UTILITAS MATHEMATICA 126 (2026) 55–81

Utilitas Mathematica
www.combinatorialpress.com/um

Combinatorial Press

Applications of infinite lower triangular matrices and their group structure in combinatorics and the theory of orthogonal polynomials

Paweł J. Szabłowski*

ABSTRACT

IOur focus is on the set of lower-triangular, infinite matrices that have natural operations like addition, multiplication by a number, and matrix multiplication. With respect to addition this set forms and abelian group while with respect to matrix multiplication, the invertivle elements of the set form a group. The set becomes an algebra (non-commutative in fact) with unity when all three operations are considered together. We indicate important properties of the algebraic structures obtained in this way. In particular, we indicate several sub-groups or sub-rings. Among sub-groups, we consider the group of Riordan matrices and indicate its several sub-groups. We show a variety of examples (approximately 20) of matrices that are composed of the sequences of important polynomial or number families as entries of certain lower-triangular infinite matrices. New, significant relationships between these families can be discovered by applying well-known matrix operations like multiplication and inverse calculation to this representation. The paper intends to compile numerous simple facts about the lower-triangular matrices, specifically the family of Rionian matrices, and briefly review their properties.

Keywords: Riordan group, lower-triangular matrix algebra, special polynomial families

2020 Mathematics Subject Classification: Primary 05A19, 33C45; Secondary 20H25.

1. Introduction, notation and elementary observations

Let $\mathbf{A} \overset{df}{=} [a_{n,j}]_{n,j\geq 0,}$ with $a_{nj} = 0$ for all $j > n \geq 0$, be a lower-triangular infinite matrix with entries belonging to $\mathbb{C}$, in general. For reasons that will be obvious in the sequel, let

* Corresponding author.

Received 24 Nov 2025; Revised 20 Jan 2026; Accepted 27 Jan 2026; Published Online 05 Feb 2026.

DOI: 10.61091/um126-03

us agree that both index entries will start from 0.

To avoid convergence issues, let us agree (following [13]), that every infinite lower-triangular matrix will be understood as a sequence $\{\mathbf{A}_n\}$ of finite matrices $\mathbf{A}_n$ organized in such a way that the matrix $\mathbf{A}_n$ is a sub-matrix of $\mathbf{A}_{n+1}$ and we have in terms of a block structure

$$\mathbf{A}_{n+1} = \begin{bmatrix} \mathbf{A}_n & 0 \\ \mathbf{a}_n^T & \alpha_{n,n} \end{bmatrix},$$

where $\mathbf{a}_n^T$ is certain row vector of dimension $n+1$ and $\alpha_{n,n}$ certain complex number. Since index n within this paper runs usually from 0 and traditionally indices of row and columns within the matrix run from 1, we notice that within this paper $\mathbf{A}_n$ will usually denote a $(n+1)\times(n+1)$ matrix.

We will denote such sequences of matrices by $\mathbf{A} = \{\mathbf{A}_n\}$, meaning that we deal with a lower-triangular infinite matrix $\mathbf{A}$ whose $(n+1)\times(n+1)$ matrices in their upper left corners are matrices $\mathbf{A}_n$, $n = 1, \ldots$. This understanding of infinite lower-triangular matrices helps address some of the reviewer's concerns about the existence and convergence of matrix operations, since all operations on lower-triangular matrices of infinite dimensions can be understood as operations on sequences of finite matrices or vectors. No convergence considerations are necessary. Notice also that in accordance with our index convention, we have $\mathbf{A}_1 = \alpha_{0,0} \overset{df}{=} \alpha_0$. Let us extend this convention and denote all diagonal elements $a_{n,n}$ by a single index that is a_n if of course $\mathbf{A} = [a_{k,j}]$. The notation $\mathbf{A} = [a_{n,j}]_{n,j\geq 0,}$ and $\mathbf{A} = \{\mathbf{A}_n\}$, where $\mathbf{A}_n = [a_{nj}]_{n,j=0,\ldots,n}$ shall be used alternatively,

In the sequel, we will use the simplified convention: $\mathbf{A}_n = [a_{k,j}]_n$. It is elementary to notice that such matrices equipped with scalar multiplication and matrix addition, constitute a linear space. Similarly, the set of infinite, lower-triangular matrices is a non-commutative, associative algebra that uses both matrix addition and multiplication, as well as scalar multiplication. More precisely, we have $\alpha\mathbf{A}+\beta\mathbf{B} = [\alpha a_{n,j} + \beta b_{n,j}]$ if $\mathbf{A} = [a_{n,j}]$ and $\mathbf{B} = [b_{n,j}]$, α, $\beta \in \mathbb{C}$ and $\mathbf{AB} = \left[\sum_{k=j}^{n} a_{n,k}b_{k,j}\right]$. Let us notice that the set of diagonal matrices makes a commutative sub-algebra of our algebra. Let us agree that diagonal matrices will be denoted in the following way. Namely, $[\{\beta_n\}]$ will denote diagonal matrix with β_n as its $(n+1)\times(n+1)$ entry. Notice that with respect to matrix addition and matrix multiplication the set of lower-triangular matrices constitutes a non-commutative ring. Among these diagonal matrices the ones with all entries equal to zero serve as the zero element of the ring and the diagonal matrix $[\{1\}]$ (i.e., with all diagonal elements equal to 1) as the ring's "one", the unity. These special matrices will be denoted respectively $\mathbf{0}$ and $\mathbf{1}$. Let us denote by $\mathcal{S}$ the whole algebra of lower-triangular matrices and by $\mathcal{D}$ the sub-algebra of all diagonal matrices.

Another important sub-algebra of $\mathcal{S}$ comprises of all lower-triangular with zeros on their diagonal. Moreover, this sub-algebra considered as a sub-ring is an ideal.

Finally, let us consider lower-triangular matrices $[a_{n,j}]$ with $a_{n,j} = 0$ whenever $n - j$ is an odd number. Let us denote the set of such matrices by $\mathcal{SE}$. To complete introducing the notation let us agree that matrices of the size $n \times 1$, i.e., columns will also be called vectors and the infinite column (raw) vector $\mathbf{a}$ will be also understood as sequence $\{\mathbf{a}_n\}_{n\geq 0}$

of $(n+1)\times 1$ vectors $\mathbf{a}_n$. All .finite and infinite vectors will be denoted generally by the bold lower case letters, , i.e., $\mathbf{a}$, $\mathbf{b}$, and so on. The symbol ^{T} denotes transposition of a matrix. For example $\mathbf{a}^T$ denotes a row matrix of the size $1\times n$.

We have an elementary observation:

Proposition 1.1. *$\mathcal{SE}$ is a sub-algebra of $\mathcal{S}$.*

Proof. Multiplication by a number retains the property of an entry of being 0 whenever $n-j$ is odd. Similarly, with the sum of such matrices. Now let us recall that $\sum_{k=j}^{n} a_{n,k}b_{k,j}$ is a $(n,j)-$th entry of a product of two matrices $[a_{n,j}]$ and $[b_{n,j}]$. Observe that $n-j=n-k+k-j$ for all $j\leq k\leq n$. Now, if $n-j$ is odd then either $n-k$ or $k-j$ must also be odd. Hence, if $[a_{n,j}]$ and $[b_{n,j}]$ both belong to $\mathcal{SE}$ their product must also belong to $\mathcal{SE}$. □

Examples of this sub-algebra are presented in Subsubsection 4.1.4.

As mentioned above, with respect to the addition of matrices, $\mathcal{S}$ is not only a commutative (i.e., Abelian) group but also a linear space, if one considers also multiplication by a number. However, with respect to the matrix multiplication those lower-triangular matrices form a non-commutative monoid (i.e., magma with associative operation and identity). Notice that the identity of this monoid it is matrix $\mathbf{1}$, i.e., the diagonal matrix with 1 on its diagonal.

Using this observation and the well-known formulae for the block multiplication of matrices we see that

$$\mathbf{A}_{n+1}\mathbf{B}_{n+1}=\begin{bmatrix}\mathbf{A}_n & 0\\ \mathbf{a}_n^T & \alpha_n\end{bmatrix}\begin{bmatrix}\mathbf{B}_n & 0\\ \mathbf{b}_n^T & \beta_n\end{bmatrix}=\begin{bmatrix}\mathbf{A}_n\mathbf{B}_n & 0\\ \mathbf{a}_n^T\mathbf{B}_n+\alpha_n\mathbf{b}_n^T & \alpha_n\beta_n\end{bmatrix}. \tag{1}$$

It is also elementary to notice that ring $\mathcal{S}$ has divisors of zero. Hence, it is not a domain. The following two matrices are the examples of the left and right divisors of zero in this ring with non-zero elements on the diagonal:

$$\begin{bmatrix}a_0 & 0 & 0\\ a_1 & 0 & 0\\ a_2 & 0 & 0\end{bmatrix},\quad \begin{bmatrix}0 & 0 & 0\\ 0 & b_1 & 0\\ b_2 & b_3 & b_4\end{bmatrix}.$$

On the other hand, matrices having a non-zero element on the $(0,0)$ position, i.e., in the upper left corner cannot be right divisors of zero.

Notice also that diagonal matrices with the same number, say α, on the diagonal, i.e., $[\{\alpha\}]$ can be identified with this number since from the formula (1) it follows that

$$[a_{n,j}][\{\alpha\}]=[\alpha a_{n,j}].$$

Now, let us discuss the set of matrices $\mathbf{A}$ which are invertible, i.e., such matrices for which there exists a matrix (called inverse) and denoted by $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{1}.$$

Notice, that if a matrix from $\mathcal{S}$, say $\mathbf{A}$ has representation $\{\mathbf{A}_n\}$ then the diagonal elements of each of the matrices $\mathbf{A}_n$ are their eigenvalues. Consequently, each matrix $\mathbf{A}_n$ having non-zero elements on the diagonal is invertible. So the set of invertible elements of $\mathcal{S}$ defined by the conditions: $\mathbf{A} = [a_{n,j}]$ is invertible iff $\forall j \geq 0$:

$$a_j \neq 0.$$

Let us denote by $\mathcal{I}$ the set of invertible matrices. It is elementary to notice that set of all invertible matrices forms a linear cone, i.e., if $\mathbf{A}$ is in $\mathcal{I}$ then all matrices of a form $\mathbf{A}[\{\beta\}] \in \mathcal{I}$ for all $\beta \neq 0$. The family of such matrices forms the so-called skew field (or a division ring) with the the skew field operations being the ordinary matrix addition (commutative) and (usually non-commutative) multiplication.

