),\ (0,1,2)(3,5,4),\ (0,2,1)(3,4,5),\ (0,2,1)(3,5,4) \Big\}. \end{aligned}$$

Here I is the identity element of the group, and $|G| = 9$.

The base blocks are:

$$\begin{aligned} \mathcal{B}_1 &= \{(0,1),\ (0,5),\ (1,2),\ (2,3),\ (4,5)\}, \\ \mathcal{B}_2 &= \{(0,3),\ (0,2),\ (0,5),\ (2,5),\ (2,3)\}, \\ \mathcal{B}_3 &= \{(0,3),\ (0,5),\ (1,3),\ (3,4),\ (3,5)\}. \end{aligned}$$

The element set of the design consists of the fifteen edges of the complete graph K_6 on six vertices, i.e.

$$\begin{aligned} X = \Big\{ &\{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \\ &\{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \Big\}. \end{aligned}$$

In order to keep the presentation consistent with the table of Clatworthy [3], we use the following transformation to denote the edges of K_6 by the numerals $1, 2, 3, 4, \ldots, 15$ in lexicographic order:

$$\begin{gathered} \{0,1\} \to 1,\ \{0,2\} \to 2,\ \{0,3\} \to 3,\ \{0,4\} \to 4,\ \{0,5\} \to 5,\ \{1,2\} \to 6, \\ \{1,3\} \to 7,\ \{1,4\} \to 8, \{1,5\} \to 9,\ \{2,3\} \to 10,\ \{2,4\} \to 11,\ \{2,5\} \to 12, \\ \{3,4\} \to 13,\ \{3,5\} \to 14,\ \{4,5\} \to 15. \end{gathered}$$

If $g = (3, 4, 5)$ then

$$\begin{aligned}
&\text{(i)} \quad g(\mathcal{B}_1) = \{(0,1),\ (0,3),\ (1,2),\ (2,4),\ (5,3)\} = \{1,3,6,11,14\},\\
&\text{(ii)} \quad g(\mathcal{B}_2) = \{(0,4),\ (0,2),\ (0,3),\ (2,3),\ (2,4)\} = \{2,3,4,10,11\},\\
&\text{(iii)} \quad g(\mathcal{B}_3) = \{(0,4),\ (0,3),\ (1,4),\ (4,5),\ (4,3)\} = \{3,4,8,13,15\}.
\end{aligned}$$

Proceeding similarly, one obtains the orbits of the remaining base blocks under the action of G. Here, G acts on the fifteen edges of K_6, and this induced action carries each base block to another block of the design.

The orbit of the base block $\mathcal{B}_1$ is

$$\Big\{(1,2,7,12,15),\ (1,5,6,10,15),\ (2,3,6,9,15),\ (1,2,8,10,14),\ (1,2,9,11,13),\\ (1,3,6,11,14),\ (1,4,6,12,13),\ (2,4,6,7,14),\ (2,5,6,8,13)\Big\}.$$

The orbit of base block $\mathcal{B}_2$ is

$$\Big\{(1,\ 3,\ 4,\ 7,\ 8),\ (1,\ 3,\ 5,\ 7, 9),\ (1,\ 4,\ 5,\ 8,\ 9), (2,\ 3,\ 4,\ 10,\ 11), (2,\ 3,\ 5,\ 10,\ 12),\\ (2,\ 4,\ 5,\ 11,\ 12), (6,\ 7,\ 8,\ 10,\ 11),\ (6,\ 7,\ 9,\ 10,\ 12), (6,\ 8,\ 9,\ 11,\ 12)\Big\}$$

The orbit of base block $\mathcal{B}_3$ is

$$\Big\{(3,\ 4,\ 8,\ 13,\ 15),\ (4,\ 10,\ 11,\ 13,\ 15),\ (4,\ 5,\ 9,\ 14,\ 15),\\ (5,\ 11,\ 12,\ 14,\ 15),\ (7,\ 8,\ 11,\ 13,\ 15),\ (8,\ 9,\ 12,\ 14,\ 15),\\ (3,\ 10,\ 12,\ 13,\ 14),\ (3,\ 5,\ 7,\ 13,\ 14),\ (7,\ 9,\ 10,\ 13,\ 14)\Big\}.$$

The design is the union of these orbits. The corresponding triangular scheme is:

×	1	2	3	4	5
1	×	6	7	8	9
2	6	×	10	11	12
3	7	10	×	13	14
4	8	11	13	×	15
5	9	12	14	15	×

The efficiency of this design is calculated as follows [see subsection 1.4]:
Here,

$$n_1 = 2\,(n-2) = 8, n_2 = \tfrac{(n-2)(n-3)}{2} = 6,\ p^1_{12} = n-3 = 3,\\ p^2_{12} = 2n-8 = 4,\ a = r\,(k-1) = 36.$$

