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Double domination number of inflated graphs

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ABSTRACT

In a graph, a vertex is said to dominate itself and all its neighbors. A subset $S \subseteq V(G)$ is a double dominating set of a graph G, if every vertex of V is dominated by at least two vertices of S. The double domination number denoted by $\gamma_{2 \times (G)}$ is the minimum cardinality of a double dominating set. Graph operations are fundamental in graph theory and have various applications across different fields including network analysis, parallel computing and electrical circuit design. This paper studies the problem of finding the double domination number under unary and binary operations of graphs. We investigate the double domination number of graphs under unary operations such as inflation and cubic inflation. Also, we introduced two new unary operations inspired from inflation operation and studied the impact of these operations on double domination number. Further, we explore the double domination number of edge corona and neighborhood corona of two graphs. Additionally, we study the double domination number of various corona operations of two graphs combined with subdivision of a graph and $R-$graph.

Keywords: double domination number, graph operations, domination in graphs

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1. Introduction

All graphs we considered are finite, simple and undirected. Let $G = (V, E)$ be a simple graph with no isolated vertices. The number of vertices, $|V(G)|$ is called the order of G and the number of edges $|E(G)|$ is called the size of G. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup v$. A vertex of degree one is called a pendant vertex and the vertex adjacent to a pendant vertex is called a support vertex. A vertex that is adjacent

to all other vertices of a graph is called a universal vertex of the graph. For graph theory terminology and notation, we in general follow [1, 25]. A dominating set of a graph G is a subset $D \subseteq V$ such that every vertex of $V - D$ is adjacent to at least one vertex in D. The domination number, $\gamma(G)$ is the minimum cardinality of a dominating set of G. For more details we refer the books, Fundamentals of domination in graphs and Topics in domination in graphs[15, 14].

Depending upon various applications, different variations of domination have appeared in the literature. Among them $k-$tuple domination is well studied variation of domination. Frank Harary and Teresa Haynes [13, 12] introduced the concept of $k-$tuple domination in graphs. The case $k = 2$ corresponds to $2-$tuple domination or double domination. A double dominating set of a graph G is a subset $D \subseteq V$ such that every vertex in V is dominated by at least two vertices in D. The double domination number denoted by $\gamma_{2\times}(G)$ is the minimum cardinality of a double dominating set of G. Equivalently, if $|N[v] \cap D| \geq 2 \forall v \in V(G)$, then D is a double dominating set of G.

A detailed survey on double domination in graphs can be found in [15]. The double domination number of trees have been explored in [3, 5]. The double domination problem has been studied under various binary operations such as corona, tensor, join, lexico-graphic, shadow and rooted products of two graphs in [7, 6, 4, 24]. It has also been studied under various unary operations in [23].

1.1. Related Work

The inflation of a graph is an unary operation which increases the order and the size of the graph but preserves the degree sequence of the graph. The cubic inflation of a graph is an unary operation that converts any connected graph with minimum degree at least two into a cubic graph. The inflation operation of a graph was introduced by Favaron in [9]. Inflation have been studied under various domination parameters in [16, 19, 18]. The cubic inflation operation was introduced in [2].

The corona of two graphs was first introduced by Frucht and Harary [10]. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices, respectively. The corona of G_1 and G_2, denoted by $G_1 \circ G_2$, is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. Hou and Shiu [17] and Gopalapillai [11] introduced the two variants of the corona operation, namely the edge corona and the neighborhood corona respectively. The subdivision operation denoted by $S(G)$ of a graph G is the graph obtained by splitting each edge of G by introducing a new vertex [1]. In [23] we obtained the double domination number of subdivision of some classes of graphs. The two new operations on subdivision graphs, namely, subdivision-vertex corona and subdivision-edge corona were introduced by Lu and Miao [22]. Liu and Lu [21] introduced two new graph operations based on subdivision graphs, namely, the subdivision vertex neighborhood corona and the subdivision edge neighborhood corona. The $R-$graph of G, denoted by $R(G)$ is the graph obtained from G by adding a vertex v to every edge e of G and joining v to the end vertices of e [8]. Observe that R(G) is just the edge corona of G and K_1. In [23] we proved that the double domination number of $R-$graph of any graph G is the order of G. Lan and nZhou[20]

defined four new graph operations based on $R-$graphs, namely the $R-$vertex corona, $R-$edge corona, $R-$vertex neighborhood corona and $R-$edge neighborhood corona.

Although the double domination number of various unary operations and vertex corona has been explored, the double domination number of inflation, cubic inflation, edge corona, neighborhood corona and various corona operations combined with subdivision and $R-$graph has not been examined so far. This inspired us to study the double domination problem under inflation and various corona operations.

The rest of the paper is organized as follows. In Section 2, we obtain the double domination number of inflation, cubic inflation and two new unary operations of graphs. In Section 3, we determine the double domination number of edge corona, neighborhood corona and various corona operations of two graphs combined with subdivision of a graph and $R-$graph respectively.

1.2.   Preliminary results

Observation 1.1. [24] *Every double dominating set of a graph must contain all its leaves and support vertices.*

Theorem 1.2. [24] *For a connected graph G, $\gamma_{2\times}(G) = 2$ if and only if G has at least two universal vertices.*

Theorem 1.3. [24] *For a connected graph G, $\gamma_{2\times}(G) = |V(G)|$ if and only if every vertex of the G is either a leaf or a support vertex.*

Theorem 1.4. [24] *If G is a graph without isolated vertices then $\gamma_{2\times}(G) \geq \frac{2n}{\Delta(G)+1}$.*

Theorem 1.5. [24] *Let G be a cycle graph of order n. Then $\gamma_{2\times}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.*

Theorem 1.6. [24] *Let G be a path graph of order n. Then $\gamma_{2\times}(P_n) = \left\lceil \frac{2(n+1)}{3} \right\rceil$.*

2.   Double domination under unary operations

2.1.   Double domination in inflated graphs

Definition 2.1. The *inflation* of G or inflated graph G_I is obtained as follows: every vertex $x \in V(G)$ of degree $d(x)$ is replaced by a clique $X = K_{d(x)}$ and each edge xy is an edge between two vertices x and y of the corresponding cliques X and Y of G_I in such a way that the edges of G_I which come from the edges of G form a matching.

As an obvious consequence of the definition of inflation, $n(G_I) = \sum\limits_{x_i \in V(G)} d(x_i) = 2m(G)$ and $\Delta(G_I) = \Delta(G)$ and $\delta(G_I) = \delta(G)$. For the notations of inflated graphs we follow [9]. A graph G and its inflated graph G_I are shown in Figure 1.
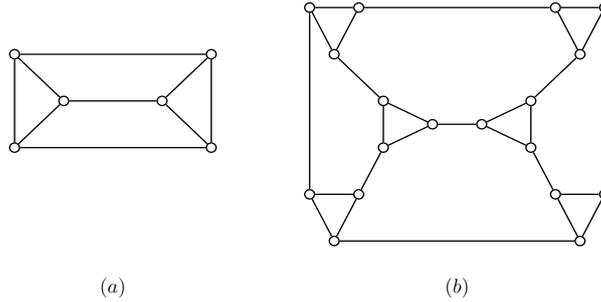
Fig. 1. (a) G and (b) G_I

Theorem 2.2. *Let G be a graph of order n with no isolated vertices. Then $n \leq \gamma_{2\times}(G_I) \leq 2n$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Let $K_{d(v_i)}; v_i \in V(G)$ be the cliques of G_I. Let S be the minimum double dominating set of G_I. Then S must contain at least one vertex from each of the cliques in G_I, because a vertex v in G_I has only one neighbor outside $K_{d(v)}$. Since, there are n cliques in G_I, to double dominate every vertex of G_I we need at least n vertices. This implies, $|S| \geq n$.

Every clique $K_{d(v_i)}$ has at least two universal vertices and thus by Theorem 1.2, we have, $\gamma_{2\times}(K_{d_{v_i}}) = 2$ for each $i; 1 \leq i \leq n$. Hence, $\gamma_{2\times}(G_I) \leq \sum_{i=1}^{n} \gamma_{2\times}(K_{d_{v_i}}) = 2n$.   □

Theorem 2.3. *Let G be a graph without isolated vertex. Then $\gamma_{2\times}(G_I) \geq 2l(G)$, where $l(G)$ is the number of leaves in G with equality if and only if every vertex of G is either a leaf or a support vertex.*

Proof. Let S be the minimum double dominating set of G_I. Let x_i be a leaf adjacent to y_i. Since every vertex $v \in V(G)$ is replaced by a clique of size $deg(v)$, a leaf in G remains as a leaf in G_I. Also each leaf in G_I is adjacent to a unique vertex in G_I. Therefore, to double dominate every leaf of G_I, we need every leaf and its support vertex to be in the minimum double dominating set S of G_I. Hence, $|S| \geq 2l(G)$.

Suppose $|S| = 2l(G)$ be a minimum double dominating set of G_I. Let u be the vertex which is neither a leaf nor a support vertex in G. Then by definition of G_I u cannot be adjacent to any of the support vertex in G_I. Thus u is not dominated even once, contradicting the fact that S is the minimum double dominating set of G_I.