The other properties of elements of $\mathcal{S}$ considered as linear operators, i.e., their eigenvectors and some decompositions at least for the special, more precisely Riordan matrices, are presented in a recently published paper [4].

Thus, naturally all elements of $\mathcal{S}$ that have non-zero elements on their diagonal, form a group if we confine ourselves to multiplication. We denote this group by $\mathcal{L}$.

As it follows from, Wikipedia (inverted blockwise formula), also e.g. from [3] or from direct calculation following (1), the formula for inversion of block matrices applied to the special case when lower-right-most corner matrix has dimension 1×1 yields

$$\mathbf{A}_{n+1}^{-1} = \begin{bmatrix} \mathbf{A}_n^{-1} & 0 \\ -\mathbf{a}_n^T \mathbf{A}_n^{-1}/\alpha_n & \alpha_n^{-1} \end{bmatrix}. \tag{2}$$

As a corollary we have the following observation.

Remark 1.2.

1. If matrix $\mathbf{A}$ has all integer entries and moreover it has 1 on its diagonal, then matrix $\mathbf{A}^{-1}$ has also integer entries and $1's$ on its diagonal.

2. If matrix $\mathbf{A}$ has polynomial entries except for the diagonal whose entries are numbers, then its inverse also has polynomial entries.

3. If $\mathbf{A} = [a_{nj}]$ and $\mathbf{A}^{-1} = [b_{nj}]$, then also the following two matrices are inverses of one another:

   3a. $\forall \lambda \in \mathbb{C} : \mathbf{F}(\lambda) = [a_{nj}\lambda^{n-j}]$ and $\mathbf{B}(\lambda) = [b_{nj}\lambda^{n-j}]$.

   3b. Suppose $\{\alpha_n\}_{n\geq 0}$, $\alpha_n \neq 0 : \tilde{\mathbf{A}} = [\alpha_n a_{n,j}]$ and $\tilde{\mathbf{B}} = [b_{n,j}/\alpha_j]$.

4. $$[\{\alpha_n\}]\,[a_{n,j}]\,[\{\beta_j\}] = [\alpha_n a_{n,j}\beta_j].$$

   In particular
   $$[\{\alpha_n\}]\,[a_{n,j}]\,[\{\alpha_j\}]^{-1} = [\alpha_n \tilde{a}_{n,j}/\alpha_j], \tag{3}$$
   where $\tilde{a}_{n,j}$ is defined by the relationship $[a_{n,j}]^{-1} = [\tilde{a}_{n,j}]$.

5. Suppose $\mathbf{A} = [d_{n-j}]$, for some sequence $\{d_j\}$, with $d_0 = 1$, then $\mathbf{A}^{-1} = [e_{n-j}]$, where sequence $\{e_j\}$ is defined by the recursion:

$$e_0 = 1, \quad e_k = -\sum_{s=0}^{k-1} d_{k-s} e_s.$$

Further the paper contains Section 2 devoted to the definition of the formal power series and presentation of some of its properties, Section 3 devoted to the presentation of the group of Riordan matrices and some of its subgroups. Finally we have Sections 4 containing many subsections in which examples are grouped with respect to their source. The last Section 5 contains glossary of various algebraic terms and symbols use in the paper.

2. Formal power series

In the sequel, we will consider also some sequences and their generating functions. In general, these sequences as well as variables in the formal power series being the expansion of these GF can be complex. However, in almost all cases, we will work with real sequences as well as real GF's. That is why we will assume that all variables and numbers will be real. We will also be aware that without any difficulty the results can be extended to the complex case.

In order to move further, let us extend the usual definition of the so-called generating function (GF) of a sequence $\{a_n\}_{n\geq 0}$. Since we will mostly deal with infinite matrices predominantly lower-triangular, let us express the notion of GF in terms of matrix operation. This perspective is not very revolutionary, however, it simplifies, in the author's opinion, many concepts and ideas. Hence, the ordinary sequence $\{a_n\}_{n\geq 0}$ will be viewed either as a column vector with a_n as its $n+1-$st entry or as the diagonal matrix $[\{a_n\}]$. The modification of the notion of the GF is as follows. First, let us fix the so-called "reference sequence" or the so-called "denominator sequence " $\{c_n\}_{n\geq 0}$, i.e., some sequence such that $c_0 = 1$ and $c_n \neq 0$, for $n \geq 1$. Then, by the generating function (GF) of a sequence $\{a_n\}$, given the reference sequence $\{c_n\}$, (or briefly $\{c_n\}$ -GF) the formal power series

$$F_a^c(x) = \sum_{n\geq 0} a_n x^n / c_n.$$

Remark 2.1. Let us fix the reference sequence $\{c_n\}$. Let us consider two sequences $\{a_n\}$ and $\{b_n\}$ with $\{c_n\}$ -GF's respectively $F_a^c(x)$ and $F_b^c(x)$, then $F_a^c(x)F_b^c(x)$ is the $\{c_n\}$ -GF of the following sequence $\left\{\sum_{j=0}^{n} \left\langle{n \atop j}\right\rangle_c a_j b_{n-j}\right\}$, where we have denoted $\left\langle{n \atop j}\right\rangle_c = \frac{c_n}{c_j c_{n-j}}$.

We will call $\left\langle{n \atop j}\right\rangle_c$ the $\{c_n\}$ -binomial coefficient. Let us set also $\left\langle{n \atop j}\right\rangle_c = 0$ for $j > n$ and $j < 0$. We will also call the following power series $C(x) = \sum_{n\geq 0} x^n/c_n$, the reference GF, that is briefly RGF.

Remark 2.2. If $c_n = 1$, $n \geq 0$ and $|x| < 1$, the RGF $C(x) = 1/(1-x)$, when $c_n = n!$ we get $C(x) = \exp(x)$ while when $c_n = 1/\prod_{j=1}^{n}(1-q^j)$, for some $q \in (-1,1)$, $n \geq 1$ and $|x| < 1$, then $C(x) = 1/\prod_{j=0}^{\infty}(1-xq^j)$ as it follows from the so-called $q-$binomial theorem.

Notice that this formal power series can be obtained as the result of the ordinary matrix operations. Namely, we have

$$F_a^c(x) = \sum_{n\geq 0} a_n x^n/c_n = \mathbf{1}^T [\{a_n\}] \, [\{1/c_n\}] \, \mathbf{x},$$

where we denoted by $\mathbf{1}$ and $\mathbf{x}$ the column vectors with $n-$th entries respectively 1 and x^n. When, $c_n = 1$ then we talk about ordinary GF or simply GF of the sequence $\{a_n\}$, when $c_n = n!$ then we talk about "exponential" GF of the sequence $\{a_n\}$, finally when $c_n = \prod_{j=1}^{n}(1-q^j)$, or $c_n = (\prod_{j=1}^{n}(1-q^j))/(1-q)^n$ for $n > 0$,and some $q \in (-1,1)$, then we talk about q GF of the sequence $\{a_n\}$. Notice, that $\lim_{q\to 1^-}(\prod_{j=1}^{n}(1-q^j))/(1-q)^n = n!$. This and many other facts concerning $q-$series can be found in first chapter of [9].

Let us notice that for every lower-triangular matrix, say $\mathbf{A} = [a_{n,i}] \in S$ and let us fix the reference sequence $\{c_n\}_{n\geq 0}$. We can define now a formal power series

$$\mathcal{A}(x,y) = \sum_{i=0}^{\infty}\sum_{n=i}^{\infty} x^i y^n a_{n,i}/(c_i c_n) = \sum_{i=0}^{\infty} x^i g_i(y)/c_i,$$

where $g_i(y) = \sum_{n\geq i} y^n a_{n,i}/c_n$ is the GF of the $i-$th column of the matrix $\mathbf{A}$. $\mathcal{A}(x,y)$ will be called a GF of the matrix $\mathbf{A}$.

The notation used in formal power series theory (FPS) will be utilized in the sequel. Namely, if $p(x) = \sum_{n\geq 0} p_n x^n$ is a FPS, then p_n is often denoted by: $[[x^n]]\, p(x)$.

In particular, we have $\sum_{n\geq 0} x^n [[x^n]](p(x) = p(x)$. We are using a double square bracket here to denote "coefficient operator" in order to minimize confusion. Recall that we use a single square bracket to denote infinite matrices.

3. Riordan arrays

One of the most important subgroups of $\mathcal{S}$ is the so-called Riordan group $\mathcal{R}$ defined in many positions of literature through the properties of generating functions (GF) of the entries of elements of $\mathcal{S}$. Let us fix the reference sequence $\{c_n\}_{n\geq 0}$. The Riordan group is characterized by the fact that the GF of the i-th column has a form $\forall i \geq 0 : g_i(y) = f(y)h^i(y)/c_i$, for some formal power series $f(y) = f_0 + \sum_{j\geq 1} f_j y^j/c_j$ and $h(y) = \sum_{j\geq 1} h_j y^j/c_j$, with $f_0, h_1 \neq 0$. For a lower-triangular infinite matrix being an element of the Riordan group, we thus have

$$\mathcal{A}(x,y) = f(y)\sum_{i=0}^{\infty} x^i h^i(y)/c_j = f(y)C(xh(y)),$$

where $C(x)$ is the RGF. Hence, the Riordan matrix is characterized by the two generating functions f and h. It is traditionally denoted as (f,h). With the Riordan matrix (f,h) we associate the following lower-triangular matrix $[d_{n,j}]$, where

$$d_{n,j} = c_n[[x^n]]f(x)h^j(x)/c_j.$$

Indeed, we get the following function as GF of the $j-$th column

$$\sum_{n\geq j} d_{n,j}x^n/c_n = \sum_{n\geq 0} x^n[[x^n]]f(x)h^j(x)/c_j = f(x)h^j(x)/c_j.$$

It is well-known that the product of two Riordan matrices is a Riordan matrix. Hence, the Riordan matrices form a subgroup $\mathcal{R}$ of group $\mathcal{S}$. Unfortunately, $\mathcal{R}$ is not a ring since in general a sum of two Riordan matrices is not Riordan. Many papers have been written about Riordan matrices over the years. They presented many features of this group. To ensure completeness in the paper, we will revisit some of these results, particularly those related to the group structure of the Riordan group.