The relation (1)$\Rightarrow k^2\triangle = (a+\lambda_1)\,(a+\lambda_2) + (\lambda_1-\lambda_2)\,\{(a\,(p^1_{12}-p^2_{12}) + \lambda_2 p^1_{12} - \lambda_1 p^2_{12}\}$

$$\Rightarrow 25\triangle = 39\times 38 + 36\,(3-4) + 2\times 3 - 3\times 4 \Rightarrow \triangle = 288/5.$$

The relation (2) $\Rightarrow k\triangle c_1 = \lambda_1\,(a+\lambda_2) + (\lambda_1-\lambda_2)\,(\lambda_2 p^1_{12} - \lambda_1 p^2_{12})$

$$\Rightarrow 5\times (288/5)\,c_1 = 3\times 38 + 2\times 3 - 3\times 4 \Rightarrow c_1 = 3/8.$$

The relation (3) $\Rightarrow$ $k\triangle c_2 = \lambda_2\,(a+\lambda_1) + (\lambda_1-\lambda_2)\,(\lambda_2 p_{12}^1 - \lambda_1 p_{12}^2)$

$$\Rightarrow 5\times(288/5)\,c_2 = 2\times 39 + 2\times 3 - 3\times 4 \Rightarrow c_2 = 1/4.$$

Substituting these values in $E = \dfrac{(k-1)(v-1)}{n_1\,(k-c_1)+n_2(k-c_2)}$, we obtain $E=0.85$.

2. T52a: v=15, r=10, k=5, b=30, n_1=8, n_2=6, λ_1=2, λ_2=4.

$$\triangle = 384/5, c_1 = 5/24, c_2 = 5/12, p_{12}^1 = 3, p_{12}^2 = 4, E{=}0.85.$$

The group G is

$$\begin{aligned} G = &\langle (0,\ 1,\ 2,\ 3,\ 4,\ 5)\,,\ (1,\ 5)\,(2,\ 4)\rangle \\ =&\Big\{ I,\ (0,\ 1,\ 2,\ 3,\ 4,\ 5)\,,\ (0,\ 2,\ 4)\,(1,\ 3,\ 5)\,,\ (0,\ 3)\,(1,\ 4)\,(2,\ 5)\,, \\ &(0,\ 4,\ 2)\,(1,\ 5,\ 3)\,,\ (0,\ 5,\ 4,\ 3,\ 2,\ 1)\,,\ (1,\ 5)\,(2,\ 4)\,, \\ &(0,\ 5)\,(1,\ 4)\,(2,\ 3)\,,\ (0,\ 4)\,(1,\ 3)\,,\ (0,\ 3)\,(1,\ 2)\,(4,\ 5)\,, \\ &(0,\ 2)\,(3,\ 5)\,,\ (0,\ 1)\,(2,\ 5)\,(3,\ 4)\Big\}. \end{aligned}$$

The order of this group is = 12 and the transformation is same as that of T51a. The base blocks are:

$$\mathcal{B}_1 = \{(0,\ 1)\,,\ (0,\ 3)\,,\ (1,\ 2)\,,\ (3,\ 4)\,,\ (4,\ 5)\}\,,$$

$$\mathcal{B}_2 = \{(0,\ 1)\,,\ (0,\ 2)\,,\ (1,\ 4)\,,\ (2,\ 5)\,,\ (3,5)\}\,,$$

$$\mathcal{B}_3 = \{(0,\ 1)\,,\ (0,\ 2)\,,\ (1,\ 3)\,,\ (2,\ 4)\,,\ (4,\ 5)\}\,.$$

The orbit of base block $\mathcal{B}_1$ is

$$\Big\{ (1,\ 3,\ 6,\ 13,\ 15)\,,\ (5,\ 6,\ 8,\ 10,\ 15)\,,\ (1,\ 5,\ 10,\ 12,\ 13)\,, \\ (3,\ 5,\ 6,\ 10,\ 15)\,,\ (1,\ 6,\ 12,\ 13,\ 15)\,,\ (1,\ 5,\ 8,\ 10,\ 13)\Big\}.$$