Conversely, suppose every vertex in the graph G is either a leaf or a support vertex. Then by observation 1, and by the fact that every leaf of G_I is adjacent to a unique support vertex, S contains every leaf and support vertex of G_I to double dominate them.   □

Theorem 2.4. *Let $G \cong C_n$ be the cycle graph of order n. Then $\gamma_{2\times}(G_I) = \left\lceil \frac{4n}{3} \right\rceil$.*

Proof. The inflation of cycle of order n is again a cycle of order $2n$. Therefore by Theorem

1.5, $\gamma_{2\times}(G_I) = \left\lceil \frac{4n}{3} \right\rceil$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 2.5. *Let $G \cong P_n$ be the path graph of order n. Then $\gamma_{2\times}(G_I) = \left\lceil \frac{4(n-2)}{3} \right\rceil$.*

Proof. The inflation of path of order n is again a path of order $2(n-1)$. Therefore by Theorem 1.6, $\gamma_{2\times}(G_I) = \left\lceil \frac{4(n-2)}{3} \right\rceil$. $\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 2.6. *Let $G \cong W_n$ be the wheel graph of order n. Then $\gamma_{2\times}(G_I) = \left\lceil \frac{2(n-1)}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Let v_n be the universal vertex of G. Then each $v_i; 1 \leq i \leq n-1$ are replaced by a clique of size 3 and v_n is replaced by a clique of size $deg(v_n)$ in G_I. Let $\{v_{i,1}, v_{i,2}, v_{i,3}; 1 \leq i \leq n-1\}$ be the vertices of the cliques, $K_{v_i}; 1 \leq i \leq n-1$ corresponding to the vertices $v_i; 1 \leq i \leq n-1$. Let $\{v_{i,3}; 1 \leq i \leq n-1\}$ be the vertices adjacent to the vertices in the clique, K_{v_n} corresponding to v_n in G_I. Then the vertices $\{v_{i,1}, v_{i,2}; 1 \leq i \leq n-1\}$ form a cycle of length $2(n-1)$. Let S be the double dominating set of G_I. We pick the minimum double dominating set $\{v_{i,j}; 1 \leq i \leq n-1; j = 1, 2; i+j \not\equiv 0 \mod 3\}$ of the cycle to be in S. Then the vertices in this cycle are dominated at least twice. Observe that, S contains two vertices from cliques $K_{v_i}; 1 \leq i \leq n-1; i \equiv 1 \mod 3$ and exactly one vertex from the cliques $K_{v_i}; 1 \leq i \leq n-1; i \not\equiv 1 \mod 3$. Hence, S dominates vertices in $K_{v_i}; 1 \leq i \leq n-1; i \equiv 1 \mod 3$ at least twice. Now there are $\left\lceil \frac{n-1}{2} \right\rceil$ vertices, $\{v_{i,3}\}$ of the cliques $K_{v_i}; 1 \leq i \leq n-1; i \not\equiv 1 \mod 3$ that are adjacent to the vertices in the clique K_{v_n} that is dominated only once by S. Hence, we include the vertices $\{v_{n,i}; 1 \leq i \leq n-1; i \not\equiv 1 \mod 3\}$ from K_{v_n} that are the neighbors of the vertices $\{v_{i,3}\}$ of the cliques $K_{v_i}; 1 \leq i \leq n-1; i \not\equiv 1 \mod 3$ in S. This set $S = \{v_{i,j}; 1 \leq i \leq n-1; j \not\equiv 0 \mod 3\} \cup \{v_{n,i}; 1 \leq i \leq n-1; i \not\equiv 1 \mod 3\}$ dominates every vertex of G_I at least twice and $|S| = \left\lceil \frac{4(n-1)}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil$. Now, we need to prove S is the minimum double dominating set of G_I. Since the double domination number is the minimum among cardinality of all the double dominating sets of G_I we have $\gamma_{2\times}(G_I) \leq |S| = \left\lceil \frac{4(n-1)}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil$.

To dominate every vertex of G_I at least twice we need to include two vertices from each clique that is $2n$ vertices which is greater than $|S|$. Otherwise we need to include the minimum dominating set $\{v_{i,j}; 1 \leq i \leq n-1; j = 1, 2; j \equiv i \mod 3\}$ of the cycle of length $2(n-1)$. Then Let $S' = S - \{u\}$ for any u in S. This set S' dominates every vertex of G_I at least twice except the neighbors of u. This implies, $\gamma_{2\times}(G_I) \geq \left\lceil \frac{2(n-1)}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil = |S|$. This proves S is the minimum double dominating set of G_I and $\gamma_{2\times}(G_I) = \left\lceil \frac{2(n-1)}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil$. $\square$

Theorem 2.7. *Let $G \cong K_n$ be the complete graph of order n. Then $\gamma_{2\times}(G_I) = 2(n-1)$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G and K_{v_i} be the cliques corresponding to the vertex v_i. Let $\{v_{j,k}; 1 \leq j \leq n; 1 \leq i, k \leq d(v_i)\}$ be the vertices of the cliques

$K_{v_i}; 1 \leq i \leq n$. Let $S = \{v_{1,1}, v_{2,1}, v_{j,k}; 3 \leq j \leq n; 1 \leq k \leq d(v_i)\}$ where $v_{j,k} \in \{N(v_{1,1}) \cup N(v_{2,1})\} - \{K_{v_1} \cup K_{v_2}\}$ and $(v_{1,1}, v_{2,1}) \in E(G_I)$. This set S dominates every vertex of G_I at least twice. Therefore, S is the double dominating set of G_I and $|S| = 2(n-1)$. Now we need to prove S is minimum. From the definition of double domination number it is obvious that $\gamma_{2\times}(G_I) \leq |S| = 2(n-1)$. Observe that every vertex in a clique has a unique neighbor from another clique. This implies, no two vertices of a clique K_{v_i} have a common neighbor outside K_{v_i}. Therefore, to dominate all the vertices of a clique at least twice, we need to include two vertices from each clique or one vertex from the clique and its neighbor from another clique. In both the cases, $\gamma_{2\times}(G_I) = 2n \geq |S| = 2(n-1)$. Thus, S is a minimum double dominating set of G_I. Hence $\gamma_{2\times}(G_I) = 2(n-1)$. $\qquad\square$

Theorem 2.8. *Let $G \cong K_{m,n}$ be the complete bipartite graph of order $m + n$. Then $\gamma_{2\times}(G_I) = 2(m + n - 1)$.*

Proof. The proof of this theorem is similar to the proof of the previous theorem. Let $\{v_i, v_j; 1 \leq i \leq m; 1 \leq j \leq n\}$ be the vertices of G and K_{v_i}, K_{v_j} be the cliques corresponding to the vertex v_i and v_j respectively. Let $\{v_{k,l}; 1 \leq k \leq m; 1 \leq l \leq d(v_i)\}$ be the vertices of the cliques K_{v_i} and $\{v_{p,q}; m + 1 \leq p \leq n; 1 \leq q \leq d(v_j)\}$ be the vertices of the cliques K_{v_j}. Let $S = \{v_{1,1}, v_{2,1}, v_{k,l}, v_{p,q}; 3 \leq k \leq m; 1 \leq l \leq 2; m + 1 \leq p \leq n; 1 \leq q \leq d(v_i)\}$ where $v_{p,q} \in \{N(v_{1,1}) \cup N(v_{2,1})\} - \{K_{v_1} \cup K_{v_2}\}$. This set S dominates every vertex of G_I at least twice. Therefore, S is the double dominating set of G_I and $|S| = 2(m + n - 1)$. Now we need to prove S is minimum. From the definition of double domination number it is obvious that $\gamma_{2\times}(G_I) \leq |S| = 2(m + n - 1)$. Observe that every vertex in a clique has a unique neighbor from another clique. This implies, no two vertices of a clique K_{v_i} have a common neighbor outside K_{v_i}. Therefore, to dominate all the vertices of a clique at least twice, we need to include two vertices from each clique or one vertex from the clique and its neighbor from another clique. In both the cases, $\gamma_{2\times}(G_I) = 2(m+n) \geq |S| = 2(m+n-1)$. Thus, S is a minimum double dominating set of G_I. Hence, $\gamma_{2\times}(G_I) = 2(m + n - 1)$. $\quad\square$

Theorem 2.9. *Let $G \cong K_{1,n}$ and $G' \cong S_{r,t}$ be a star graph and a double star graph of order $n + 1$ and $r + t$ respectively. Then $\gamma_{2\times}(G_I) = 2l(G)$ and $\gamma_{2\times}(G'_I) = 2l(G')$ where $l(G)$ is the number of leaves in G.*

**Proof.** The proof follows from the Theorem 2.3. $\qquad\qquad\qquad\qquad\square$

Observe that the inflation operation increases the double domination number as the number of vertices and edges increases. Also the gap between $\gamma_{2\times}(G)$ and $\gamma_{2\times}(G_I)$ can be arbitrarily large. For example, $\gamma_{2\times}(K_n) = 2$ but $\gamma_{2\times}(K_{nI}) = 2(n-1)$.

2.2. Double domination in cubic inflated graphs

Definition 2.10. Let G be a connected graph without isolated vertices. The *cubic inflation* of graph G denoted by $CI(G)$ is obtained as follows: First we replace every vertex

v with a cycle Q_v of length twice the degree of v and then replace every edge uv of G by two edges joining Q_u and Q_v.

A graph G and its cubic inflation $CI(G)$ are shown in Figure 2. From the definition we have $n(CI(G)) = \sum\limits_{v \in V(G)} 2deg(v) = 4m(G)$ [2].
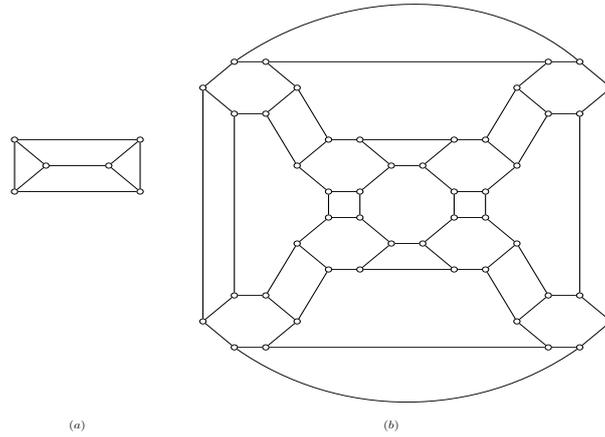

Fig. 2. (a) G and (b) $CI(G)$

Theorem 2.11. *Let G be a connected graph of order n and size m, with $\delta(G) \geq 2$. Then $\gamma_{2\times}(CI(G)) = 2m$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of $CI(G)$. Let C_{v_i} be the cycles of order $2deg(v_i)$ in $CI(G)$ corresponding to the vertices v_i. Let $\{v_{i,j}; 1 \leq i \leq n; 1 \leq j \leq 2deg(v_i)\}$ be the vertices of the cycles C_{v_i}. Let S be the double dominating set of $CI(G)$. We pick the vertices $\{v_{i,j}; 1 \leq i \leq n; j = even\}$ from each cycle C_{v_i} to S in such a way that $(v_{i,j}, v_{k,j}) \in E(CI(G)); 1 \leq i, k \leq n; i \neq k$. That is we pick $deg(v_i)$ vertices from each cycle and there are n cycles. Therefore, $|S| = \sum\limits_{i=1}^{n} deg(v_i) = 2m$. From Theorem 1.4, we have $\gamma_{2\times}(CI(G)) \geq \frac{2n(CI(G))}{\Delta(CI(G))+1} = \frac{8m}{4} = 2m$. This implies, S is the minimum double dominating set of $CI(G)$ and hence $\gamma_{2\times}(CI(G)) = 2m$. $\qquad\square$

2.3. Double domination under α-operation

Definition 2.12. Let G be a connected graph of order n. Let G_α be the graph obtained after applying α - operation on G. The operation is defined as follows: every vertex $v \in V(G)$ is replaced by the independent set, I_v of cardinality $deg(v)$ and every vertex of I_v is adjacent to every vertex of I_u if $uv \in E(G)$. G_α has $\sum\limits_{i=1}^{n} deg(v_i)$ vertices.