We start with the following well-known, old result.

Theorem 3.1 (Roman)**.** *Suppose we have two Riordan matrices:* $(a(x), b(x))$ *and* $(c(x), d(x))$ *where functions* a, b, c, d *are such that* $a(0) \neq 0 \neq c(0)$, $b(0) = d(0) = 0$ *and* $b'(0) \neq 0 \neq d'(0)$. *Then their product is a Riordan matrix* $(a(x)c(b(x)), d(b(x)))$ *and its inverse the Riordan matrix* $(a(x), b(x))$ *is equal to* $\left(1/a(\bar{b}(x), \bar{b}(x)\right)$, *where the function* $\bar{b}$ *is defined by the relationship:* $\bar{b}(b(x)) = b\left(\bar{b}\left(x\right)\right) = x$.

Proof. The proof for $c_n = 1$ is presented in [11], p.43. The proof for the general reference sequence $\{c_n\}$ is presented in [6]. □

Below, we present a list of known and newly identified subgroups of the $\mathcal{R}$ group. The most common and the most important reference sequence is $\{1\}$. That is why it will be considered in the sequel. The other cases of the reference sequence will be clearly underlined. One of the nontrivial examples of usage of the "exponential" reference sequence is given in Proposition 4.7 in Section 4. The facts presented in the list below are predominantly known and scattered through the literature. We recall them for completeness of the paper. It should be noted that the subgroup $\mathcal{IP}(p, \beta, \alpha)$ and several of its subgroups listed in the initial positions below are believed to be unknown and are being introduced for the first time.

List of subgroups of some subgroups of the Riordan group

(a) Riordan matrices of the form $(p(x), \beta x/(1-\alpha x))$, for some reals α, β, $\beta \neq 0$ and a formal power series $p\left(x\right)$ such that $p(0) \neq 0$ forms a subgroup of $\mathcal{R}$. The multiplication rule within this subgroup is the following:

$$\left(p_1(x), \frac{\beta_1 x}{(1-\alpha_1 x)}\right)\left(p_2(x), \frac{\beta_2 x}{(1-\alpha_2 x)}\right) = \left(p_1(x)p_2\left(\frac{\beta_1 x}{(1-\alpha_1 x)}\right), \frac{\beta_1\beta_2 x}{(1-(\alpha_1+\beta_1\alpha_2)x)}\right).$$

We will denote this subgroup by $\mathcal{IP}(p,\beta,\alpha)$.

(b) Notice, that Riordan matrices from $\mathcal{IP}(p,1,\alpha)$, i.e., Riordan matrices of the form $(p(x), x/(1-\alpha x))$, for some real α, and a formal power series $p(x)$ such that $p(0)\neq 0$, form another subgroup of $\mathcal{R}$. The multiplication rule within this subgroup is the following:

$$\left(p_1(x), \frac{x}{(1-\alpha_1 x)}\right)\left(p_2(x), \frac{x}{(1-\alpha_2 x)}\right) = \left(p_1(x)p_2\left(\frac{x}{(1-\alpha_1 x)}\right), \frac{x}{(1-(\alpha_1+\alpha_2)x)}\right).$$

We will denote this subgroup by $\mathcal{P}(p,\alpha)$ and call matrices of this form a *generalized Pascal* matrices (briefly GP). Recall that the name of a *Pascal* matrix is attributed traditionally to the matrix $(1/(1-x), x/(1-x))$, hence a particular element of the group $\mathcal{P}(p,\alpha)$.

(c) Notice that also matrices from $\mathcal{IP}(p,a,0)$, i.e., Riordan matrices of the form $(p(x), ax)$, for some real a and a formal power series $p(x)$ such that $p(0)\neq 0$, form another subgroup of $\mathcal{R}$. The multiplication rule within this subgroup is the following:

$$(p_1(x), a_1x)(p_2(x), a_2x) = (p_1(x)p_2(a_1x), a_1a_2x).$$

We will denote this subgroup by $\mathcal{L}(p,a)$. Obviously, $\mathcal{IP}(p,a,0) = \mathcal{L}(p,a)$. It has a subgroup $\mathcal{L}(p,1)$ which we will denote by $\mathcal{A}$ (called *Appell* subgroup) or $\mathcal{A}(p)$, if one wants to underline the dependence on the formal power series p. Moreover, $\mathcal{A}$ has a subgroup consisting of elements such that $p(0)=1$. Let us denote it by $\mathcal{O}$. Notice that, if $p(x)=1$ then this particular element, i.e., matrix $(1,x)$ plays the rôle of 1 in all these groups and subgroups.

(d) Riordan matrices of the form $(1,p(x))$ make another subgroup of $\mathcal{R}$ (called *associated*, sometimes called also iterations (see [6]) subgroup or simply *Lagrange* (denoted by $\mathcal{C}$)) with the following multiplication rule:

$$(1,p_1(x))(1,p_2(x)) = (1,p_2(p_1(x))).$$

(e) The *Bell* subgroup can be formed by Riordan matrices of the form $(g(x), xg(x))$ or $(f(x)/x, f(x))$, with the following multiplication rule:

$$(g_1(x), xg_1(x))\,(g_2(x), xg_2(x)) = (g_1(x)g_2(xg_1(x)), xg_1(x)g_2(xg_1(x))).$$

(f) The Riordan matrices of the form $(f'(x), f(x))$ make also the subgroup (called the *derivative* subgroup with the following multiplication rule:

$$(f'(x), f(x))\,(g'(x), g(x)) = (f'(x)\,g(f'(x)), g(f(x)))\,.$$

(g) The Riordan matrices of the form (f,g), where f is even while g odd function form another sub-group (called *Checkerboard*) with the following multiplication rule:

$$(f_1,g_1)\,(f_2,g_2) = (f_1f_2(g_1), g_2(g_1(x)))\,.$$

This is so, since $f_1(-x))\,f_2(g_1(-x)) = f_1(x)\,f_2(-g_1(x)) = f_1(x)\,f_2(g_1(x))$ is even if only f_1, f_2 are even and g_1 is odd. Similarly, we have $g_2(g_1(-x)) = g_2(-g_1(x)) = -g_2(g_1(x))$. Hence, $g_2(g_1)$ is odd if only g_2 and g_1 are odd.

We have the following couple of simple observations concerning subgroups of $\mathcal{R}$.

Remark 3.2.

1. We have:
$$\mathcal{S} \supset \mathcal{R} \supset \mathcal{IP}(p,a,\alpha) \supset \mathcal{P}(p,\alpha) \supset \mathcal{A}(p) \supset \mathcal{O}(p),\ \ \mathcal{R} \supset \mathcal{C}.$$
Notice also that in general $\mathcal{O} \not\supseteq \mathcal{D}$ unless elements of $\mathcal{D}$ are of the form $[\{r^n\}]$ for some r.
2. We have immediately
$$\left(a\left(x\right),x\right)\left(1,b\left(x\right)\right)=\left(a\left(x\right),b\left(x\right)\right).$$
That is, every Riordan matrix can be decomposed as a product of an Appell and an associated matrices.

Proof. The proof is elementary, based entirely on the Theorem 3.1. □

Remark 3.3. In [7], it was shown that Appell subgroup is normal in $\mathcal{R}$.

Remark 3.4. Notice also, following [7], that every Riordan matrix $(g(x), f(x))$ can be presented as the product of an Appell matrix and a Bell one. Namely, we have
$$(xg(x)/f(x), x)(f(x)/x, f(x)) = (g(x), f(x)).$$

Remark 3.5. Since the matrix $(1, x)$ plays the rôle of a neutral element in the Riordan group we see that the inverse of $(p(x), x/(1-\alpha x))^{-1}$ is equal to $(p_1(x), x/(1+\alpha x))$ where $p_1(x)$ is such that $p_1(x/(1-\alpha x)) = 1/p(x)$.

Remark 3.6. All subgroups mentioned on the List of subgroups ...3 except for the generalized Pascal subgroup are known and mentioned, e.g., in [7] or even earlier, like, e.g., the associated subgroup is mentioned in [12].

Hence, let us analyze subgroup $\mathcal{P}(p,\alpha)$ in more detail.

Proposition 3.7. *Let us consider a generalized Pascal matrix $(p(x), x/(1-\alpha x))$. Let us take $\alpha \in \mathbb{R}$ and that $p(x)$ has the following expansion in the FPS $p(x) = \sum_{n\geq 0} p_n x^n$ with $p_0 \neq 0$. We have then*

1. $$[[x^n]]\,p(x)\,(x/(1-\alpha x))^k = \sum_{j=0}^{n-k} p_j \alpha^{n-k-j}\binom{n-j-1}{k-1},$$
2. $$p(x/(1-\alpha x)) = p_0 + \sum_{s=1}^{\infty} x^s \sum_{j=0}^{s-1} p_{j+1}\alpha^{s-1-j}\binom{s-1}{j}.$$

Proof. 1. Let us find $[[x^n]]\,p(x)\,(x/(1-\alpha x))^k$. To do this, let us expand $(x/(1-\alpha x))^k$ as the power series. It is easy, noticing that for $\alpha=0$ the series is finite and consists of only one element, while for $\alpha\neq 0$ we have

$$(x/(1-\alpha x))^k=\frac{1}{\alpha^k}\left(\frac{\alpha x}{1-\alpha x}\right)^k=\frac{1}{\alpha^k}(1-\alpha x)\sum_{n\geq k}\binom{n}{k}(\alpha x)^n,$$

by the well-known binomial theorem. Now applying Cauchy rule of multiplication of series, we get

$$\begin{aligned}p(x)\,(x/(1-\alpha x))^k&=p\left(x\right)\left(\frac{1}{\alpha^k}(1-\alpha x)\sum_{n\geq 0}\binom{n}{k}(\alpha x)^n\right)\\&=(1-\alpha x)x^k\sum_{s=0}^{\infty}x^s\sum_{j=0}^{s}p_j\alpha^{s-j}\binom{s-j+k}{k}.\end{aligned}$$

Hence, further we get, after some simple algebra:

$$\begin{aligned}[[x^n]]\,p(x)\,(x/(1-\alpha x))^k=&\sum_{j=0}^{n-k}p_j\alpha^{n-k-j}\binom{n-k-j+k}{k}\\&-\alpha\sum_{j=0}^{n-k-1}p_j\alpha^{n-k-1-j}\binom{n-k-1-j+k}{k}\\=&\sum_{j=0}^{n-k}p_j\alpha^{n-k-j}\binom{n-j-1}{k-1}.\end{aligned}$$

2. Let us analyze also what is the formal power series (FPS) of $p(x/(1-\alpha x))$ provided FPS of $p(x)$ is given by: $\sum_{j\geq 0}p_jx^j$.