The orbit of base block $\mathcal{B}_2$ is

$$\Big\{ (1,\ 2,\ 8,\ 12,\ 14)\,,\ (3,\ 4,\ 6,\ 7,\ 12)\,,\ (3,\ 8,\ 9,\ 10,\ 11)\,, \\ (2,\ 8,\ 12,\ 13,\ 14)\,,\ (3,\ 4,\ 7,\ 12,\ 15)\,,\ (3,\ 5,\ 8,\ 9,\ 11)\,, \\ (4,\ 5,\ 7,\ 8,\ 12)\,,\ (2,\ 3,\ 8,\ 14,\ 15)\,,\ (3,\ 9,\ 11,\ 12,\ 13)\,, \\ (4,\ 7,\ 8,\ 10,\ 12)\,,\ (2,\ 3,\ 6,\ 8,\ 14)\,,\ (1,\ 3,\ 9,\ 11,\ 12)\Big\}.$$

The orbit of base block $\mathcal{B}_3$ is

$$\Big\{ (1,\ 2,\ 7,\ 11,\ 15)\,,\ (5,\ 6,\ 7,\ 11,\ 14)\,,\ (1,\ 4,\ 10,\ 11,\ 14)\,, \\ (4,\ 6,\ 9,\ 13,\ 14)\,,\ (2,\ 4,\ 9,\ 10,\ 15)\,,\ (2,\ 5,\ 7,\ 9,\ 13)\,, \\ (4,\ 5,\ 6,\ 11,\ 14)\,,\ (1,\ 7,\ 11,\ 14,\ 15)\,,\ (2,\ 5,\ 7,\ 11,\ 13)\,, \\ (2,\ 7,\ 9,\ 10,\ 15)\,,\ (2,\ 4,\ 6,\ 9,\ 13)\,,\ (1,\ 4,\ 9,\ 10,\ 14)\Big\}.$$

The design is the union of these orbits. The corresponding triangular scheme is the same as that of T51a.

3. T53a: v=21, r=10, k=5, b=42, n_1=n_2=10, λ_1=3, λ_2=1.

$$\triangle = 63, c_1 = 19/63, c_2 = 1/21, p^1_{12} = 4, p^2_{12} = 6, E{=}0.83.$$

The group G is

$$\begin{aligned} G = &\langle (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6),\ (1,\ 2,\ 4)(3,\ 6,\ 5)\rangle \\ = &\Big\{ I,\ (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6),\ (0,\ 2,\ 4,\ 6,\ 1,\ 3,\ 5),\ (0,\ 3,\ 6,\ 2,\ 5,\ 1,\ 4), \\ &(0,\ 4,\ 1,\ 5,\ 2,\ 6,\ 3),\ (0,\ 5,\ 3,\ 1,\ 6,\ 4,\ 2),\ (0,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1), \\ &(1,\ 2,\ 4)(3,\ 6,\ 5),\ (1,\ 4,\ 2)(3,\ 5,\ 6),\ (0,\ 2,\ 6)(1,\ 4,\ 3), \\ &(0,\ 4,\ 5)(1,\ 6,\ 2),\ (0,\ 6,\ 4)(2,\ 3,\ 5),\ (0,\ 1,\ 3)(2,\ 5,\ 4), \\ &(0,\ 3,\ 2)(1,\ 5,\ 6),\ (0,\ 5,\ 1)(3,\ 4,\ 6),\ (0,\ 4,\ 6)(2,\ 5,\ 3), \\ &(0,\ 1,\ 5)(3,\ 6,\ 4),\ (0,\ 5,\ 4)(1,\ 2,\ 6),\ (0,\ 2,\ 3)(1,\ 6,\ 5), \\ &(0,\ 6,\ 2)(1,\ 3,\ 4),\ (0,\ 3,\ 1)(2,\ 4,\ 5) \Big\}. \end{aligned}$$

The order of this group is 21.
The base blocks are:

$$\mathcal{B}_1 = \{(0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (3,\ 4),\ (0,\ 5)\},$$
$$\mathcal{B}_2 = \{(0,\ 6),\ (0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (1,\ 2)\}.$$

The transformation is:

$$\begin{gathered} \{0,\ 1\} \to 1,\ \{0,\ 2\} \to 2,\ \{0,\ 3\} \to 3,\ \{0,\ 4\} \to 4,\ \{0,\ 5\} \to 5,\ \{0,\ 6\} \to 6, \\ \{1,\ 2\} \to 7,\ \{1,\ 3\} \to 8,\ \{1,\ 4\} \to 9,\ \{1,\ 5\} \to 10,\ \{1,\ 6\} \to 11, \\ \{2,\ 3\} \to 12,\ \{2,\ 4\} \to 13,\ \{2,\ 5\} \to 14,\ \{2,\ 6\} \to 15, \\ \{3,\ 4\} \to 16,\ \{3,\ 5\} \to 17,\ \{3,\ 6\} \to 18,\ \{4,\ 5\} \to 19,\ \{4,\ 6\} \to 20,\ \{5,\ 6\} \to 21. \end{gathered}$$