A graph G and the graph G_α obtained after applying α - operation on G are shown in Figure 3.

(a) (b)
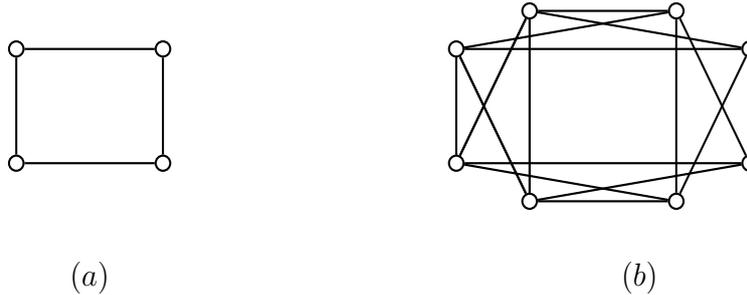
Fig. 3. (a) G and (b) G_α

Theorem 2.13. *Let G be a connected graph of order n and with a universal vertex or two vertices u and v such that $N[u] \cup N[v] = V(G)$. Then $\gamma_{2\times}(G_\alpha) = 4$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Then each vertex v_i is replaced by an independent set I_{v_i} of size $deg(v_i)$ in G_α. Let v_n be the universal vertex of G and I_{v_n} be the independent set corresponding to the vertex v_n. Since v_n is a universal vertex, every vertex in I_{v_n} is connected to every vertex in $I_{v_i}; 1 \leq i \leq n - 1$. Therefore, any two vertices from the independent set I_{v_n} will dominate every vertex of G_α at least twice except the vertices in I_{v_n}. Since vertices in I_{v_n} share no edges between them, in order to double dominate the vertices in I_{v_n} choose any two vertices from any of the independent sets $I_{v_i}; 1 \leq i \leq n - 1$. This implies, $\gamma_{2\times}(G_\alpha) = 4$.

Similarly, let u and v be two vertices of G such that $N[u] \cup N[v] = V(G)$ and I_u and I_v be the independent sets corresponding to the vertices u and v respectively. In this case, any two vertices from each of the independent sets I_u and I_v will dominate every vertex of G_α at least twice. Hence, $\gamma_{2\times}(G_\alpha) = 4$. $\square$

Theorem 2.14. *Let $G \cong C_n$ be the cycle graph of order n. Then $\gamma_{2\times}(G_\alpha) = n$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Each v_i is replaced by an independent set I_{v_i} of size 2 in G_α. Let $\{v_{i,1}, v_{i,2}; 1 \leq i \leq n\}$ be the vertices of G_α. Let the set $S = \{v_{i,1}; 1 \leq i \leq n\}$. This set S dominates every vertex of G_α at least twice because these vertices form a cycle of length n and the vertices $\{v_{i,2}; 1 \leq i \leq n\}$ are adjacent to exactly two vertices of S. Therefore S is a double dominating set of G_α. Now we need to prove S is minimum. From the definition of double domination number it is obvious that $\gamma_{2\times}(G_\alpha) \leq |S| = n$. In G_α, each vertex is replaced by an independent set of size 2 and the resulting graph is a 4-regular graph. The vertices in each independent set have four neighbors in common. But these four neighbors share no edges among them. Therefore, to dominate every vertex at least twice we need to include the vertex and one of its neighbors or two of the neighbors of each vertex and one more vertex common to these two neighbors. In both the cases, we need at least n vertices to dominate every vertex of G_α at least twice. This implies, S is the minimum double dominating set of G_α and hence $\gamma_{2\times}(G_\alpha) = n$. $\square$

Theorem 2.15. *Let $G \cong P_n$ be the path graph of order n. Then ,*

$$\gamma_{2\times}(G_\alpha) = \begin{cases} n-1, & n \equiv 1 \mod 3, \\ n, & otherwise. \end{cases}$$

Proof. Let $\{v_i; 1 \le i \le n\}$ be the vertices of G. Each $v_i; 2 \le i \le n-1$ is replaced by an independent set I_{v_i} of size 2 and the vertices v_1, v_n are replaced by the independent sets I_{v_1}, I_{v_n} of size 1 in G_α. Let $\{v_{1,1}, v_{n,1}, v_{i,1}, v_{i,2}; 2 \le i \le n-1\}$ be the vertices of G_α.

Case 1. When $n \not\equiv 1 \mod 3$.

Let us consider the set $S = \{v_{2,1}, v_{2,2}, v_{n-1,1}, v_{n-1,2}, v_{i,1}, v_{i+1,2}; 2 \le i, \le n-3; i \equiv 2 \mod 3\}$. This set S dominates every vertex of G_α at least twice. Therefore, S is a double dominating set of G_α. Now we need to prove S is minimum. From the definition of double domination number it is obvious that $\gamma_{2\times}(G_\alpha) \le |S| = n$. In G_α, each vertex is replaced by an independent set of size 2 except for the vertices v_1 and v_n. The vertices in each independent set $I_{v_i}; 3 \le i \le n-2$ have four neighbors in common. But these four neighbors share no edges among them. Therefore, to dominate every vertex at least twice we need to include the vertex and one of its neighbors or two of the neighbors of each vertex and one more vertex common to these two neighbors. In both the cases, we need at least n vertices to dominate every vertex of G_α at least twice. This implies, S is the minimum double dominating set of G_α. Hence, $\gamma_{2\times}(G_\alpha) = |S| = n$.

Case 2. When $n \equiv 1 \mod 3$.

In this case consider the set $S = \{v_{2,1}, v_{2,2}, v_{n-1,1}, v_{n-1,2}, v_{i,1}, v_{i+1,2}; 2 \le i, j \le n-3; i \equiv 2 \mod 3\} - \{v_{n-3,2}\}$. This set S dominates every vertex of G_α at least twice. Therefore, S is a double dominating set of G_α. Now we need to prove S is minimum. The same set as in the previous case without the vertex $v_{n-3,2}$ acts as the minimum double dominating set for this case. This implies, S is the minimum double dominating set of G_α with $|S| = n-1$. Hence, $\gamma_{2\times}(G_\alpha) = |S| = n-1$. $\square$

2.4. Double domination under β - operation

Definition 2.16. Let G be a connected graph of order n. Let G_β be the graph obtained after applying β - operation on G. The operation is defined as follows: every vertex $v \in V(G)$ is replaced by the path, P_v of order $deg(v)$ and every vertex of P_v is adjacent to every vertex of P_u if $uv \in E(G)$. G_β has $\sum_{i=1}^{n} deg(v_i)$ vertices.

In Figure 4, we present a graph G and the graph G_β obtained after applying β - operation on G.

Theorem 2.17. *Let G be a connected graph of order n and with a universal vertex or two vertices u and v such that $N[u] \cup N[v] = V(G)$. Then for $n \ge 6$, $\gamma_{2\times}(G_\beta) = 4$.*

Proof. The proof is similar to the proof of Theorem 2.13. Let $\{v_i; 1 \le i \le n\}$ be the vertices of G. Then each vertex v_i is replaced by a path P_{v_i} of order $deg(v_i)$ in G_β. Let v_n

be the universal vertex of G and P_{v_n} be the path corresponding to the vertex v_n. Since v_n is a universal vertex, every vertex in P_{v_n} is connected to every vertex in $P_{v_i}; 1 \leq i \leq n-1$. Therefore, any two vertices from the path P_{v_n} will dominate every vertex of G_β at least twice except the vertices in P_{v_n}. Since vertices in P_{v_n} is a path of order $deg(v_i)$, any two vertices of P_{v_n} is not sufficient to double dominate the vertices in P_{v_n}. Therefore, choose any two vertices from any of the paths $P_{v_i}; 1 \leq i \leq n-1$. This implies, $\gamma_{2\times}(G_\beta) = 4$.

Similarly, let u and v be two vertices of G such that $N[u] \cup N[v] = V(G)$ and P_u and P_v be the paths corresponding to the vertices u and v respectively. In this case, any two vertices from each of the paths P_u and P_v will dominate every vertex of G_β at least twice. Hence, $\gamma_{2\times}(G_\beta) = 4$. $\qquad\square$

(a) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (b)
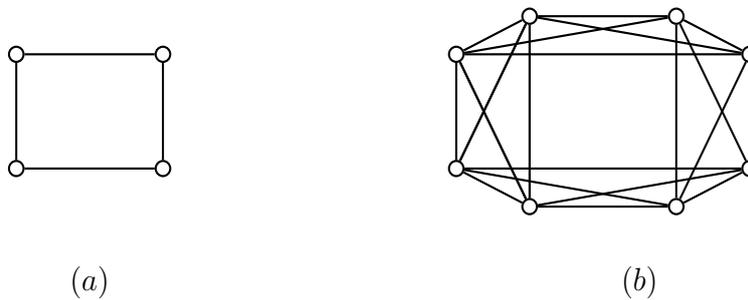
Fig. 4. (a) G and (b) G_β

Corollary 2.18. *Let G be a connected graph of order n and with a universal vertex or two vertices u and v such that $N[u] \cup N[v] = V(G)$. Then for $n < 6$, $\gamma_{2\times}(G_\beta) = 3$.*

Proof. Since $n < 6$, the degree of the universal vertex u will be less than or equal to 4. Therefore, the order of the path P_u replaced instead of u will be less than or equal to 4. Therefore, any two vertices from P_u will dominate every vertex of G_β at least twice but the vertices in the path P_u corresponding to u is dominated only once by those two vertices. To dominate these vertices at least twice its enough to pick any vertex of G_β other than the vertices in P_u. Therefore, for $n < 6$, $\gamma_{2\times}(G_\beta) = 4$.