We have, after some elementary algebra:

$$\begin{aligned}\sum_{j=0}^{\infty}p_j\alpha^{-j}\left(\frac{\alpha x}{1-\alpha x}\right)^j=&(1-\alpha x)\sum_{j=0}^{\infty}p_jx^j\sum_{k=0}^{\infty}x^k\alpha^k\binom{j+k}{k}\\=&(1-\alpha x)\sum_{s=0}^{\infty}x^s\sum_{j=0}^{s}p_j\alpha^{s-j}\binom{s}{s-j}\\=&p_0+\sum_{s=1}^{\infty}x^s\sum_{j=0}^{s}p_j\alpha^{s-j}\binom{s}{s-j}-\alpha\sum_{s=0}^{\infty}x^{s+1}\sum_{j=0}^{s}p_j\alpha^{s-j}\binom{s}{s-j}.\end{aligned}$$

Further, we get also after some simple algebra:

$$\begin{aligned}p(x/1-\alpha x))=&p_0+\sum_{s=1}^{\infty}x^s\sum_{j=0}^{s}p_j\alpha^{s-j}\binom{s}{j}-\sum_{m=1}^{\infty}x^m\sum_{j=0}^{m-1}p_j\alpha^{m-j}\binom{m-1}{j}\\=&p_0+\sum_{s=1}^{\infty}x^s\sum_{j=0}^{s-1}p_{j+1}\alpha^{s-1-j}\binom{s-1}{j}.\end{aligned}$$

□

We have also the following immediate observations:

Remark 3.8. 1. If we set $\alpha = 0$ in assertion 1. of the Proposition 3.7, then obviously $[[x^n]]\,p(x)x^k = p_{n-k}$. Further, if we set $\alpha = 1 = p_j$ for all $j \geq 0$ then $[[x^n]]\,p(x)\,(x/(1-x))^k = \sum_{j=0}^{n-k} \binom{n-j-1}{k-1} = \binom{n}{k}$, i.e., we deal with the so-called Pascal matrix. That's why we call the Riordan matrix $(p(x), x/(1-\alpha x))$ a generalized Pascal matrix.

2. Given an infinite sequence $\{p_j\}_{j\geq 0}$ the following infinite sequence $\left\{\sum_{j=0}^{n} \binom{n}{j} p_j \alpha^{n-j}\right\}_{n\geq 0}$ is called the sequence of generalized binomial transforms (GBT(α)) of the sequence $\{p_j\}_{j\geq 0}$. In the case of $\alpha = 1$, simply the binomial transform. It is well known that GBT(α) and GBT($-\alpha$) are mutually inverse, that is GBT($-\alpha$) applied to GBT(α) of a sequence $\{p_j\}$ recovers this sequence. Besides GF of the sequence $\left\{\hat{p}_n \stackrel{def}{=} \sum_{j=0}^{n} \binom{n}{j} p_j \alpha^{n-j}\right\}_{n\geq 0}$ is equal to

$$\frac{1}{1-\alpha x} p\left(\frac{x}{1-\alpha x}\right),$$

hence

$$p\left(\frac{x}{1-\alpha x}\right) = (1-\alpha x)\sum_{j\geq 0} x^j \hat{p}_j,$$

and consequently

$$[[x^n]]p\left(\frac{x}{1-\alpha x}\right) = \hat{p}_n - \alpha \hat{p}_{n-1}.$$

3. Observe that

$$\begin{aligned}[[x^n]]\,p(x)\,(x/(1-\alpha x))^k &= \sum_{j=0}^{n-k} p_j \alpha^{n-k-j}\binom{n-j-1}{k-1} \\ &= \binom{n-1}{k-1}\sum_{j=0}^{n-k}\binom{n-k}{j} p_j \alpha^{n-k-j} / \binom{n-1}{j}.\end{aligned}$$

As a corollary, we have the following observations:

Corollary 3.9. *1. Suppose we have two Riordan matrices $(a(x), b(x))$ and $(d(x), x)$. Then, we have*

$$(a(x), b(x))(d(x), x) = (a(x)d((b(x)), b(x)),$$

and

$$(d(x), x))\,(a(x), b(x)) = (d(x)a(x), b(x)).$$

In particular, we have the following form of multiplication by the Pascal matrix:

$$\left(\frac{1}{1-\alpha x}, \frac{x}{1-\alpha x}\right)(d(x), x) = \left(D(x,\alpha), \frac{x}{1-\alpha x}\right), \tag{4}$$

$$(d(x),x)\left(\frac{1}{1-\alpha x},\frac{x}{1-x}\right)=\left(\frac{1}{1-\alpha x}d(x),\frac{x}{1-x}\right), \tag{5}$$

where we denoted $d_n=[[x^n]]d(x)$, $\hat{d}_n(\alpha)=\sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}d_j$, $D(x,\alpha)=\sum_{j=0}^{\infty}\hat{d}_j(\alpha)x^j$.

2. *For* $\alpha\in\mathbb{R}$:

$$\left(\frac{1}{1-\alpha x},\frac{x}{1-\alpha x}\right)\left(d(x),\frac{x}{1+\alpha x}\right)=\left(D(x,\alpha),x\right),$$

where $D(x)$ *is defined as above.*

Proof. The only thing that requires justification is (4). Following Theorem of Roman, we have

$$\left(\frac{1}{1-x},\frac{x}{1-x}\right)(d(x),x)=\left(\frac{1}{1-x}d\left(\frac{x}{1-x}\right),\frac{x}{1-x}\right).$$

Now, let us recall that

$$\sum_{n=0}^{\infty}x^n\binom{n}{j}=\frac{x^j}{(1-x)^{j+1}},$$

consequently we have

$$\begin{aligned}d\left(\frac{x}{1-x}\right)&=\sum_{j=0}^{\infty}d_j\left(\frac{x}{1-x}\right)^j=\sum_{j=0}^{\infty}d_j(1-x)\sum_{n=j}^{\infty}x^n\binom{n}{j}\\&=(1-x)\sum_{n=0}^{\infty}x^n\sum_{j=0}^{n}d_j\binom{n}{j}.\end{aligned}$$

Now it is elementary to notice that $\frac{1}{1-x}d(\frac{x}{1-x})=\sum_{n=0}^{\infty}x^n\hat{d}_n\overset{def}{=}D(x)$. □

Remark 3.10. As a by-product of assertion 2. of the Lemma 3.9 we see that every Appell matrix can be decomposed as the product of two generalized Pascal matrices. Namely, we have for all α.

$$\begin{aligned}\left(d(x),\frac{x}{1-\alpha x}\right)\left(1,\frac{x}{1+\alpha x}\right)&=\left(d(x),x\right),\\\left(d(x),\frac{x}{1-\alpha x}\right)&=\left(d(x),x\right)\left(1,\frac{x}{1-\alpha x}\right).\end{aligned}$$

Remark 3.11. Since we have

$$\left(1,\frac{x}{1-\alpha x}\right)\left(1,\frac{x}{1-bx}\right)=\left(1,\frac{x}{1-(\alpha+b)x}\right),$$

as it follows from assertion 2. of the List of subgroups ...3, we see that there exists a homomorphism between the group of Riordan matrices $\left\{\left(1,\frac{x}{1-\alpha x}\right)\right\}_{a\in\mathbb{C}}$ with matrix multiplication as group operation and the group of complex numbers with addition as group operation.

Besides notice that

$$\left[a^{n-j}\binom{n-1}{j-1}\right]=\left(1,\frac{x}{1-\alpha x}\right),$$

as it follows from assertion 1. of Proposition 3.7 with $p_0=1$ and $p_j=0$ for $j>0$.

Remark 3.12. To complete presentation of simple subgroups of the Riordan group let us notice that $[\{a^n\}]=(1,ax)$. Consequently we observe that $(1,\gamma x)(a(x),b(x))=(a(\gamma x),b(\gamma x))$, and $(a(x),b(x))(1,\gamma x)=(a(x),\gamma b(x))$.

Why do we study the lower-triangular representation of some sequences? Well, to get, for example, some nontrivial identities.

To see what we mean let us consider several examples. Most of them would concern Riordan matrices of the form $(d(x),x)$, i.e., Appell matrices, since these matrices have a form easy to deal with. Besides, there exist in the literature many examples of matrices of this form whose entries are polynomials. Moreover, there are also ready-to-use formulas for the inverses of such matrices. In the sequel, we often use matrix operation as defined by (3). We we will denote the result of such operation on the matrix, say $\mathbf{A}$ as $[\{\alpha_n\}]$ transform of $\mathbf{A}$.

4. Examples

This section focuses on analyzing well-known identities and sequences of numbers or polynomials using lower-triangular matrices, with most of them being of Riordan type. One has to underline that this is not the only approach to this problem. One can use simply the GF techniques like it was done in [2] or linear difference equations as it was presented in [8] and [5]. Most of the examples will be Riordan matrices of the form $(d(x,y),y)$, with $d(x,y)$ being a GF of the sequence $\{d_n(x)\}_{n\geq0}$ consisting either of numbers or polynomials in x. By assertion 1. of the Remark 3.8 we see that $[d_{n-j}(x)]$ is the Riordan matrix $(d(x,y),y)$. Two examples will refer to the concept of $\mathcal{IP}(p,\beta,\alpha)$ class of Riordan matrices introduced at position (1) of the List of subgroups ...3. The sub-algebra $\mathcal{SE}$ introduced in the paper's first section is demonstrated in the subsubsection 4.1.4.