The orbit of base block $\mathcal{B}_1$ is

$$\begin{gathered} \Big\{ (1,\ 2,\ 3,\ 5,\ 16),\ (7,\ 8,\ 9,\ 11,\ 19),\ (2,\ 12,\ 13,\ 14,\ 21),\ (6,\ 8,\ 16,\ 17,\ 18), \\ (1,\ 4,\ 13,\ 19,\ 20),\ (5,\ 7,\ 10,\ 17,\ 21),\ (6,\ 11,\ 12,\ 15,\ 20),\ (2,\ 3,\ 4,\ 6,\ 11), \\ (1,\ 4,\ 5,\ 6,\ 14),\ (7,\ 8,\ 13,\ 14,\ 15),\ (4,\ 9,\ 16,\ 17,\ 20),\ (5,\ 11,\ 15,\ 18,\ 21), \\ (1,\ 2,\ 8,\ 9,\ 10),\ (3,\ 12,\ 13,\ 17,\ 18),\ (5,\ 10,\ 14,\ 19,\ 20),\ (9,\ 13,\ 15,\ 16,\ 19), \\ (1,\ 7,\ 10,\ 11,\ 18),\ (3,\ 14,\ 17,\ 19,\ 21),\ (2,\ 4,\ 7,\ 12,\ 15),\ (6,\ 9,\ 18,\ 20,\ 21), \\ (3,\ 8,\ 10,\ 12,\ 16) \Big\}. \end{gathered}$$

The orbit of base block $\mathcal{B}_2$ is

$$\begin{gathered} \Big\{ (1,\ 2,\ 3,\ 6,\ 7),\ (1,\ 7,\ 8,\ 9,\ 12),\ (7,\ 12,\ 13,\ 14,\ 16),\ (12,\ 16,\ 17,\ 18,\ 19), \\ (4,\ 16,\ 19,\ 20,\ 21),\ (5,\ 6,\ 10,\ 19,\ 21),\ (1,\ 6,\ 11,\ 15,\ 21),\ (2,\ 4,\ 5,\ 6,\ 13), \\ (1,\ 3,\ 4,\ 5,\ 9),\ (2,\ 7,\ 13,\ 15,\ 20),\ (9,\ 11,\ 13,\ 16,\ 20),\ (8,\ 11,\ 18,\ 20,\ 21), \\ (1,\ 8,\ 10,\ 11,\ 17),\ (3,\ 5,\ 8,\ 12,\ 17),\ (2,\ 5,\ 14,\ 17,\ 19),\ (4,\ 9,\ 10,\ 13,\ 19), \\ (7,\ 9,\ 10,\ 11,\ 14),\ (10,\ 14,\ 15,\ 17,\ 21),\ (2,\ 12,\ 14,\ 15,\ 18),\ (3,\ 6,\ 15,\ 18,\ 20), \\ (3,\ 4,\ 8,\ 16,\ 18) \Big\}. \end{gathered}$$

The design is the union of these orbits. The corresponding triangular scheme is given below

$\times$	1	2	3	4	5	6
1	$\times$	7	8	9	10	11
2	7	$\times$	12	13	14	15
3	8	12	$\times$	16	17	18
4	9	13	16	$\times$	19	20
5	10	14	17	19	$\times$	21
6	11	15	18	20	21	$\times$

A direct count of the displayed blocks shows that each element occurs five times in the orbit of $\mathcal{B}_1$ and also five times in the orbit of $\mathcal{B}_2$. Hence the design is 5–resolvable.

4. T67a: v=b=21, r=k=6, n_1=n_2=10, λ_1=2, λ_2=1.

$$\triangle = 77/3, c_1 = 27/77, c_2 = 12/77, p^1_{12} = 4, p^2_{12} = 6, E{=}0.87.$$

The group G is:

$$G = \langle (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6),\ (1,\ 2,\ 4)(3,\ 6,\ 5)\rangle .$$

The order of this group is 21.

The base block is:

$$\mathcal{B} = \{(0,\ 1),\ (0,\ 6),\ (0,\ 5),\ (0,\ 2),\ (2,\ 5),(4,\ 5)\} .$$

The transformation is same as in T53a.