Similarly, let u and v be two vertices such that $N[u] \cup N[v] = V(G)$. Then for $n < 6$, $\gamma_{2\times}(G_\beta) = 3$. $\qquad\square$

Theorem 2.19. *Let $G \cong C_n$ be the cycle graph of order n. Then $\gamma_{2\times}(G_\beta) = \gamma_{2\times}(G)$.*

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Each v_i is replaced by a path P_{v_i} of order 2 in G_β. Let $\{v_{i,1}, v_{i,2}; 1 \leq i \leq n\}$ be the vertices of G_β. Note that the set $S_G = \{v_i; 1 \leq i \leq n; i \not\equiv 0 \mod 3\}$ is the minimum double dominating set of G and $\gamma_{2\times}(G) = \left\lceil \frac{2n}{3} \right\rceil$. Now consider the set $S_{G_\beta} = \{v_{i,1}; 1 \leq i \leq n; i \not\equiv 0 \mod 3\}$. This set dominates every vertex of G_β at least twice because, these vertices form a cycle of length n and the vertices $\{v_{i,2}; 1 \leq i \leq n\}$ are adjacent to exactly two vertices of S_{G_β} and therefore S_{G_β} is a double dominating set of G_β and $|S| = \left\lceil \frac{2n}{3} \right\rceil = \gamma_{2\times(G)}$. Now we need to prove S_{G_β}

is minimum. From Theorem 1.4, we have $\gamma_{2\times}(G_\beta) \geq \frac{2(n(G_\beta))}{\Delta(G_\beta)+1} = \frac{4n}{6} = \frac{2n}{3}$. This implies, S_{G_β} is the minimum double dominating set of G_β and hence $\gamma_{2\times}(G_\beta) = \gamma_{2\times}(G)$. $\square$

Theorem 2.20. *Let $G \cong P_n$ be the path graph of order n. Then,*

$$\gamma_{2\times}(G_\beta) = \begin{cases} \left\lceil \frac{2(n+1)}{3} \right\rceil - 1, & n \equiv 0 \mod 3, \\ \left\lceil \frac{2(n+1)}{3} \right\rceil, & otherwise. \end{cases}$$

Proof. Let $\{v_i; 1 \leq i \leq n\}$ be the vertices of G. Each $v_i; 2 \leq i \leq n-1$ is replaced by a path P_{v_i} of order 2 and the vertices v_1, v_n are replaced by the path P_{v_1}, P_{v_n} of size 1 in G_β. Let $\{v_{1,1}, v_{n,1}, v_{i,1}, v_{i,2}; 2 \leq i \leq n-1\}$ be the vertices of G_β.

Case 1. When $n \not\equiv 0 \mod 3$.

Let us consider the set $S = \{v_{2,1}, v_{2,2}, v_{n-1,1}, v_{n-1,2}, v_{i,1}, v_{i,2}; 3 \leq i \leq n-3; i \equiv 2 \mod 3\}$. This set S dominates every vertex of G_β at least twice. Therefore, S is a double dominating set of G_β. Now to prove S is minimum. From the definition of double domination number it is obvious that $\gamma_{2\times}(G_\beta) \leq |S| = \left\lceil \frac{2(n+1)}{3} \right\rceil$. In G_β, each vertex is replaced by a path of order 2 except for the vertices v_1 and v_n. The vertices in each path $P_{v_i}; 3 \leq i \leq n-2$ have five neighbors in common. But these five neighbors share no edges among them. Therefore, to dominate every vertex at least twice we need to include the vertex and one of its neighbors or two of the neighbors of each vertex and one more vertex common to these two neighbors. In both the cases, we need at least n vertices to dominate every vertex of G_β at least twice. This implies, S is the minimum double dominating set of G_β. Hence, $\gamma_{2\times}(G_\beta) = |S| = \left\lceil \frac{2(n+1)}{3} \right\rceil$.

Case 2. When $n \equiv 0 \mod 3$.

In this case, the same set $S = \{v_{2,1}, v_{2,2}, v_{n-1,1}, v_{n-1,2}, v_{i,1}, v_{i,2}; 3 \leq i, j \leq n-3; i \equiv 2 \mod 3\}$ dominates every vertex of G_β at least twice but $|S| = \left\lceil \frac{2(n+1)}{3} \right\rceil - 1$. Therefore, S is a double dominating set of G_β which is minimum. Hence, $\gamma_{2\times}(G_\beta) = |S| = \left\lceil \frac{2(n+1)}{3} \right\rceil - 1$. $\square$

3. Double domination under binary operations

3.1. Edge corona of two graphs

Definition 3.1. [17] Let G_1 and G_2 be two graphs on disjoint sets n_1 and n_2 vertices, m_1 and m_2 edges respectively. The edge corona $G_1 \diamond G_2$ is defined as the graph obtained by taking one copy of G_1 and m_1 copies of G_2 and joining two end vertices of ith edge of G_1 to every vertex in the ith copy of G_2. $G_1 \diamond G_2$ has $n_1 + m_1 n_2$ vertices and $m_1 + 2m_1 n_2 + m_1 m_2$ edges. For example consider the edge corona graph of a cycle of order 3 and a path of order 2, $C_3 \diamond P_2$, see Figure 5.

Theorem 3.2. *Let G_1 and G_2 be two graphs of order n_1 and n_2 and size m_1 and m_2 respectively. Then $\gamma_{2\times}(G_1 \diamond G_2) = n_1$.*
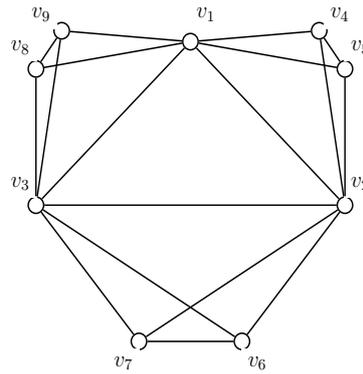
Fig. 5. $C_3 \diamond P_2$

Proof. Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then $G_1 \diamond G_2$ has $n_1 + m_1 n_2$ vertices. Let $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$. Let $\{v_1, v_2, \cdots, v_{n_1}\}$ be the vertices of G_1 and $\{e_1, e_2, \cdots, e_{m_1}\}$ be the edges of G_1. Consider the set $S = \{v_1, v_2, \cdots, v_{n_1}\}$. This set dominates every vertex of $G_1 \diamond G_2$ at least twice. Therefore, S is a double dominating set of $G_1 \diamond G_2$ and $|S| = n_1$. Now we need to prove S is minimum. Since S is a double dominating set of $G_1 \diamond G_2$ and $|S| = n_1$ and the minimum double dominating set of a graph is the minimum cardinality among all the double dominating sets of a graph, we have, $\gamma_{2\times}(G_1 \diamond G_2) \leq n_1 = |S|$. In $G_1 \diamond G_2$, to double dominate every vertex of each copy of G_2 we either need to include the minimum double dominating set of each copy of G_2 or the minimum dominating set of each copy of G_2 and one of the endpoints of each edge of G_1. In both the cases, $|S| \geq n_1$. Observe that every vertex of each copy of G_2 is adjacent to exactly two vertices of G_1 in $G_1 \diamond G_2$. Therefore, we need all the vertices of G_1 to double dominate every vertex of each copy of G_2. This proves $S = \{v_1, v_2, \cdots, v_{n_1}\}$ is minimum. Hence, $\gamma_{2\times}(G_1 \diamond G_2) = |S| = n_1$. $\square$

3.2.   Neighborhood corona of two graphs

Definition 3.3. [11] Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then the neighborhood corona $G_1 \star G_2$ is the graph obtained by taking n_1 copies of G_2 and for each i, making all vertices in the ith copy of G_2 adjacent with neighbors of v_i, $i = 1, 2, ..., n_1$.

 $G_1 \star G_2$ has $n_1 + n_1 n_2$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges. For example, consider the neighborhood corona of cycle of order 3 and path of order 3, $C_3 \star P_3$, see Figure 6.