4.1. Bernoulli and Euler polynomials and numbers

4.1.1. Bernoulli & Euler Polynomials. Let us recall that Bernoulli and Euler polynomials (denoted respectively $B_n(x)$ and $E_n(x)$, for $n-$th polynomial) are defined respectively by the expansions:

$$t\exp(xt)/(\exp(t)-1)=\sum_{n\geq0}\frac{t^n}{n!}B_n(x), \tag{6a}$$

$$2\exp(xt)/(\exp(t)+1)=\sum_{n\geq0}\frac{t^n}{n!}E_n(x). \tag{6b}$$

Hence, we have the following identity

$$\begin{aligned}\sum_{n\geq 0}\frac{2^n t^n}{n!}B_n(x) &=2t\exp(2xt)/(\exp(2t)-1)\\ &=\left(\sum_{n\geq 0}\frac{t^n}{n!}B_n(x)\right)\left(\sum_{n\geq 0}\frac{t^n}{n!}E_n(x)\right)\\ &=\sum_{k\geq 0}\frac{t^k}{k!}\sum_{n=0}^{k}\binom{k}{n}B_n(x)E_{k-n}(x).\end{aligned}$$

Comparing polynomials by t^n and dividing both sides by $k!$ we end up with the identity.

$$2^k B_k(x)/k! = \sum_{n=0}^{k}(B_n(x)/n!\,)\,(E_{k-n}(x)/(k-n)!\,)\,. \tag{7}$$

Let us define the following lower-triangular infinite matrices $[B_{n-j}(x)/(n-j)!\,]$ and $[E_{n-j}(x)/(n-j)!\,]$. Hence, the Eq. (7) is now reflected in the following relationship between these matrices:

$$[B_{n-j}(x)/(n-j)!\,]\,[E_{n-j}(x)/(n-j)!\,] = [\{2^n\}]\,[B_{n-j}(x)/(n-j)!\,]\left[\{2^{-j}\}\right]. \tag{8}$$

Remark 4.1. Notice that, following definition of Appell matrix, we deduce that $[B_{n-j}(z)/(n-j)!\,]$ and $[E_{n-j}(z)/(n-j)!\,]$ are two Riordan matrix of an Appell type. More precisely, for every $z\in\mathbb{R}$:

$$\begin{aligned}[B_{n-j}(z)/(n-j)!\,] &= \left(\frac{x\exp(zx)}{\exp(x)-1},x\right),\\ [E_{n-j}(z)/(n-j)!\,] &= \left(\frac{2\exp(zx)}{\exp(x)+1},x\right).\end{aligned}$$

Moreover, (8) is just the example of the position (3). of the List of subgroups ...3, above. Namely, we have

$$\begin{aligned}[B_{n-j}(z)/(n-j)!\,]\,[E_{n-j}(z)/(n-j)!\,] &= \left(\frac{x\exp(zx)}{\exp(x)-1},x\right)\left(\frac{2\exp(zx)}{\exp(x)+1},x\right)\\ &= \left(\frac{2\exp(2zx)}{\exp(2x)-1},x\right),\end{aligned}$$

where $\frac{2\exp(2zx)}{\exp(2x)-1} = \sum_{n>j}x^{n-j}2^{n-j}B_{n-j}(z)/(n-j)!$.

4.1.2. Laguerre polynomials. Let $L_n^{(\alpha)}(x)$ denote $n-$th Laguerre polynomial, the member of the family of polynomials which are orthogonal with respect to measure $x^{-\alpha}\exp(-x)$ for $x>0$ and $\alpha\neq 1,2,\ldots$. Let us define the following family of orthogonal polynomials:

$$\mathcal{L}_n^{(\alpha)}(x) = L_n^{(-\alpha-2)}(-x)\,.$$

Proposition 4.2. *For all $\alpha \neq 1, 2, \ldots, n \geq m \geq 0$: $\sum_{j=m}^{n} L_{n-j}^{(\alpha)}(x)\mathcal{L}_{j-m}^{(\alpha)}(x) = 0$, Consequently, we have*

$$\left[L_{n-j}^{(\alpha)}(x)\right]^{-1} = \left[\mathcal{L}_{n-j}^{(\alpha)}(x)\right].$$

Besides, we know that we deal with the Riordan matrix of the Appell type

$$\left[L_{n-j}^{(a)}(x)\right] = \left(\exp\left(-\frac{tx}{1-t}\right)/(1-t)^{a+1}, 1\right),$$

since $\sum_{j\geq 0} t^n L_n(x) = \exp\left(-\frac{tx}{1-t}\right)/(1-t)$.

Proof. First, notice that the we can take $k = j-m$ and prove the identity $\sum_{k=0}^{n-m} L_{n-m-k}^{(a)}(x)$ $\mathcal{L}_k^{(a)}(x) = 0$. We are now using the concept of generating functions and asserting Remark 3.5. Consequently, we realize that our identity is equivalent to the fact that the generating function of the polynomials $\left\{\mathcal{L}_k^{(\alpha)}(x)\right\}$ is equal to the inverse of the generating function of the Laguerre polynomials. But it is well-known that

$$\sum_{n\geq 0} t^n L_n^{(\alpha)}(x) = \frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{tx}{1-t}\right).$$

Hence, it remains to notice that :

$$\sum_{n\geq 0} t^n \mathcal{L}_n^{(\alpha)}(x) = (1-t)^{\alpha+1} \exp\left(\frac{tx}{1-t}\right).$$

□

4.1.3. Hermite polynomials. Following an identity presented in, say [14](unnumbered formula below (8.17)), we have

$$\left[He_{n-j}(x)/(n-j)!\right]^{-1} = \left[i^{n-j} He_{n-j}(ix)/(n-j)!\right],$$

where i is an imaginary unit, and $He_n(x)$ $n-$th so-called probabilistic Hermite polynomial, i.e., the ones orthogonal with respect to the measure with the density $\exp(-x^2/2)/\sqrt{2\pi}$.

Recall also that the GF function of the Hermite polynomials is $\exp(yx - y^2/2)$ that is we have (see e.g. Wikipedia, [9])

$$\sum_{n\geq 0} \frac{y^n}{n!} He_n(x) = \exp(yx - y^2/2).$$

Remark 4.3. Now we see that $[He_{n-j}(x)/(n-j)!]$ is an Riordan matrix of the Appell type given by the formula $(\exp(yx - y^2/2), x)$.

4.1.4. Numbers. We will also use the following notation:

$$\varepsilon(n)=\begin{cases}0 & \text{if } \quad n \text{ is odd}\\ 1 & \text{if } \quad n \text{ is even}\end{cases},\quad H(x)=\begin{cases}0 & \text{if } \quad x<0,\\ 1 & \text{if } \quad x\geq 0,\end{cases}$$

$$H2(x)=H(x)+H(x-2)-2H(x-1)=\begin{cases}1 & \text{if } \quad x=0,\\ -1 & \text{if } \quad x=1,\\ 0 & \text{if } \quad \text{otherwise},\end{cases}$$

$$H3(x)=H(x)-H(x-1)-H(x-2)+H(x-3).$$

and $\{E_n\}$, and $\{B_n\}$ denote respectively $n-$th Euler and Bernoulli numbers.

We have

$$\left[\binom{n}{j}\right]^{-1}=\left[(-1)^{n-j}\binom{n}{j}\right],\ \left[\lambda^{n-j}\binom{n}{j}\right]^{-1}=\left[(-\lambda)^{n-j}\binom{j}{n}\right],$$
$$[H(n-j)]^{-1}=[H2(n-j)],\ [\varepsilon(n-j)H(n-j)]^{-1}=[H3(n-j)],$$

for some real λ. Let us recall that matrix $\left[\binom{n}{j}\right]$ it is a well know Pascal matrix mentioned already above. Hence, we have for example, for all $n>j$

$$\sum_{k=j}^{n}(-1)^k\binom{n}{k}\binom{k}{j}=0.$$

Remark 4.4. Notice also that $[H[n-j]=(1/(1-x),x)$, hence naturally we have $[H\,(n-j)]^{-1}=(1-x,x)$, as it follows from the List of subgroups ...3, 3.

As shown in [13] we have

$$\left[\binom{n}{j}\frac{1}{n-j+1}\right]^{-1}=\left[\binom{n}{j}B_{n-j}\right],\ \left[\varepsilon(n-j)\binom{n}{j}\right]^{-1}=\left[\binom{n}{j}E_{n-j}\right]. \tag{9}$$

Thus we have for all $n>j$:

$$\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}\frac{B_{k-j}}{n-k+1}=0,$$
$$\sum_{k=j}^{n}e(n-k)\binom{n}{k}\binom{k}{j}E_{k-j}=0.$$

Notice that performing the following matrix multiplications on the first of the identities (9):

$$[\{1/n!\}]\left[\binom{n}{j}B_{n-j}\right][\{j!\}],\ [\{1/n!\}]\left[\binom{n}{j}\frac{1}{n-j+1}\right]^{-1}[\{j!\}],$$

we end up with the rather unexpected identities:

$$[1/(n-j)!]^{-1}=\left[(-1)^{n-j}/(n-j)!\right], \tag{10}$$
$$[1/(n-j+1)!]^{-1}=[B_{n-j}/(n-j)!], \tag{11}$$
$$[\varepsilon(j-i)/(j-i)!]^{-1}=[E_{j-i}/(j-i)!]. \tag{12}$$

Remark 4.5. Let us notice that $[1/(n-j)!\,]$, $[1/(n-j+1)!\,]$, $[\varepsilon(j-i)/(j-i)!\,]$, $[B_{n-j}/(n-j)!\,]$, $[E_{j-i}/(j-i)!\,]$ are Riordan matrices and $[1/(n-j)!\,] = (\exp(x), x)$, $[1/(n-j+1)!\,] = (\exp(x)-1, x)$, $[\varepsilon(j-i)/(j-i)!\,] = (\cosh(x), x)$, $[B_{n-j}/(n-j)!\,] = (x\exp(x)/(\exp(x)-1), x)$, $[E_{j-i}/(j-i)!\,] = (2\exp(x)/(\exp(x)+1), x)$. Besides we see that $\left[\binom{n}{j}E_{n-j}\right]$, $\left[\varepsilon(n-j)\binom{n}{j}\right]$, $[\varepsilon(j-i)/(j-i)!\,]$, $[E_{j-i}/(j-i)!\,]$ are the examples of the elements of the sub-algebra $\mathcal{SE}$.