The orbit of base block $\mathcal{B}$ is

$$\Big\{ (1,\ 2,\ 5,\ 6,\ 14,\ 19),\ (1,\ 7,\ 8,\ 11,\ 18,\ 21),\ (2,\ 4,\ 6,\ 7,\ 12,\ 13),$$
$$(1,\ 8,\ 10,\ 12,\ 16,\ 17),\ (7,\ 13,\ 15,\ 16,\ 19,\ 20),\ (3,\ 5,\ 12,\ 17,\ 19,\ 21),$$
$$(6,\ 9,\ 11,\ 16,\ 20,\ 21),\ (1,\ 3,\ 4,\ 6,\ 11,\ 15),\ (2,\ 3,\ 4,\ 5,\ 8,\ 16),$$
$$(2,\ 13,\ 14,\ 15,\ 17,\ 21),\ (1,\ 4,\ 5,\ 9,\ 13,\ 20),\ (2,\ 11,\ 12,\ 15,\ 18,\ 20),$$
$$(8,\ 9,\ 10,\ 11,\ 13,\ 19),\ (3,\ 6,\ 8,\ 17,\ 18,\ 20),\ (4,\ 9,\ 16,\ 17,\ 18,\ 19),$$
$$(1,\ 2,\ 3,\ 7,\ 9,\ 10),\ (4,\ 10,\ 14,\ 19,\ 20,\ 21),\ (7,\ 8,\ 9,\ 12,\ 14,\ 15),$$
$$(5,\ 6,\ 10,\ 15,\ 18,\ 21),\ (3,\ 12,\ 13,\ 14,\ 16,\ 18),\ (5,\ 7,\ 10,\ 11,\ 14,\ 17) \Big\},$$

which represents the design. The corresponding triangular scheme is the same as that of T53a.

5. T68a: v=21, r=8, k=6, b=28, n_1=n_2=10, λ_1=3, λ_2=1.

$$\triangle = 175/4, c_1 = 38/125, c_2 = 6/125, p^1_{12} = 4, p^2_{12} = 6, E{=}0.86.$$

The group G is:

$$G = \langle (0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6),\ (1,\ 2,\ 4)(3,\ 6,\ 5)\rangle .$$

The order of this group is 21.

The base blocks are:

$$\mathcal{B}_1 = \{(0,\ 1),\ (0,\ 6),\ (0,\ 5),\ (0,\ 2),\ (2,\ 5),\ (4,\ 5)\},$$
$$\mathcal{B}_2 = \{(0,\ 6),\ (0,\ 1),\ (0,\ 2),\ (0,\ 3),\ (0,\ 4),\ (0,\ 5)\}.$$

The transformation is same as in T53a.
The orbit of base block $\mathcal{B}_1$ is

$$\Big\{(1,\ 2,\ 5,\ 6,\ 14,\ 19),\ (1,\ 7,\ 8,\ 11,\ 18,\ 21),\ (2,\ 4,\ 6,\ 7,\ 12,\ 13),\ (1,\ 8,\ 10,\ 12,\ 16,\ 17),\ (7,\ 13,\ 15,\ 16,\ 19,\ 20),\ (3,\ 5,\ 12,\ 17,\ 19,\ 21),\ (6,\ 9,\ 11,\ 16,\ 20,\ 21),\ (1,\ 3,\ 4,\ 6,\ 11,\ 15),\ (2,\ 3,\ 4,\ 5,\ 8,\ 16),\ (2,\ 13,\ 14,\ 15,\ 17,\ 21),\ (1,\ 4,\ 5,\ 9,\ 13,\ 20),\ (2,\ 11,\ 12,\ 15,\ 18,\ 20),\ (8,\ 9,\ 10,\ 11,\ 13,\ 19),\ (3,\ 6,\ 8,\ 17,\ 18,\ 20),\ (4,\ 9,\ 16,\ 17,\ 18,\ 19),\ (1,\ 2,\ 3,\ 7,\ 9,\ 10),\ (4,\ 10,\ 14,\ 19,\ 20,\ 21),\ (7,\ 8,\ 9,\ 12,\ 14,\ 15),\ (5,\ 6,\ 10,\ 15,\ 18,\ 21),\ (3,\ 12,\ 13,\ 14,\ 16,\ 18),\ (5,\ 7,\ 10,\ 11,\ 14,\ 17)\Big\}.$$

The orbit of base block $\mathcal{B}_2$ is

$$\Big\{(1,\ 2,\ 3,\ 4,\ 5,\ 6),\ (1,\ 7,\ 8,\ 9,\ 10,\ 11),\ (2,\ 7,\ 12,\ 13,\ 14,\ 15),\ (3,\ 8,\ 12,\ 16,\ 17,\ 18),\ (4,\ 9,\ 13,\ 16,\ 19,\ 20),\ (5,\ 10,\ 14,\ 17,\ 19,\ 21),\ (6,\ 11,\ 15,\ 18,\ 20,\ 21)\Big\}.$$

The design is the union of these orbits. The corresponding triangular scheme is the same as that of T53a.

A direct count of the displayed blocks shows that each element occurs six times in the orbit of $\mathcal{B}_1$ and two times in the orbit of $\mathcal{B}_2$. Hence the design is (2, 6)–resolvable.