Theorem 3.4. *Let G_1 be a graph of order n_1 whose vertices are either a pendant or a support vertex and let G_2 be a graph of order n_2. Then $\gamma_{2\times}(G_1 \star G_2) = n_1 + l.\gamma(G_2)$ where l is the number of pendant vertices in G_1.*

Proof. Let G_1 be a graph whose vertices are either pendant or a support vertex. Observe that the pendant vertices have only one neighbor and so every vertex in the copies of G_2 with respect to the pendant vertices of G_1 are adjacent to exactly one vertex of G_1 in $G_1 \star G_2$. But the vertices in the copies of G_2 with respect to the support vertices of G_1 are

adjacent to at least two vertices of G_1. Therefore, the set $S = \{v_i; 1 \le i \le n_1\}$ dominates every vertex of $G_1 \star G_2$ at least twice but the vertices in the copies of G_2 with respect to the pendant vertices are dominated only once by S. Therefore, we include the minimum dominating set of each copy of G_2 with respect to the pendant vertices to S. This S dominates every vertex of $G_1 \star G_2$ at least twice. Therefore, S is the double dominating set of $G_1 \star G_2$ which is minimum. Hence, $\gamma_{2\times}(G_1 \star G_2) = n_1 + l.\gamma(G_2)$. $\qquad\square$
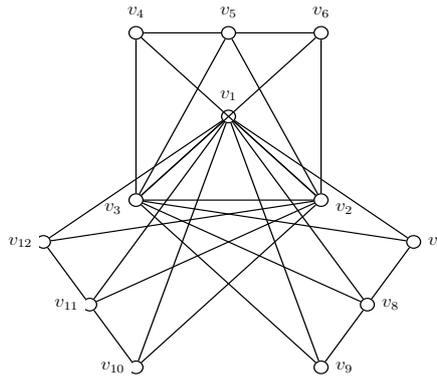

Fig. 6. $C_3 \star P_3$

Theorem 3.5. *Let G_1 be a graph with at least two universal vertices and of order n_1 and G_2 be a graph of order n_2. Then $\gamma_{2\times}(G_1 \star G_2) = 3$.*

Proof. Let $\{v_i; 1 \le i \le n_1\}$ be the vertices of G_1 and let v_i, v_j for any $i, j; 1 \le i, j \le n_1; i \ne j$ be the two universal vertices. Observe that every vertex of each copy of G_2 is adjacent to two universal vertices of G_1 in $G_1 \star G_2$. But the vertices in the copy of G_2 with respect to the universal vertices are adjacent to only one vertex of G_1. Note that every vertex of $G_1 \star G_2$ are dominated at least twice by the universal vertices itself but the vertices in the copy of G_2 with respect to the two universal vertices are dominated only once by the universal vertices. Consider the set $S = \{v_i, v_j, v_k; 1 \le i, j, k \le n_1; i \ne j \ne k\}$ where v_i and v_j are two universal vertices and v_k is any vertex of G_1 which is included in S to double dominate the vertices in the copy of G_2 with respect to the universal vertices. This set dominates every vertex of $G_1 \star G_2$ at least twice. Therefore, S is a double dominating set of $G_1 \star G_2$ which is minimum. Hence, $\gamma_{2\times}(G_1 \star G_2) = |S| = 3$. $\quad\square$

Theorem 3.6. *Let G_1 be a cycle of order n_1 and G_2 be any graph of order n_2. Then $\gamma_{2\times}(G_1 \star G_2) = n_1$.*

Proof. Let G_1 be a cycle of order n_1 and G_2 be any graph of order n_2. Let $\{v_i; 1 \le i \le n_1\}$ be the vertices of G_1. Observe that every vertex of ith copy of G_2 is adjacent to exactly two vertices of G_1. Consider the set $S = \{v_1, v_2, \cdots, v_{n_1}\}$. This set dominates every vertex of $G_1 \star G_2$ at least twice. Therefore, S is a double dominating set of $G_1 \star G_2$. It is clear that S is minimum because, to dominate every vertex of each copy of G_2 at least twice

we need to include one of their neighbors from G_1 and minimum dominating set of each copy of G_2 or minimum double dominating set of each copy of G_2. In both the cases $\gamma_{2\times}(G_1 \star G_2) \geq n_1 = |S|$. Hence, $\gamma_{2\times}(G_1 \star G_2) = |S| = n_1$. $\square$

Theorem 3.7. *Let G_1 be a path graph of order n_1 and G_2 be a graph of order n_2. Then $\gamma_{2\times}(G_1 \star G_2) = n_1 + 2\gamma(G_2)$.*

Proof. The proof is similar to the proof of previous theorem. Let G_1 be a path graph of order n_1 and G_2 be any graph of order n_2. Let $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1 where v_1 and v_n are the pendant vertices of G_1. Observe that every vertex of ith copy of G_2 is adjacent to exactly two vertices of G_1 except the first and last copy of G_2 which are adjacent to only one vertex of G_1. Consider the set $S = \{v_1, v_2, \cdots, v_{n_1}\}$. This set dominates every vertex of $G_1 \star G_2$ at least twice but the vertices in first and last copy of G_2 are dominated only once by S. In order to double dominate them we include the minimum double dominating sets of first and last copies of G_2 to S. This set S will dominate every vertex of $G_1 \star G_2$ at least twice. Thus S is a double dominating set of $G_1 \star G_2$. This set S is minimum because to dominate every vertex of each copy of G_2 at least twice we need to include one of their neighbors from G_1 and minimum dominating set of each copy of G_2 or minimum double dominating set of each copy of G_2. In both the cases, $\gamma_{2\times}(G_1 \star G_2) \geq n_1 + 2\gamma(G_2) = |S|$.

Hence, $\gamma_{2\times}(G_1 \star G_2) = |S| = n_1 + 2\gamma(G_2)$. $\square$

Theorem 3.8. *Let G_1 be a complete bipartite graph of order $m + n$; $m, n \geq 2$ and G_2 be any graph of order n'. Then $\gamma_{2\times}(G_1 \star G_2) = 4$.*

Proof. Let G_1 be a complete bipartite graph of order $m + n$ and G_2 be any graph of order n'. Then $G_1 \star G_2$ has $(m + n) + (m + n)n'$ vertices. Observe that every vertex of each copy of G_2 is adjacent to at least m vertices of G_1. Let $\{v_1, v_2, \cdots, v_m, u_1, u_2, \cdots, u_n\}$ be the vertices of G_1. Then $G_{2,1}, G_{2,2}, \cdots, G_{2,m}, G_{2,1}, \cdots, G_{2,n}$ are the $m + n$ copies of G_2 in $G_1 \star G_2$. Observe that every vertex of $G_{2,i}; \forall i, 1 \leq i \leq m$ is adjacent to n vertices of G_1 and every vertex of $G_{2,i}; \forall i, 1 \leq i \leq n$ is adjacent to m vertices of G_1 in $G_1 \star G_2$. Let $S = \{v_i, v_j, u_k, u_l\}$ for any $i, j | i \neq j; 1 \leq i, j \leq m$ and for any $k, l | 1 \leq k, l \leq n; k \neq l$. This set dominates every vertex of $G_1 \star G_2$ at least twice. Therefore, S is a double dominating set of $G_1 \star G_2$ which is minimum.

Hence, $\gamma_{2\times}(G_1 \star G_2) = |S| = 4$. $\square$

Theorem 3.9. *Let G_1 be a wheel graph of order $n_1 + 1$ and G_2 be any graph of order n_2. Then $\gamma_{2\times}(G_1 \star G_2) = \left\lceil \frac{n_1}{2} \right\rceil + 1$.*

Proof. Let G_1 be a wheel graph of order $n_1 + 1$ and G_2 be any graph of order n_2. Then $G_1 \star G_2$ has $(n_1 + 1) + (n_1 + 1)n_2$ vertices. Let u be the universal vertex of G_1. Here, every vertex of each copy of G_2 has at least three neighbors in G_1 in $G_1 \star G_2$. Also,

u is the common neighbor to every vertex of each copy of G_2 except the copy of G_2 corresponding to u in $G_1 \star G_2$. Let $\{u, v_1, v_n, \cdots, v_{n_1}\}$ be the vertices of G_1. Consider the set $S = \{u, v_i, v_{i+1} \forall i = 2k+1; k = 0, 2, ..., \lfloor \frac{n_1}{2} \rfloor \}$. This set dominates every vertex of $G_1 \star G_2$ at least twice. Therefore, S is a double dominating set of $G_1 \star G_2$ and $|S| = 1 + \lceil \frac{n_1}{2} \rceil$. Now we need to prove S is minimum. From the definition of double domination number $\gamma_{2\times}(G_1 \star G_2) \leq \lceil \frac{n_1}{2} \rceil + 1$. The vertex v_n dominates every vertex of $G_1 \star G_2$ exactly once except the copy of G_2 corresponding to v_n. The vertices in copies of G_2 corresponding to the vertices $v_i; 1 \leq i \leq n-1$ is adjacent to exactly two vertices other than v_n. To dominate them at least twice we need to include for each copy of G_2, a vertex $v_i; 1 \leq i \leq n-1$. Also, observe that two copies of G_2 have exactly one common neighbor. This implies, to dominate every vertex of each copy of G_2 at least twice, we need at least $\lceil \frac{n_1}{2} \rceil + 1$ vertices. Thus, $\gamma_{2\times}(G_1 \star G_2) \geq \lceil \frac{n_1}{2} \rceil + 1$. Therefore, S is a minimum double dominating set of $G_1 \star G_2$. Hence, $\gamma_{2\times}(G_1 \star G_2) = |S| = \lceil \frac{n_1}{2} \rceil + 1$. $\qquad \square$

4. Double domination number of variants of corona combined with subdivision of a graph

Let G_1 and G_2 be two vertex disjoint graphs of order n_1 and n_2 and size m_1 and m_2 respectively. Let $I(G)$ be the new vertices added in the $S(G)$ to divide each edge of G. The *subdivision-vertex corona* of G_1 and G_2, denoted by $G_1^{(S)} \odot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex disjoint, and joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The *subdivision-edge corona* of G_1 and G_2, denoted by $G_1^{(S)} \ominus G_2$, is the graph obtained from $S(G_1)$ and m_1 copies of G_2, all vertex disjoint, and joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2, where $I(G_1)$ is the set of new vertices added to $S(G_1)$. The *subdivision-vertex neighborhood corona* of G_1 and G_2, denoted by $G_1^{(S)} \boxdot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex disjoint, and joining the neighbors of the ith vertex of G_1 to every vertex in the ith copy G_2.

The *subdivision-edge neighborhood corona* of G_1 and G_2, denoted by $G_1^{(S)} \boxminus G_2$, is the graph obtained from $S(G_1)$ and m_1 copies of G_2, all vertex disjoint, and joining the neighbors of the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2. Figure 7 show the examples of Subdivision vertex corona, Subdivision- edge corona, subdivision vertex neighbourhood corona and subdivision edge neighbourhood corona of graphs C_4 and P_2.