As shown also in [13] we have further

$$\left[\varepsilon(n-j)\binom{n}{j}\frac{1}{n-j+1}\right]^{-1} = \left[\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}2^k B_k\right],$$

$$\left[\binom{2n}{2j}\right]^{-1} = \left[\binom{2n}{2j}E_{2(n-j)}\right],$$

$$\left[\binom{2n}{2j}\frac{1}{2(n-j)+1}\right]^{-1} = \left[\binom{2n}{2j}\sum_{k=0}^{2(n-j)}\binom{2n-2j}{k}2^k B_k\right].$$

Hence, in particular we have $\forall s \geq 1$:

$$\sum_{k=0}^{s}\binom{2s}{2k}E_{2k} = 0,$$

$$\sum_{k=0}^{2s}\binom{2s}{k}B_k/(2s-k+1) = 0.$$

Notice that we have changed the order of summation to get the second identity and use the following, elementary to prove, identity:

$$\sum_{m=\lfloor l/2\rfloor}^{s}\binom{2s-l}{2m-l}/(2s-2m+1) = \frac{2^{2s-l}}{2s-l+1}.$$

Again, performing right hand side and left-hand side multiplications by matrices $[\{1/n!\,\}]$ and its inverse we get the following also unexpected identities:

$$[\varepsilon(j-i)/(j-i+1)!\,]^{-1} = \left[\sum_{k=0}^{j-i}2^k B_{j-i-k}/(k!\,(j-i-k)!\,)\right], \quad (13)$$

$$[1/(2j-2i)!\,]^{-1} = [E_{2j-2i}/(2j-2i)!\,], \quad (14)$$

$$[1/(2j-2i+1)!\,]^{-1} = \left[\sum_{k=0}^{2j-2i}2^k B_k/(k!\,(2j-2i-k)!\,)\right]. \quad (15)$$

Remark 4.6. Again notice that $[\varepsilon(n-j)/(n-j+1)!\,]$, $[1/(2n-2j)!\,]$, $[1/(2n-2i+1)!\,]$, $[E_{2j-2i}/(2j-2i)!\,]$ are Riordan matrices respectively: $(\sinh(x)/x, x)$, $(\cosh(\sqrt{x}), x)$, $(\sinh(\sqrt{x})\,/\sqrt{x}, x)$, $(\sqrt{x}/\sinh(\sqrt{x}), x)$. Consequently we deduce that $\Big[\sum_{k=0}^{j-i}2^k B_{j-i-k}/(k!\,(j-$

$i-k)!\,)\Big]$, $\left[\sum_{k=0}^{2j-2i} 2^k B_k/(k!\,(2j-2i-k)!\,)\right]$ are also Riordan matrices respectively: $(x/\sinh(x), x)$ and $(\sqrt{x}/\sinh(x), x)$. Again notice that the lower-triangular infinite matrices $\Big[\varepsilon(n-j)\binom{n}{j}\frac{1}{n-j+1}\Big]$ and $[\varepsilon(j-i)/(j-i+1)!\,]$ belong to the sub-algebra $\mathcal{SE}$.

4.2. Pochhammer symbol - rising factorials

We will use the following notation. For $x \in \mathbb{C}$ let us denote

$$(x)_{(n)} = x(x-1)\ldots(x-n+1). \tag{16}$$

This polynomial in x will be called falling factorial while the following polynomial

$$(x)^{(n)} = x(x+1)\ldots(x+n-1), \tag{17}$$

will be called rising factorial. The $n-$th rising factorial is also called the Pochhammer symbol. In both cases, we set 1 when $n = 0$.

It is also well-know that

$$(x)_{(n)} = (-1)^n(-x)^{(n)}, \text{and } (x)^{(n)} = (-1)^n(-x)_{(n)}. \tag{18}$$

Recall, that we have also the so-called binomial theorem stating that for all complex $|x| < 1$ we have

$$(1-x)^{\alpha} = \sum_{j\geq 0}(-x)^j(\alpha)_{(j)}/j! = \sum_{j\geq 0} x^j(-\alpha)^{(j)}/j!\,. \tag{19}$$

Notice that from this expansion, by the (so-called exponential) generating function method we get the following identity which is true for all $\alpha, \beta \in \mathbb{C}$:

$$(\alpha+\beta)^{(n)} = \sum_{j=0}^{n}\binom{n}{j}(\alpha)^{(n-j)}(\beta)^{(j)}, \tag{20}$$

$$(\alpha+\beta)_{(n)} = \sum_{j=0}^{n}\binom{n}{j}(\alpha)_{(n-j)}(\beta)_{(j)}. \tag{21}$$

The following is our initial observation:

Proposition 4.7. i) $\forall\mathbb{C} \ni x \neq 0:$

$$\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]^{-1} = \left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]. \tag{22}$$

ii) $\forall\mathbb{C} \ni x \neq 0:$

$$\left[\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]^{-1} = \left[(-1)^{n-j}\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]. \tag{23}$$

iii) *For the reference sequence $c_n = n!$, we have*

$$\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right] = \left((1-y)^{-x}, \frac{-y}{1-y}\right).$$

Thus, it belongs to a class $\mathcal{IP}(p,-1,1)$ (compare position (1) of the List of subgroups ...3.

Proof. To prove this identity, we have to show that $\forall n>j$:

$$\begin{aligned} 0 &= \sum_{k=j}^{n}(-1)^k\binom{n}{k}\frac{(x)^{(n)}}{(x)^{(k)}}(-1)^j\binom{k}{j}\frac{(x)^{(k)}}{(x)^{(j)}} \\ &= \binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\sum_{k=j}^{n}(-1)^{k-j}\frac{(n-j)!}{(n-k)!\,(k-j)!}. \end{aligned}$$

But this is obvious in the face of the above calculations.

ii) We apply right-hand side multiplication of a lower triangular infinite matrix by a diagonal matrix $\{(-1)^j\}$ as described in Remark 1.2,4. getting on one hand $\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]\left[\left\{(-1)^j\right\}\right]=\left[\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]$. On the other hand since

$$\begin{aligned} \left(\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right]\left[\left\{(-1)^j\right\}\right]\right)^{-1} &= \left[\left\{(-1)^j\right\}\right]^{-1}\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right] \\ &= [\{(-1)^n\}]\left[(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right] \\ &= \left[(-1)^{n-j}\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\right] \end{aligned}$$

iii) Following (19), we get for the $j-$th column

$$\begin{aligned} g_j(x,y) &= \sum_{n\geq j}(-1)^j\binom{n}{j}\frac{(x)^{(n)}}{(x)^{(j)}}\frac{y^n}{n!} = (-1)^j\frac{y^j}{j!}\sum_{n\geq j}y^{n-j}\frac{(x+j)^{(n-j)}}{(n-j)!} \\ &= (-1)^j\frac{y^j}{j!}(1-y)^{-x-j} = (1-y)^{-x}\left(-\frac{y}{1-y}\right)^j/j!\,. \end{aligned}$$

□

Now, taking $\alpha=x$ and $\beta=-x$ in (20) and then in (21) and finally using the standard trick with multiplication by the diagonal matrix $[\{n!\}]$ and its inverse (compare Remark 1.2,4.), we arrive at the following identities

$$\begin{aligned} \left[(x)^{(n-j)}/(n-j)!\right]^{-1} &= \left[(-x)^{(n-j)}/(n-j)!\right], \\ \left[(x)_{(n-j)}/(n-j)\,!\right]^{-1} &= \left[(-x)_{(n-j)}/(n-j)!\right]. \end{aligned}$$

Similarly, setting $x=1$ in (22,i)), we get

$$\left[(-1)^j(n-j)!\binom{n}{j}^2\right]^{-1} = \left[(-1)^j(n-j)!\binom{n}{j}^2\right].$$

Similarly, setting $x=1$ in (23), we get

$$\left[(n-j)!\binom{n}{j}^2\right]^{-1} = \left[(-1)^{n-j}(n-j)!\binom{n}{j}^2\right].$$

Remark 4.8. It is elementary to notice that $\left[(x)^{(n-j)}/(n-j)!\right]$ is equal to the Riordan matrix of Appell type$((1-y)^{-x}, y)$. Further let us note, that $\left[(-1)^j(n-j)!\binom{n}{j}^2\right]$ is a Riordan matrix $(\frac{1}{1-y}, \frac{-y}{1-y})$. Finally, combining the fact that $\left[(-1)^j(n-j)!\binom{n}{j}^2\right]$ is a Riordan matrix $(\frac{1}{1-y}, \frac{-y}{1-y})$, the fact that

$$\left[(n-j)!\binom{n}{j}^2\right] = \left[(-1)^j(n-j)!\binom{n}{j}^2\right]\left[\{(-1)^j\}\right],$$

and the observations contained in the Remark 3.12, we deduce the $\left[(n-j)!\binom{n}{j}^2\right]$ is the Riordan matrix of the Pascal type equal to $(\frac{1}{1-y}, \frac{y}{1-y})$.

Recently, in [16] several lower-triangular matrices involving rising factorials of two variables have been defined. Let us present these matrices and some relationships they are involved in. Let us remark that the matrices presented below are not Riordan matrices. However, since they have rather complicated structure, they can be used to obtain non-trivial relationships involving Pochhammer symbols of two variables.