6. T70a: v=21, r=10, k=6, b=35, n_1=n_2=10, λ_1=2, λ_2=3.

$$\triangle = 238/3, c_1 = 53/238, c_2 = 39/119, p^1_{12} = 4, p^2_{12} = 6, E\text{=}0.87.$$

Cyclically develop the five initial blocks written columnwise as:

1	3	8	2	1
2	4	9	5	2
4	8	11	10	4
8	13	15	11	15
9	16	16	15	16
11	19	18	20	18

Using partial cycles of length 7 and cycling within the three subsets of elements $1 \longleftrightarrow 7,\ 8 \longleftrightarrow 14,\ 15 \longleftrightarrow 21$ where $(1 \leftrightarrow q) \iff 1 \to 2,\ 2 \to 3,\ \ldots, (q-1) \to q,\ q \to 1$ [see Saurabh [10]]. For example, the seven blocks generated using the initial block (1, 2, 4, 8, 9, 11) are:

$$(1,\ 2,\ 4,\ 8,\ 9,\ 11),\ (2,\ 3,\ 5,\ 9,\ 10,\ 12),\ (3,\ 4,\ 6,\ 10,\ 11,\ 13),\ (4,\ 5,\ 7,\ 11,\ 12,\ 14),\ (5,\ 6,\ 1,\ 12,\ 13,\ 8),\ (6,\ 7,\ 2,\ 13,\ 14,\ 9),\ (7,\ 1,\ 3,\ 14,\ 8,\ 10)$$

The corresponding association scheme is:

$$\begin{array}{ccccccc} \times & 1 & 6 & 9 & 12 & 17 & 18 \\ 1 & \times & 14 & 3 & 20 & 11 & 19 \\ 6 & 14 & \times & 15 & 4 & 16 & 10 \\ 9 & 3 & 15 & \times & 21 & 5 & 13 \\ 12 & 20 & 4 & 21 & \times & 8 & 2 \\ 17 & 11 & 16 & 5 & 8 & \times & 7 \\ 18 & 19 & 10 & 13 & 2 & 7 & \times \end{array}$$

The incidence matrix N of this design admits a decomposition into 7×7 sub–matrices using the method given in Saurabh [12].

$$N = (N_1 \mid N_2 \mid N_3) = \left(\begin{array}{c|c|ccc} \alpha^3+\alpha^4 & \alpha^2+\alpha^5 & 0_7 & M & M \\ \alpha+\alpha^6 & \alpha^3+\alpha^4 & M & 0_7 & M \\ \alpha^2+\alpha^5 & \alpha+\alpha^6 & M & M & 0_7 \end{array} \right),$$

where $\alpha = circ.\ (0\ 1\ 0\ 0 \ldots 0)$ is a permutation circulant matrix of order 7 such that $\alpha^7 = I_7$, I_7 is the identity matrix, 0_7 is a null matrix of order 7 and $M = \alpha + \alpha^2 + \alpha^4$ is the incidence matrix of a symmetric BIB design: $v = b = 7, r = k = 3, \mu = 1$.

Since each row sum of N_1, N_2 and N_3 is 2, 2 and 6, respectively, the design is $(2,\ 2,\ 6)$ −resolvable. The resolution classes are given below in Table 3, where RC stands for a resolution class. A direct count of the listed blocks shows that each element of the design occurs two times in RC –I and RC –II and six times in RC –III. Also, since each smaller block sub–matrix in N has size 7, the design is a square tactical decomposable design, STD (7) [see the subsection on tactical decomposable designs].

Table 3. Resolution classes of T70a

RC –I	RC– II	RC–III		
(3, 4, 8, 13, 16, 19)	(2, 5, 10, 11, 15, 20)	(1, 2, 4, 8, 9, 11)	(1, 2, 4, 15, 16, 18)	(8, 9, 11, 15, 16, 18)
(4, 5, 9, 14, 17, 20)	(3, 6, 11, 12, 16, 21)	(2, 3, 5, 9, 10, 12)	(2, 3, 5, 16, 17, 19)	(9, 10, 12, 16, 17, 19)
(5, 6, 10, 8, 18, 21)	(4, 7, 12, 13, 17, 15)	(3, 4, 6, 10, 11, 13)	(3, 4, 6, 17, 18, 20)	(10, 11, 13, 17, 18, 20)
(6, 7, 11, 9, 19, 15)	(5, 1, 13, 14, 18, 16)	(4, 5, 7, 11, 12, 14)	(4, 5, 7, 18, 19, 21)	(11, 12, 14, 18, 19, 21)
(7, 1, 12, 10, 20, 16)	(6, 2, 14, 8, 19, 17)	(5, 6, 1, 12, 13, 8)	(5, 6, 1, 19, 20, 15)	(12, 13, 8, 19, 20, 15)
(1, 2, 13, 11, 21, 17)	(7, 3, 8, 9, 20, 18)	(6, 7, 2, 13, 14, 9)	(6, 7, 2, 20, 21, 16)	(13, 14, 9, 20, 21, 16)
(2, 3, 14, 12, 15, 18)	(1, 4, 9, 10, 21, 19)	(7, 1, 3, 14, 8, 10)	(7, 1, 3, 21, 15, 17)	(14, 8, 10, 21, 15, 17)