$C_4^{(S)} \odot P_2$ $C_4^{(S)} \ominus P_2$ $C_4^{(S)} \boxdot P_2$ $C_4^{(S)} \boxminus P_2$
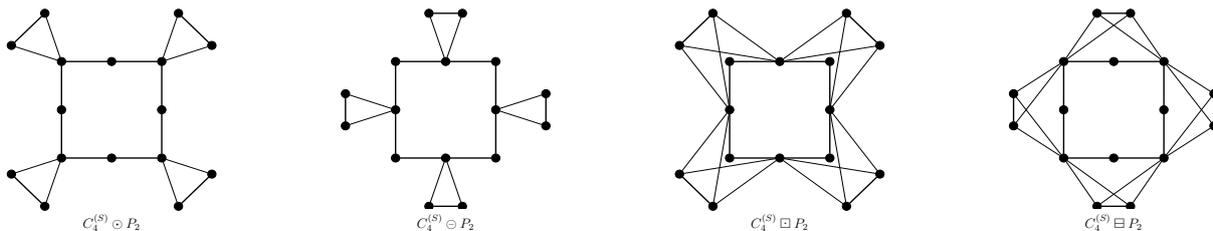
Fig. 7. Subdivision-vertex corona, Subdivision-edge corona, Subdivision-vertex neighbourhood corona and Subdivision-edge neighbourhood corona of graphs C_4 and P_2

Theorem 4.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then $\gamma_{2\times}(G_1^{(S)} \odot G_2) = n_1(1 + \gamma(G_2))$.*

Proof. Let $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1. Let S be the double dominating set of $G_1^{(S)} \odot G_2$. Consider the set $S_1 = \{v_i; 1 \leq i \leq n_1\}$. This set S_1 will dominate every vertex in $I(G_1)$ exactly twice but the vertices of G_1 and the vertices in each copy of G_2 is dominated only once by S_1. Since every i^{th} vertex of G_1 is adjacent to every vertex in the ith copy of G_2, we include the minimum dominating set $\{D_i; 1 \leq i \leq n_1\}$ of each copy of G_2 in S to double dominate every vertex of $(G_1^{(S)} \odot G_2)$. Therefore, the set $S = \{S_1 \cup D_i; 1 \leq i \leq n_1\}$ will dominate every vertex of $G_1^{(S)} \odot G_2$ at least twice and S is a double dominating set and $|S| = n_1(\gamma(G_2) + 1)$. Now we need to prove S is the minimum double dominating set. Observe that every vertex in $I(G_1)$ is adjacent to exactly two vertices of G_1 and every vertex in the each copy of G_2 is adjacent to exactly one vertex of G_1. To dominate them at least twice we need to include either both the neighbors of vertices in $I(G_1)$ and the minimum dominating set of each copy of G_2 or every vertex in $I(G_1)$ and one of their neighbors and minimum double dominating set of each copy of G_2. In both the cases $\gamma_{2\times}(G_1^{(S)} \odot G_2) \geq |S| = n_1(\gamma(G_2) + 1)$. This proves S is the minimum double dominating set of $(G_1^{(S)} \odot G_2)$. Hence, $\gamma_{2\times}(G_1^{(S)} \odot G_2) = n_1(1 + \gamma(G_2))$. $\square$

Theorem 4.2. *Let G_1 and G_2 be two graphs of order n_1 and n_2 and of size m_1 and m_2 respectively. Then*

$$\gamma_{2\times}(G_1^{(S)} \ominus G_2) = \begin{cases} m_1(\gamma(G_2) + 1) + l_1; & \text{if } \delta(G_1) = 1, \\ m_1(\gamma(G_2) + 1); & \text{otherwise}, \end{cases}$$

where l_1 is the number of leaves in G_1

Proof. The proof is similar to the proof of previous theorem.

Case 1. When $\delta(G) \geq 2$, in $(G_1^{(S)} \ominus G_2)$ every vertex in each copy of G_2 is adjacent to exactly one vertex of $I(G_1)$. Let S be the double dominating set of $(G_1^{(S)} \ominus G_2)$. Let $I(G_1) \subseteq S$. Now S dominates every vertex of G_1 at least twice but the vertices in $I(G_1)$ and the vertices of each copy of G_2 is dominated only once by S. Since every i^{th} vertex of $I(G_1)$ is adjacent to every vertex of ith copy of G_2, we include the minimum dominating set $\{D_i; 1 \leq i \leq m_1\}$ of each copy of G_2 in S to double dominate every vertex of $(G_1^{(S)} \ominus G_2)$ at least twice and thus $S = I(G_1) \cup D_i$ for $1 \leq i \leq n$ is a double dominating set of $(G_1^{(S)} \ominus G_2)$ which is minimum. Hence, $\gamma_{2\times}(G_1^{(S)} \ominus G_2) = m_1(\gamma(G_2) + 1)$.

Case 2. When $\delta(G_1) = 1$, a pendant vertex of G_1 remains as a pendant vertex in $(G_1^{(S)} \ominus G_2)$. Therefore, the set $S = I(G_1) \cup D_i$ for $1 \leq i \leq n$ dominates every vertex of $(G_1^{(S)} \ominus G_2)$ at least twice, but the pendant vertices are dominated only once by S. By observation 1, we need to include all the pendant vertices to S. Hence, $\gamma_{2\times}(G_1^{(S)} \ominus G_2) = m_1(\gamma_{2\times(G)} + 1) + l_1$ where, l_1 is the number of pendant vertices in G_1. $\square$

Theorem 4.3. *Let G_1 be a graph of order n_1 whose vertices are either a pendant vertex or*

a support vertex and G_2 be a graph of order n_2. Then $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = m_1 + l(1 + \gamma(G_2))$ where l is the number of pendant vertices in G_1.

Proof. In $G_1^{(S)} \boxdot G_2$ the pendant vertices of G_1 remain as pendant vertices. Let $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1 and $\{u_j; 1 \leq j \leq m_1\}$ be the vertices of $I(G_1)$. Consider the set $S = \{u_j, v_k; 1 \leq j \leq m_1; 1 \leq k \leq l\}$. This set dominates every vertex of $G_1^{(S)} \boxdot G_2$ at least twice. Therefore, S is the double dominating set of $G_1^{(S)} \boxdot G_2$. Now we need to prove S is minimum. From the definition of double domination number, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) \leq |S|$. By Observation 1.1, we need to include all the pendant vertices and support vertices of $G_1^{(S)} \boxdot G_2$ in the double dominating set. This dominates every vertex of $G_1^{(S)} \boxdot G_2$ at least twice but the vertices in copies of G_2 corresponding to the pendant vertices are dominated only once by S. To dominate them at least twice we need to include the minimum dominating set of each copy of G_2 corresponding to the pendant vertices. Hence, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = m_1 + l(1 + \gamma(G_2))$. $\square$

Theorem 4.4. *Let G_1 be a cycle graph of order n_1 and size m_1 and G_2 be a graph of order n_2 and size m_2. Then $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = m_1 + \left\lceil \frac{n_1}{2} \right\rceil$.*

Proof. In $G_1^{(S)} \boxdot G_2$ every vertex in each copy of G_2 is adjacent to exactly two vertices of G_1. Let S be the double dominating set of $G_1^{(S)} \boxdot G_2$. Let $I(G_1) \subseteq S$. Now S dominates every vertex of $G_1^{(S)} \boxdot G_2$ at least twice but the vertices in $I(G_1)$ are dominated only once by S. Let $\{u_j; 1 \leq j \leq m_1\}$ be the vertices in $I(G_1)$ and $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1. Each vertex in $I(G_1)$ has exactly two neighbors in G_1 and two consecutive vertices u_i, u_{i+1} of $I(G_1)$ have exactly one common neighbor. Therefore, to dominate the vertices in $I(G_1)$ at least twice we include the vertices $\{v_i; i \equiv 1 \mod 2; 1 \leq i \leq n_1\}$ to S. Thus the set $S = I(G_1) \cup \{v_i; i \equiv 1 \mod 2; 1 \leq i \leq n_1\}$ is a double dominating set of $G_1^{(S)} \boxdot G_2$ which is minimum. Hence, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = |S| = m_1 + \left\lceil \frac{n_1}{2} \right\rceil$. $\square$

Theorem 4.5. *Let G_1 be a path graph of order n_1 and size m_1 and G_2 be a graph of order n_2 and size m_2. Then $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = m_1 + \left\lceil \frac{n_1}{2} \right\rceil + 2\gamma(G_2)$.*

Proof. The proof is similar to the previous theorem. In $G_1^{(S)} \boxdot G_2$ every vertex in each copy of G_2 is adjacent to exactly two vertices of G_1 but the two copies of G_2 corresponding to the pendant vertices of G_1 is adjacent to only one vertex of G_1. Let v_1 and v_{n_1} be the pendant vertices of G_1 and $G_2^{(v_1)}$ and $G_2^{(v_{n_1})}$ be the copies of G_2 corresponding to the pendant vertices respectively. Let S be the double dominating set of $G_1^{(S)} \boxdot G_2$. Let $I(G_1) \subseteq S$. Now S dominates every vertex of $G_1^{(S)} \boxdot G_2$ at least twice but the vertices in $I(G_1)$ and vertices in $G_2^{(v_1)}$ and $G_2^{(v_{n_1})}$ are dominated only once by S. Let $\{u_i; 1 \leq i \leq m_1\}$ be the vertices in $I(G_1)$ and $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1. To dominate the vertices in $I(G_1)$ and vertices in $G_2^{(v_1)}$ and $G_2^{(v_{n_1})}$ at least twice, we include the vertices $\{v_i; i \equiv 1 \mod 2; 1 \leq i \leq n_1\}$ and the minimum dominating sets D_1 and D_{n_1} of $G_2^{(v_1)}$ and $G_2^{(v_{n_1})}$ to S. Thus the set $S = I(G_1) \cup \{v_i; 1 \leq i \leq \left\lceil \frac{n_1}{2} \right\rceil\} \cup D_1 \cup D_{n_1}$ is a double dominating

set of $G_1^{(S)} \square G_2$ which is minimum. Hence, $\gamma_{2\times}(G_1^{(S)} \square G_2) = |S| = m_1 + \lceil \frac{n_1}{2} \rceil + 2\gamma(G)$. $\square$

Theorem 4.6. *Let G_1 be a wheel graph of order n_1 and size m_1 and G_2 be a graph of order n_2 and size m_2. Then $\gamma_{2\times}(G_1^{(S)} \square G_2) = \frac{m_1}{2} + \lceil \frac{n_1 - 1}{2} \rceil + \lfloor \frac{n_1 - 1}{2} \rfloor + 1$.*