1.

$$E(a,b) = \left[\frac{(a+b+n-1)^{(j)}}{j!}\frac{(b+j)^{(n-j)}}{(n-j)!}\right]^{-1}$$

$$\bar{E}(a,b) = E^{-1}(a,b) = \left[(-1)^{n-j}\frac{n!\,(b+j)^{(n-j)}\,(a+b+2j-1)}{(n-j)!\,(a+b+j-1)^{(n+1)}}\right].$$

2.

$$E(a,b)\bar{E}(b,b) = \left[(-1)^{n-j}\frac{(b+j)^{(n-j)}(a-b)^{(n-j)}(a+b+n-1)^{(j)}(2b+2j-1)}{(n-j)!\,(2b+j-1)^{(n+1)}}\right],$$

$$E(b,a)\bar{E}\,(b,b) = [(-1)^n]\,E(a,b)\bar{E}(b,b)\left[(-1)^j\right]$$
$$= \left[\frac{(b+j)^{(n-j)}(a-b)^{(n-j)}(a+b+n-1)^{(j)}(2b+2j-1)}{(n-j)!\,(2b+j-1)^{(n+1)}}\right],$$

$$E(b,b)\bar{E}(a,b) = \left(E(a,b)\bar{E}(b,b)\right)^{-1}$$
$$= \left[(-1)^{n-j}\frac{(b+j)^{(n-j)}(2b+n-1)^{(j)}(b-a)^{(n-j)}\,(a+b+2j-1)}{(n-j)!\,(a+b+j-1)^{(n+1)}}\right],$$

$$E\,(b,b)\,\bar{E}(b,a) = [(-1)^n]\,E(b,b)\bar{E}(a,b)\left[(-1)^j\right]$$
$$= \left[\frac{(b+j)^{(n-j)}(2b+n-1)^{(j)}(b-a)^{(n-j)}\,(a+b+2j-1)}{(n-j)!\,(a+b+j-1)^{(n+1)}}\right].$$

3.

$$E\left(a,a\right)\bar{E}\left(b,b\right)=$$
$$\left[\frac{\varepsilon\left(n-j\right)\left(2b+2j-1\right)\left(2a+n-1\right)^{(j)}\left(a-b\right)^{((n-j)/2)}\left(b+j\right)^{((n-j)/2)}\left(a+\left(n+j\right)/2\right)^{((n-j)/2)}}{\left(\left(n-j\right)/2\right)!\left(2b+j-1\right)^{(n+1)}}\right].$$

4.3. Examples coming from q−series theory

4.3.1. Introduction and notation. Let us introduce a few elementary notions of the so-called $q-$series theory.

q is a parameter. Generally, we will assume that $-1<q<1$, unless otherwise stated. However, sometimes we will consider the case $q=1$, not always directly, but as a left-hand side limit (i.e.,$q\longrightarrow 1^{-}$). We will point out these cases.

We will use traditional notations of the $q-$series theory i.e.,

$$\left[0\right]_q=0,\ \left[n\right]_q=1+q+\ldots+q^{n-1},\left[n\right]_q!=\prod_{j=1}^{n}\left[j\right]_q,\text{with }\left[0\right]_q!=1,$$

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\frac{[n]_q!}{[n-k]_q![k]_q!} & ,\quad n\geq k\geq 0,\\ 0 & ,\quad \text{otherwise.}\end{cases}$$

$\binom{n}{k}$ will denote the ordinary, well known binomial coefficient. It turns out that $\begin{bmatrix}n\\k\end{bmatrix}_q$ are polynomials in q (called Gauss polynomials).

It is useful to use the so-called $q-$Pochhammer symbol for $n\geq 1$:

$$\left(a|q\right)_n=\prod_{j=0}^{n-1}\left(1-aq^j\right),\quad \left(a_1,a_2,\ldots,a_k|q\right)_n=\prod_{j=1}^{k}\left(a_j|q\right)_n, \tag{24}$$

with $\left(a|q\right)_0=1$.

Although the formula below was known much earlier we cite [10] because of its nice proof. Namely, we have

$$\left(a|q\right)_n=\sum_{j=0}^{n}(-a)^j q^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q. \tag{25}$$

This formula is often referred to as a finite $q-$binomial formula, that will be generalized just below.

Often $\left(a|q\right)_n$ as well as $\left(a_1,a_2,\ldots,a_k|q\right)_n$ will be abbreviated to $\left(a\right)_n$ (not to be confused with the falling factorial defined above) and $\left(a_1,a_2,\ldots,a_k\right)_n$, if it will not cause misunderstanding.

We will also use the following symbol $\lfloor n\rfloor$ to denote the largest integer not exceeding n.

It is worth to mention the following two formulae, that are well known. Namely, the following formulae are true for $|t|<1$, $|q|<1$ (derived already by Euler, see [1] Corollary 10.2.2)

$$\frac{1}{(t)_\infty}=\sum_{k\geq 0}\frac{t^k}{(q)_k}, \tag{26}$$

$$(t)_\infty = \sum_{k\geq 0}(-1)^k q^{\binom{k}{2}}\frac{t^k}{(q)_k}. \tag{27}$$

It is easy to notice that $(q)_n = (1-q)^n [n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{28}$$

The above-mentioned formula is just an example where direct setting $q = 1$ is senseless however, the passage to the limit $q \longrightarrow 1^-$ makes sense.

Notice that, in particular,

$$[n]_1 = n,\ [n]_1! = n!,\ \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k},\ (a)_1 = 1-a,\ (a|1)_n = (1-a)^n, \tag{29}$$

and

$$[n]_0 = \begin{cases} 1 & \text{if} \quad n \geq 1 \\ 0 & \text{if} \quad n = 0 \end{cases},\ [n]_0! = 1,\ \begin{bmatrix} n \\ k \end{bmatrix}_0 = 1,\ (a|0)_n = \begin{cases} 1 & \text{if} \quad n = 0, \\ 1-a & \text{if} \quad n \geq 1. \end{cases} \tag{30}$$

4.3.2. Examples concerning important in $q-$series theory families of polynomials. To proceed further, let us prove some auxiliary results.

Proposition 4.9. 1. $\forall x, y, q \in \mathbb{C}, |q|, |y| < 1:$

$$\sum_{j=0}^{\infty} \frac{y^j}{(q)_j}(x)_j = \frac{(yx)_\infty}{(y)_\infty},$$

2. $\forall x, q \in \mathbb{C}, x \neq 0, |q| < 1, n \geq 1$

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (x)_{n-j}\, x^j\, (1/x)_j = 0. \tag{31}$$

Proof. 1. Using (25), we get

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{y^j}{(q)_j}(x)_j &= \sum_{j=0}^{\infty} \frac{y^j}{(q)_j} \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{\binom{k}{2}}(-x)^k \\ &= \sum_{k=0}^{\infty} \frac{(yx)^k}{(q)_k}(-1)^k q^{\binom{k}{2}} \sum_{j=k}^{\infty} \frac{y^{j-k}}{(q)_{j-k}} = \frac{(yx)_\infty}{(y)_\infty},\end{aligned}$$

by (26) and (27).

2. Now notice that using assertion 1. with y substituted by xy we get

$$\frac{(y)_\infty}{(yx)_\infty} = \frac{(yx/x)_\infty}{(yx)_\infty} = \sum_{j=0}^{\infty} \frac{y^j}{(q)_j} x^j\, (1/x)_j\,.$$

Let us denote $d_n(x|q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (x)_{n-j}\, x^j\, (1/x)_j$ and let us find its generating function. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{y^n}{(q)_n} d_n(x|q) &= \sum_{n=0}^{\infty} y^n \sum_{j=0}^{n} \frac{(x)_{n-j}}{(q)_{n-j}} \frac{x^j\,(1/x)_j}{(q)_j} \\
&= \sum_{j=0}^{\infty} \frac{y^j x^j\,(1/x)_j}{(q)_j} \sum_{n=j}^{\infty} \frac{y^{n-j}}{(q)_{n-j}} (x)_{n-j} \\
&= \frac{(y)_\infty}{(yx)_\infty} \frac{(yx)_\infty}{(y)_\infty} = 1.
\end{aligned}$$

Hence $d_n(x|q) = 0$ for $n \geq 1$. □

We also have the following almost elementary observation:
Following (25) for $n > 0$ and $a = 1$ we have he following identity, true for $n \geq 1$:

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}} = 0.$$

As a corollary we have the following relationship:

Corollary 4.10. 1. $\forall x, q \in \mathbb{C}, |q| \neq 1$:

$$\left[x^{n-j}/(q)_{n-j}\right]^{-1} = \left[(-1)^{n-j} x^{n-j} q^{\binom{n-j}{2}}/(q)_{n-j}\right].$$

In particular, taking $x = 1 - q$, we have

$$\left[1/[n-j]_q!\right]^{-1} = \left[(-1)^{n-j} q^{\binom{n-j+1}{2}}/[n-j]_q!\right].$$

Further multiplying from the left-hand side by the matrix $[\{(q)_n\}]$ and from the right-hand side by the matrix $\left[\left\{(q)_j^{-1}\right\}\right]$, we get the following relationship:

$$\left[x^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q\right]^{-1} = \left[(-x)^{n-j}\, q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q\right]. \tag{32}$$

2. *Following (31), we get for all $x, y \in \mathbb{C}$:*

$$\left[y^{n-j}\,(x)_{n-j}\,/(q)_{n-j}\right]^{-1} = \left[y^{n-j} x^{n-j}\,(1/x)_{n-j}\,/(q)_{n-j}\right],$$

and after applying similar trick with diagonal matrix multiplication we get

$$\left[y^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q (x)_{n-j}\right]^{-1} = \left[y^{n-j} x^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_q (1/x)_{n-j}\right].$$

Proof. We start with the assertion of Proposition 32 and multiply both sides of this identity from the left-hand side by $\left[\{(q)_n^{-1}\}\right]$ and from the right-hand side by $\left[\left\{(q)_j\right\}\right]$ and then apply assertion 4. of Remark 1.2. On the way we utilize the definition of $\begin{bmatrix} n \\ j \end{bmatrix}_q$ given by (28). □

Remark 4.11. Notice that when passing with q to 1 in the last assertion of the Corollary 4.10 we get (10).