7. T71a: v=21, r=10, k=7, b=30, n_1=n_2=10, λ_1=2, λ_2=4.

$$\triangle = 600/7, c_1 = 1/5, c_2 = 2/5, p^1_{12} = 4, p^2_{12} = 6, E\text{=}0.90.$$

The group G is:

$$G = \langle (0,\ 1,\ 2,\ 3,\ 4,\ 5)\ (\infty),\ (0)\ (1,\ 5)\,(2,\ 4)\ (3)\ (\infty) \rangle .$$

The order of this group is 12.

The base blocks are:

$$\mathcal{B}_1 = \{(0,\ 1),\ (0,\ 3),\ (1,\ 2),\ (3,\ 4),\ (4,\ 5),\ (2,\ \infty),\ (5,\ \infty)\},$$

$$\mathcal{B}_2 = \{(0,\ 1),\ (0,\ 2),\ (1,\ 4),\ (2,\ 5),\ (3,5),\ (3,\ \infty),\ (4,\ \infty)\},$$
$$\mathcal{B}_3 = \{(0,\ 1),\ (0,\ 2),\ (1,\ 3),\ (2,\ 4),\ (4,\ 5),\ (3,\ \infty),\ (5,\ \infty)\}.$$

The transformation is:

$$\{0,\ 1\} \to 1,\ \{0,\ 2\} \to 2,\ \{0,\ 3\} \to 3,\ \{0,\ 4\} \to 4,\ \{0,\ 5\} \to 5,\ \{0,\ \infty\} \to 6,$$
$$\{1,\ 2\} \to 7,\ \{1,\ 3\} \to 8,\ \{1,\ 4\} \to 9,\ \{1,\ 5\} \to 10,\ \{1,\ \infty\} \to 11,$$
$$\{2,\ 3\} \to 12,\ \{2,\ 4\} \to 13,\ \{2,\ 5\} \to 14,\ \{2,\ \infty\} \to 15,$$
$$\{3,\ 4\} \to 16,\ \{3,\ 5\} \to 17,\ \{3,\ \infty\} \to 18,\ \{4,\ 5\} \to 19,\ \{4,\ \infty\} \to 20,\ \{5,\ \infty\} \to 21.$$

The orbit of base block $\mathcal{B}_1$ is

$$\Big\{(1,\ 3,\ 7,\ 15,\ 16,\ 19,\ 21),\ (5,\ 6,\ 7,\ 9,\ 12,\ 18,\ 19), (1,\ 5,\ 11,\ 12,\ 14,\ 16,\ 20),$$
$$(3,\ 5,\ 7,\ 11,\ 12,\ 19,\ 20), (1,\ 6,\ 7,\ 14,\ 16,\ 18,\ 19),\ (1,\ 5,\ 9,\ 12,\ 15,\ 16,\ 21)\Big\}.$$

The orbit of base block $\mathcal{B}_2$ is

$$\Big\{(1,\ 2,\ 9,\ 14,\ 17,\ 18,\ 20),\ (3,\ 4,\ 7,\ 8,\ 14,\ 20,\ 21),\ (3,\ 6,\ 9,\ 10,\ 12,\ 13,\ 21),$$
$$(2,\ 6,\ 9,\ 11,\ 14,\ 16,\ 17),\ (3,\ 4,\ 8,\ 11,\ 14,\ 15,\ 19),\ (3,\ 5,\ 9,\ 10,\ 13,\ 15,\ 18),$$
$$(4,\ 5,\ 8,\ 9,\ 14,\ 15,\ 18),\ (2,\ 3,\ 9,\ 11,\ 15,\ 17,\ 19),\ (3,\ 6,\ 10,\ 11,\ 13,\ 14,\ 16),$$
$$(4,\ 6,\ 8,\ 9,\ 12,\ 14,\ 21),\ (2,\ 3,\ 7,\ 9,\ 17,\ 20,\ 21),\ (1,\ 3,\ 10,\ 13,\ 14,\ 18,\ 20)\Big\}.$$