Proof. In $G_1^{(S)} \square G_2$, every vertex in each copy of G_2 is adjacent to exactly three vertices of G_1. Let $\{v_i; 1 \le i \le n_1\}$ be the vertices of G_1 where v_{n_1} is the universal vertex of G_1. Let $\{u_j; 1 \le j \le m_1\}$ be the vertices in $I(G_1)$ where $\{u_j; 1 \le j \le \frac{m_1}{2}\}$ are the vertices corresponding to the edges in the outer cycle of G_1 and $\{u_j; \frac{m_1}{2} + 1 \le j \le m_1\}$ be the vertices corresponding to the edges inside the cycle of G_1. Consider the set $S = \{u_j, v_i; 1 \le i \le \frac{m_1}{2}; i \equiv 0 \mod 2; 1 \le i \le n_1 - 1\} \cup \{u_j; \frac{m_1}{2} + 1 \le j \le m_1; j \equiv 1 \mod 2\} \cup \{v_{n_1}\}$. This set S dominates every vertex of $G_{n_1}^{(S)} \square G_2$ at least twice. Thus S is a double dominating set of $G_{n_1}^{(S)} \square G_2$. Now we need to prove S is the minimum double dominating set. Observe that the vertices in each copy of G_2 are adjacent to the vertices in $I(G_1)$ only. Also, the every vertex of $I(G_1)$ is adjacent to exactly two vertices in G_1. Therefore, to dominate the vertices of $I(G_1)$ and every vertex of each copy of G_2 at least twice we need to include every vertex of $I(G_1)$ and one of the neighbors of each vertex in $I(G_1)$ or both the neighbors of each vertex of $I(G_1)$ and the minimum double dominating set of each copy of G_2. In both the cases $\gamma_{2\times}(G_{n_1}^{(S)} \square G_2) \ge |S| = \frac{m_1}{2} + \lceil \frac{n_1 - 1}{2} \rceil + \lfloor \frac{n_1 - 1}{2} \rfloor + 1$. Hence, S is the minimum double dominating set and $\gamma_{2\times}(G_1^{(S)} \square G_2) = \frac{m_1}{2} + \lceil \frac{n_1 - 1}{2} \rceil + \lfloor \frac{n_1 - 1}{2} \rfloor + 1$. $\square$

Theorem 4.7. *Let G_1 be a complete graph of order n_1 and size m_1 and G_2 be a graph of order n_2 and size m_2. Then $\gamma_{2\times}(G_1^{(S)} \square G_2) = 2n_1$.*

Proof. Let $\{v_i; 1 \le i \le n_1\}$ be the vertices of G_1 and $\{u_j; 1 \le j \le m_1\}$ be the vertices in $I(G_1)$ where $\{u_j; 1 \le j \le n_1\}$ be the vertices dividing the edges in outer cycle of G_1. Consider the set $S = \{v_i, u_j; 1 \le i \le n_1; 1 \le j \le n_1\}$. This set dominates every vertex of $G_1^{(S)} \square G_2$ at least twice. Therefore, S is a double dominating set of $G_1^{(S)} \square G_2$. We need to prove S is minimum. In $G_1^{(S)} \square G_2$, every vertex in each copy of G_2 is adjacent to exactly $n - 1$ vertices of $I(G_1)$ and each vertex in $I(G_1)$ is adjacent to exactly two neighbors of G_1. Observe that every vertex dividing the edges inside of the cycle $\{u_j; n_1 + 1 \le j \le m_1\}$ share no common vertices of G_1. To dominate every vertex of $G_1^{(S)} \square G_2$ at least twice we need to include either two of the neighbors of copies of G_2 from $I(G_1)$ and two neighbors of $\{u_j; n_1 + 1 \le j \le m_1\}$ or two of the neighbors of copies of G_2 from $I(G_1)$ and the vertices $\{u_j; n_1 + 1 \le j \le m_1\}$ and one of their neighbors from G_1. In both the cases, $\gamma_{2\times}(G_1^{(S)} \square G_2) \ge 2n_1 = |S|$. Hence $\gamma_{2\times}(G_1^{(S)} \square G_2) = 2n_1$. $\square$

Theorem 4.8. *Let G_1 be a complete bipartite graph of order $n_1 + n_2$ and size m_1 and G_2 be a graph of order n_3 and size m_2. Then $\gamma_{2\times}(G_1^{(S)} \square G_2) = n_1 n_2 + min\{n_1, n_2\}$.*

Proof. Let $\{v_i; 1 \le i \le n_1 + n_2\}$ be the vertices of G_1 and $\{u_j; 1 \le j \le m_1\}$ be the vertices in $I(G_1)$. Consider the set $\{u_j; 1 \le j \le m_1\}$. This set of vertices will

dominate every vertex of each copy of G_2 at least twice but the vertices $\{u_j, v_i; 1 \leq j \leq m_1; 1 \leq i \leq n_1 + n_2\}$ are dominated only once by $I(G_1)$. Now consider the set $S = \{v_i, u_j; 1 \leq i \leq min\{n_1, n_2\}\}$. This set will dominate every vertex of $G_1^{(S)} \boxdot G_2$ at least twice. Therefore, S is the double dominating set of $G_1^{(S)} \boxdot G_2$. We need to prove S is minimum. From definition of double domination number, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) \leq |S| = n_1 n_2 + min\{n_1, n_2\}$. In $G_1^{(S)} \boxdot G_2$, the vertices in $I(G_1)$ are adjacent to exactly two vertices of G_1. To dominate these vertices at least twice we need to include all the vertices in $I(G_1)$ and one of their neighbors or both the neighbors of $I(G_1)$. In first case, those vertices dominates every vertex of $G_1^{(S)} \boxdot G_2$ at least twice, but in the latter case, only the vertices of $I(G_1)$ are dominate twice and remaining vertices are not dominated at least twice. Therefore, we need to include two of the vertices from $I(G_1)$ for each copy of G_2. In both the cases, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) \geq n_1 n_2 + min\{n_1, n_2\} = |S|$. Hence, $\gamma_{2\times}(G_1^{(S)} \boxdot G_2) = n_1 n_2 + min\{n_1, n_2\}$. $\qquad\square$

Theorem 4.9. *Let G_1 and G_2 be the two graphs of order n_1 and n_2 respectively. Then $\gamma_{2\times}(G_{n_1}^{(S)} \boxminus G_2) = n_1 + \left\lceil \frac{n_1}{2} \right\rceil$.*

Proof. Let $\{v_i; 1 \leq i \leq n_1\}$ be the vertices of G_1 and $\{u_j; 1 \leq j \leq m_1\}$ be the vertices of $I(G_1)$. In $G_{n_1}^{(S)} \boxminus G_2$ every vertex in each copy of G_2 is adjacent to exactly two vertices of G_1. Consider the set $S = \{v_i; 1 \leq i \leq n_1\}$. This set dominates every vertex of $(G_{n_1}^{(S)} \boxminus G_2)$ at least twice except the vertices of G_1. In order to dominate them at least twice we include the vertices $\{u_j; j \equiv 1 \mod 2; 1 \leq j \leq m_1\}$ to S. Thus the set $S = \{v_i, u_j; 1 \leq i \leq n_1; 1 \leq j \leq m_1; j \equiv 1 \mod 2\}$ dominates every vertex of $(G_{n_1}^{(S)} \boxminus G_2)$ at least twice. This S is the minimum double dominating set because every vertex of each copy of G_2 and $I(G_1)$ are adjacent to exactly two vertices of G_1. But the vertices of G_1 are adjacent to exactly one vertex of $I(G_1)$ and adjacent to the vertices of G_2. To dominate them we have to include one of the neighbors of the vertices of G_1. Since, two vertices of G_1 have a common neighbor from $I(G_1)$ to dominate them at least twice, we need to include at least $\left\lceil \frac{n_1}{2} \right\rceil$ vertices from $I(G_1)$. This implies, $\gamma_{2\times}(G_{n_1}^{(S)} \boxminus G_2) \geq n_1 + \left\lceil \frac{n_1}{2} \right\rceil = |S|$. Hence, $\gamma_{2\times}(G_{n_1}^{(S)} \boxminus G_2) = |S| = n_1 + \left\lceil \frac{n_1}{2} \right\rceil$. $\qquad\square$

Theorem 4.10. *Let G_1 be a graph of order n_1 where every vertex of G_1 is either a pendant or a support vertex and G_2 be a graph of order n_2. Then $\gamma_{2\times}(G_{n_1}^{(S)} \boxminus G_2) = n_1 + l$ where l is the number of pendant vertices in G_1.*

Proof. In $G_{n_1}^{(S)} \boxminus G_2$, every vertex in each copy of G_2 is adjacent to exactly two vertices of G_1. Let $S = \{v_i; 1 \leq i \leq n_1\}$. This set dominates every vertex of $G_{n_1}^{(S)} \boxminus G_2$ at least twice except the vertices of G_1. Let $\{e_j; 1 \leq j \leq l\}$ be the pendant edges of G_1 and $\{u_j; 1 \leq j \leq l\}$ be the vertices of $I(G_1)$ corresponding to the pendant edges. In order to dominate the vertices of G_1 at least twice we include the vertices $\{u_j; 1 \leq j \leq l\}$ to S because these vertices are adjacent to both pendant and support vertices of G_1. Therefore, the set $S = \{v_i, u_j; 1 \leq i \leq n_1; 1 \leq j \leq l\}$ is a double dominating set which is

also minimum. Hence, $\gamma_{2\times}(G_{n_1}^{(S)} \boxminus G_2) = |S| = n_1 + l$. $\qquad\square$

5. Corona based on R-graph

Let G_1 and G_2 be two vertex disjoint graphs. The *R-vertex corona* of G_1 and G_2, denoted by $G_1^{(R)} \odot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $V(G_1)$ copies of G_2 by joining the ith vertex of G_1 to every vertex in the ith copy of G_2 . The *R-edge corona* of G_1 and G_2, denoted by $G_1^{(R)} \ominus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $I(G_1)$ copies of G_2 by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2, where $I(G_1) = V(R(G_1)) - V(G_1)$. The *R-vertex neighborhood corona* of G_1 and G_2, denoted by $G_1^{(R)} \boxdot G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $V(G_1)$ copies of G_2 by joining the neighbors of the ith vertex of G_1 in $R(G_1)$ to every vertex in the ith copy of G_2. The *R-edge neighborhood corona* of G_1 and G_2, denoted by $G_1^{(R)} \boxminus G_2$, is the graph obtained from vertex disjoint $R(G_1)$ and $I(G_1)$ copies of G_2 by joining the neighbors of the ith vertex of $I(G_1)$ in $R(G_1)$ to every vertex in the ith copy of G_2. Figure 8 show the examples of R-vertex corona, R-edge corona, R-vertex neighbourhood corona, R-edge neighbourhood corona of graphs C_4 and P_2.