Remark 4.12. Notice also that some of the relationships mentioned in the Corollary 4.10 refer to Riordan matrices of the Appell type. Namely, we have $[x^{n-j}/(q)_{n-j}] = \left((tx)_{\infty}^{-1},t\right)$ and obviously $\left[(-1)^{n-j}x^{n-j}q^{\binom{n-j}{2}}/(q)_{n-j}\right] = ((tx)_{\infty},t)$. $\left[y^{n-j}\left(x\right)_{n-j}/(q)_{n-j}\right]$ $= ((txy)_{\infty}/(ty)_{\infty},t)$ as it follows from Proposition 4.9, 1. somewhat less obvious is the relationship

$$\left[y^{n-j}x^{n-j}\left(1/x\right)_{n-j}/(q)_{n-j}\right] = ((ty)_{\infty}/(tyx)_{\infty},t).$$

Recall, e.g., following [14], that the following polynomials

$$R_n(x|q) = \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q x^j,$$

are called Rogers-Szegö polynomials and they play an important rôle in the $q-$series theory. Following, e.g., [14], we know that the generating function of the polynomials $\{R_s\}$ are given by the following formula:

$$\sum_{s\geq 0}\frac{t^s}{(q)_s}R_s(x|q) = \frac{1}{(t)_{\infty}(tx)_{\infty}}.$$

Hence, by simple operation of series multiplication and making use of the formula (27) we deduce that the following family of polynomials:

$$\hat{R}_n(x|q) = (-1)^n\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q x^s q^{\binom{s}{2}+\binom{n-s}{2}},$$

defined for $n\geq 0$ satisfy the following identity:

$$\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q R_j(x|q)\hat{R}_{n-j}(x|q) = \delta_{n,0}.$$

Consequently, we have

$$\left[R_{n-j}(x|q)/(q)_{n-j}\right]^{-1} = \left[\hat{R}_{n-j}(x|q)/(q)_{n-j}\right].$$

Following [14], formulae (3.16), (4.10), and (5.15), and the proceeding each of these formulae definitions, we have

$$\left[h_{n-j}(x|q)/[n-j]_q!\right]^{-1} = \left[b_{n-j}(x|q)/[n-j]_q!\right],$$
$$\left[h_{n-j}(x|a,q)/[n-j]_q!\right]^{-1} = \left[\hat{h}_{n-j}(x|a,q)/[n-j]_q!\right],$$
$$\left[Q_{n-j}(x|a,b,q)/[n-j]_q!\right]^{-1} = \left[\hat{Q}_{n-j}(x|a,b,q)/[n-j]_q!\right],$$

where $\{h_n(x|q\}$, $\{h_n(x|a,q)\}$, $\{Q_n(x|a,b,q)\}$, are respectively the so-called $q-$Hermite, big $q-$Hermite, Al-Salam–Chihara polynomials. They constitute a part of the so-called Askey-Wilson scheme of orthogonal polynomials defined by their three-term recurrences given by the formulae respectively (3.1), (4.1), (5.1) in [14]. The families of polynomials $\{b_n(x|q)\}$, $\left\{\hat{h}_n(x|a,q)\right\}$, $\left\{\hat{Q}_n(x|a,b,q)\right\}$ are defined by their three-term recurrences given by the formulae respectively (3.14), (4.9) and unnumbered formula proceeding (5.15). Although these families of polynomials were defined for real x,a,b,q all from the segment $[-1,1]$, we can extend their ranges to all complex numbers since they are polynomials.

Remark 4.13. Let us denote the following infinite product

$$\varphi_h(x|a) = 1/\prod_{j=0}^{\infty} v(t|aq^j),$$

defined for all $|t|,|a|,|q|<1$. where $v(x|a) = 1-2ax+a^2$. It is known (see e.g. [9], or [15], (3.6)) that GF of the sequence $\{h_n(x|q\}$ is equal to $\varphi_h(x|t)$. That is we have

$$\sum_{n\geq 0} \frac{t^n}{(q)_n} h_n(x|q) = \varphi_h(x|t),$$

defined for all $|t|,|x|,|q|<1$.Similarly, following either [9], or [15](4.7) we see that the GF of the family $\{h_n(x|a,q)\}$ is $(at)_\infty \varphi_h(x|t)$. Finally following the same references we observe that the GF of the family $\{Q_n(x|a,b,q)\}$ is $(at,bt)_\infty \varphi_h(x|t)$. Hence, we observe that $\left[h_{n-j}(x|q)/[n-j]_q!\right] = \left[h_{n-j}(x|q)/(q)_{n-j}\right][\{(1-q)^j\}]$ and consequently that is a Riordan matrix having the form of a product $(\varphi_h(x|t),t)(1,(1-q)t)$. Similarly $\left[h_{n-j}(x|a,q)/[n-j]_q!\right]$ is a Riordan matrix of a form $((at)_\infty \varphi_h(x|t),t))(1,(1-q)t)$ and $\left[Q_{n-j}(x|a,b,q)/[n-j]_q!\right]$ is also a Riordan matrix of a form $((at,bt)_\infty \varphi_h(x|t),t)$ $(1,(1-q)t)$. All these considerations are true for $|t|,|x|,|a|,|b|,|q|<1$.

5. Glossary

1. $[a_{n,j}]_{n,j\geq 0,}$ with $a_{nj}=0$ for all $j>n\geq 0$, lower-triangular infinite matrix with entries belonging to $\mathbb{C}$,
2. $\mathcal{S}$ -the algebra of lower- triangular matrices .
3. $\mathcal{D}$ -the sub-algebra of all diagonal matrices.
4. $[\{\beta_n\}]$ -diagonal matrix with β_n as its $(n+1)\times(n+1)$ entry.
5. $\mathcal{SE}$ -sub-ring of lower-triangular matrices $[a_{n,j}]$ with $a_{n,j}=0$ whenever $n-j$ is an odd number.
6. $\mathcal{L}$ -the group of lower-triangular matrices.
7. $\mathcal{R}$ -the sub-group of Riordan matrices.
8. $\mathcal{IP}(p,\beta,\alpha)$, $\mathcal{P}(p,\alpha)$, $\mathcal{L}(p,a)$, $\mathcal{A}$, $\mathcal{C}$ -various Riordan subgroups defined in the List of subgroups ...3 ,

9. $B_n(x)$, $E_n(x)$ -respectively Bernoulli and Euler polynomials defined by (6a) and (6b).
10. $L_n(x)$, $He_n(x)$ -Laguerre and Hermite (more precisely the so-called probabilistic Hermite polynomials)
11. $(x)_{(n)}$, $(x)^{(n)}$ respectively falling and rising factorials.
12. $(x|q)_n$ the so-called $q-$Pochhammer symbol (defined by (24), often also denoted by $(x)_n$ when q is well defined.
13. GF-generating functions.
14. RGF -reference generating function.
15. FPS -formal power series.
16. GBT(α) -generalized binomial transformation defined in Remark 3.8.

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999. MR1688958 (2000g:33001).

[2] J. F. Barbero, J. Salas, and E. J. S. Villaseñor. Bivariate generating functions for a class of linear recurrences: general structure. *Journal of Combinatorial Theory. Series A*, 125:146–165, 2014. https://doi.org/10.1016/j.jcta.2014.02.007. MR3207469.

[3] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas*. Princeton University Press, second edition, 2009.

[4] G.-S. Cheon, M. M. Cohen, and N. Pantelidis. Decompositions and eigenvectors of riordan matrices. *Linear Algebra and its Applications*, 642:118–138, 2022. https://doi.org/10.1016/j.laa.2022.02.023. MR4387299.

[5] A. M. Encinas and M. J. Jiménez. Triangular sequences, combinatorial recurrences and linear difference equations. *Linear Algebra and its Applications*, 576:301–323, 2019. https://doi.org/10.1016/j.laa.2018.10.015. MR3958150.

[6] S. Goodenough and C. Lavault. Overview of the heisenberg-weyl algebra and subsets of riordan subgroups. *Electronic Journal of Combinatorics*, 22(4):Paper 4.16, 2015. https://doi.org/10.37236/5264. 39 pp.; MR3425607.

[7] C. Jean-Louis and A. Nkwanta. Some algebraic structure of the riordan group. *Linear Algebra and its Applications*, 438(5):2018–2035, 2013. https://doi.org/10.1016/j.laa.2012.10.027. MR3005272.

[8] M. J. Jiménez and A. M. Encinas. Combinatorial recurrences and linear difference equations. *Electronic Notes in Discrete Mathematics*, 54:313–318, 2016. https://doi.org/10.1016/j.endm.2016.09.054. ENDM.

[9] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder; xx+578 pp.; MR2656096 (2011e:33029).

[10] G. Pólya and G. L. Alexanderson. Gaussian binomial coefficients. *Elemente der Mathematik*, 26:102–109, 1971. MR0299490.

[11] S. Roman. *The Umbral Calculus*, volume 111 of *Pure and Applied Mathematics*. Academic Press, Inc., New York, 1984. [Harcourt Brace Jovanovich, Publishers]; 193 pp.; MR0741185.

[12] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson. The riordan group. *Discrete Applied Mathematics*, 34(1-3):229–239, 1991. https://doi.org/10.1016/0166-218X(91)90088-E. Combinatorics and theoretical computer science (Washington, DC, 1989); MR1137996.

[13] P. J. Szabłowski. A few remarks on euler and bernoulli polynomials and their connections with binomial coefficients and modified pascal matrices. *Mathematica Aeterna*, 4(1-2):83–88, 2014. MR3176129.

[14] P. J. Szabłowski. On the families of polynomials forming a part of the askey-wilson scheme and their probabilistic applications. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 25(1):2230001, 2022. https://doi.org/10.1142/S0219025722300018. Paper No. 2230001, 57 pp.; MR4408180.

[15] P. J. Szabłowski. On the families of polynomials forming a part of the askey-wilson scheme and their probabilistic applications. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 25(1):2230001, 2022. https://doi.org/10.1142/S0219025722300018. Paper No. 2230001, 57 pp.

[16] P. J. Szabłowski. A few finite and infinite identities involving pochhammer and q-pochhammer symbols obtained via analytical methods. submitted, 2024. arXiv:2411.00647.

Paweł J. Szabłowski
Department of Mathematics and Information Sciences,
Warsaw University of Technology ul Koszykowa 75, 00-662 Warsaw, Poland
E-mail pawel.szablowski@gmail.com