The orbit of base block $\mathcal{B}_3$ is

$$\Big\{(1,\ 2,\ 8,\ 13,\ 18,\ 19,\ 21),\ (5,\ 6,\ 7,\ 8,\ 13,\ 17,\ 20),\ (1,\ 4,\ 11,\ 12,\ 13,\ 17,\ 21),$$
$$(4,\ 6,\ 7,\ 10,\ 15,\ 16,\ 17),\ (2,\ 4,\ 10,\ 11,\ 12,\ 18,\ 19),\ (2,\ 5,\ 8,\ 10,\ 15,\ 16,\ 20),$$
$$(4,\ 5,\ 7,\ 11,\ 13,\ 17,\ 18),\ (1,\ 6,\ 8,\ 13,\ 15,\ 17,\ 19),\ (2,\ 5,\ 8,\ 11,\ 13,\ 16,\ 21),$$
$$(2,\ 6,\ 8,\ 10,\ 12,\ 19,\ 20),\ (2,\ 4,\ 7,\ 10,\ 16,\ 18,\ 21),\ (1,\ 4,\ 10,\ 12,\ 15,\ 17,\ 20)\Big\}.$$

The design is the union of these orbits. The corresponding triangular scheme is the same as that of T53a.

Remark 1.1. In the solutions of the designs T51a, T53a, and T67a, there are no short orbits, i.e. every orbit has the same length as the order of the underlying group, whereas the designs T52a, T68a, and T71a contain short orbits. This distinction is useful when comparing the compactness of the orbit descriptions.

Remark 1.2. Here, an explicit solution of the design T70a is presented by means of a $(0,\ 1)$ −linear combination of permutation circulant matrices, together with its resolvability and STD (7) description. Combined with the orbit tables recorded above, this yields a concise and verifiable supplement to Clatworthy's tabulation and suggests a possible starting point for more unified matrix-based constructions of triangular designs.

Acknowledgement

The author is thankful to Dr. D. L. Kreher for helpful suggestions in generating the blocks of the designs. The author is also grateful to the anonymous reviewers for their constructive comments on a previous draft.

References

[1] K. R. Aggarwal. A note on construction of triangular P.B.I.B. design with parameters: $v = 21$, $b = 35$, $r = 10$, $k = 6$, $\lambda_1 = 1$, $\lambda_2 = 2$. *Annals of Mathematical Statistics*, 43:371–371, 1972. https://doi.org/10.1214/aoms/1177692736.

[2] H. Bekar, C. Mitchell, and F. Piper. Tactical decomposition of designs. *Aequationes Mathematicae*, 25:123–152, 1982. https://doi.org/10.1007/BF02189612.

[3] W. H. Clatworthy. *Tables of Two–Associate–Class Partially Balanced Designs*, volume 63 of *Applied Mathematics Series*. 1973.

[4] A. Dey. *Theory of Block Designs*. Wiley Eastern, New Delhi, 1986.

[5] G. H. Freeman. A cyclic method of constructing regular group divisible incomplete block designs. *Biometrika*, 63:555–558, 1976. https://doi.org/10.2307/2335735.

[6] G. Ge and Y. Miao. PBDs, frames, and resolvability. In C. J. Colbourn and J. H. Dinitz, editors, *Handbook of Combinatorial Designs*, pages 261–265. Chapman & Hall/CRC, New York, 2nd edition, 2007.

[7] M. Greig, D. L. Kreher, and A. C. H. Ling. On PBIBD designs based on triangular schemes. *Annals of Combinatorics*, 2:147–155, 2002. https://doi.org/10.1007/PL00012581.

[8] S. Kageyama. Resolvability of block designs. *The Annals of Statistics*, 4:655–661, 1976.

[9] D. Raghavarao. *Constructions and Combinatorial Problems in Design of Experiments*. Dover, Mineola, New York, 1971.

[10] S. Saurabh. A note on cyclic and partial cyclic solutions of block designs. *Gujarat Journal of Statistics and Data Science*, 40:45–50, 2024. b.

[11] S. Saurabh. Resolvability of a BIB design of takeuchi (1962). *Statistics and Applications*, 22(2):361–362, 2024. a.

[12] S. Saurabh. Tactical decomposable regular group divisible designs and their threshold schemes. *Utilitas Mathematica*, 122:109–116, 2025. https://www.doi.org/10.61091/um122-08.

[13] S. Saurabh, S. N. Topno, and D. Prasad. Resolvability of some BIB and regular group divisible designs. *Statistics and Applications*, 23(2):317–321, 2025.

Shyam Saurabh
Department of Mathematics, Tata College, Kolhan University, Chaibasa, India
E-mail shyamsaurabh785@gmail.com