$C_4^{(R)} \odot P_2$ $\qquad$ $C_4^{(R)} \ominus P_2$ $\qquad$ $C_4^{(R)} \boxdot P_2$ $\qquad$ $C_4^{(R)} \boxminus P_2$
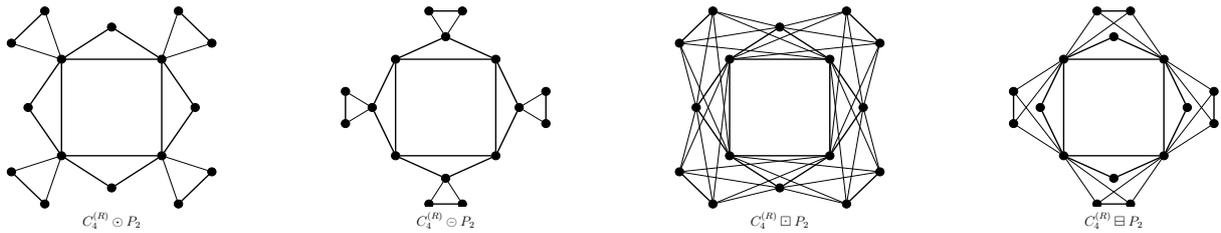
Fig. 8. R-vertex corona, R-edge corona, R-vertex neighbourhood corona, R-edge neighbourhood corona of graphs C_4 and P_2

Theorem 5.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then $\gamma_{2\times}(G_1^{(R)} \odot G_2) = n_1(1 + \gamma(G_2))$.*

Proof. Let $\{v_i, u_j; 1 \leq i \leq n_1; 1 \leq j \leq m_1\}$ be the vertices of $G_1^{(R)}$. Consider the set $S = \{v_i; 1 \leq i \leq n_1\}$. This set dominates every vertex of $(G_1^{(R)} \odot G_2)$ at least twice, but the vertices in G_1 and vertices in each copy of G_2 are dominated only once by S. In order to double dominate them we include minimum dominating set $D_i; 1 \leq i \leq n_1$ of each copy of G_2 to S. Therefore, the set $S = \{v_i; 1 \leq i \leq n_1\} \cup \{D_i; 1 \leq i \leq n_1\}$ dominates every vertex of $G_1^{(R)} \odot G_2$ at least twice and thus a double dominating set of $G_1^{(R)} \odot G_2$. This set S is a minimum double dominating set because, each u_j is adjacent to exactly two vertices of G_1. In $(G_1^{(R)} \odot G_2)$ every vertex of each copy of G_2 is adjacent to exactly one vertex of G_1. The vertices of $I(G_1)$ are dominated exactly twice by S but the vertices in each copy of G_2 are dominated only once by S. In order to dominate them at least twice, we need to include the neighbors of those vertices from G_1 and the minimum dominating set of each copy of G_2. Hence, $\gamma_{2\times}(G_1^{(R)} \odot G_2) = n_1(1 + \gamma(G_2))$. $\qquad\square$

Theorem 5.2. *Let G_1 and G_2 be two graphs of order n_1 and n_2 and size m_1 and m_2 respectively. Then*

$$\gamma_{2\times}(G_1^{(R)} \ominus G_2) = \begin{cases} m_1(\gamma(G_2) + 1) + s_1; & \text{if } \delta(G_1) = 1, \\ m_1(\gamma(G_2) + 1); & \text{otherwise,} \end{cases}$$

where s_1 is the number of support vertices in G_1.

Proof. *Case 1.* When $\delta(G_1) \geq 2$.

The proof is similar to the previous theorem. Let $\{v_i, u_j; 1 \leq i \leq n_1; 1 \leq j \leq m_1\}$ be the vertices of $G_1^{(R)}$. Consider the set $S = \{u_j; 1 \leq i \leq m_1\}$. This set dominates every vertex of $(G_1^{(R)} \odot G_2)$ at least twice, but the vertices $\{u_j; 1 \leq j \leq m_1\}$ and vertices in each copy of G_2 are dominated only once by S. In order to double dominate them we include minimum dominating set $D_i; 1 \leq i \leq n_1$ of each copy of G_2 to S. Therefore, the set $S = \{u_j; 1 \leq j \leq m_1\} \cup \{D_i; 1 \leq i \leq n_1\}$ dominates every vertex of $G_1^{(R)} \odot G_2$ at least twice and thus a double dominating set of $G_1^{(R)} \odot G_2$. Now we need to prove S is minimum. From definition of double domination number $\gamma_{2\times}(G_1^{(R)} \odot G_2) \leq |S|$. Observe that each u_j is adjacent to exactly two vertices of G_1. In $(G_1^{(R)} \odot G_2)$ every vertex of each copy of G_2 is adjacent to exactly one vertex of $I(G_1)$. To dominate them at least twice we need to include the vertices of $I(G_1)$ and the minimum dominating set of each copy of G_2. Hence, $\gamma_{2\times}(G_1^{(R)} \odot G_2) = |S| = m_1(1 + \gamma(G_2))$.

Case 2. When $\delta(G_1) = 1$.

In this case the set $S = \{u_j; 1 \leq j \leq m_1\} \cup \{D_i; 1 \leq i \leq n_1\}$ defined in the previous case dominates every vertex of $G_1^{(R)} \ominus G_2$ at least twice but the pendant vertices of G_1 are dominated only once by S. Therefore, by observation 1, we include the support vertices of G_1 to S. Let $SP(G_1)$ be the set of all support vertices of G_1. The set $S = \{u_j; 1 \leq j \leq m_1\} \cup \{D_i; 1 \leq i \leq n_1\} \cup \{SP(G_1)\}$ dominates every vertex of $G_1^{(R)} \ominus G_2$ at least twice. Thus S is a double dominating set of $G_1^{(R)} \ominus G_2$ which is minimum. Hence, $\gamma_{2\times}(G_1^{(R)} \ominus G_2) = |S| = m_1(1 + \gamma(G_2)) + s_1$. $\square$

Theorem 5.3. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then $\gamma_{2\times}(G_1^{(R)} \boxdot G_2) = n_1$.*

Proof. Let $\{v_i, u_j; 1 \leq i \leq n_1; 1 \leq j \leq m_1\}$ be the vertices of $G_1^{(R)}$. Consider the set $S = \{v_i; 1 \leq i \leq n_1\}$. This set S dominates every vertex of $G_1^{(R)} \boxdot G_2$ at least twice. Therefore, S is a double dominating set of $G_1^{(R)} \boxdot G_2$. This set S is the minimum double dominating set because every vertex of $G_1^{(R)} \boxdot G_2$ is adjacent to exactly two vertices of G_1 and to dominate these vertices at least twice, we need at least n_1 vertices. Hence, $\gamma_{2\times}(G_1^{(R)} \boxdot G_2) = n_1$. $\square$

Theorem 5.4. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively. Then $\gamma_{2\times}(G_1^{(R)} \boxminus G_2) = n_1$.*

Proof. The proof is similar to the previous theorem. Let $\{v_i, u_j; 1 \le i \le n_1; 1 \le j \le m_1\}$ be the vertices of $G_1^{(R)}$. Consider the set $S = \{v_i; 1 \le i \le n_1\}$. This set S dominates every vertex of $G_1^{(R)} \boxminus G_2$ at least twice. Therefore, S is a double dominating set of $G_1^{(R)} \boxminus G_2$. This set S is minimum double dominating set because, in $G_1^{(R)} \boxminus G_2$ every vertex in each copy of G_2 and the vertices $\{u_j; 1 \le j \le m_1\}$ are adjacent to exactly two vertices of G_1. To double dominate them we need at least n_1. Hence, $\gamma_{2\times}(G_1^{(R)} \boxminus G_2) = n_1$. $\qquad\square$

6. Summary of the main results

The impact of unary operations on double domination number of standard graphs is given below:

Graph class	Operation	$\gamma_{2\times}$
C_n	Inflation	$\lceil \frac{4n}{3} \rceil$
P_n	Inflation	$\frac{4(n-2)}{3}$
W_n	Inflation	$\lfloor \frac{2(n-1)}{3} \rfloor + \lceil \frac{n-1}{2} \rceil$
K_n	Inflation	$2(n-1)$
$K_{m,n}$	Inflation	$2(m+n-1)$
$K_{1,n}$	Inflaton	$2l(K_{1,n})$
$S_{r,t}$	Inflation	$2l(S_{r,t})$
Any graph G	Cubic inflation	$2m(G)$
C_n	$\alpha-$operation	n
P_n	$\alpha-$operation	$n-1, n \equiv 1 \mod 3; n$, otherwise
W_n	$\alpha-$operation	4
K_n	$\alpha-$operation	4
$K_{m,n}$	$\alpha-$operation	4
$K_{1,n}$	$\alpha-$operation	4
$S_{r,t}$	$\alpha-$operation	4
C_n	$\beta-$operation	$\lceil \frac{2n}{3} \rceil$
P_n	$\beta-$operation	$\lfloor \frac{2(n+1)}{3} \rfloor - 1, n \equiv 1 \mod 3; \lfloor \frac{2(n+1)}{3} \rfloor$, otherwise
W_n	$\beta-$operation	4
K_n	$\beta-$operation	4
$K_{m,n}$	$\beta-$operation	4
$K_{1,n}$	$\beta-$operation	4
$S_{r,t}$	$\beta-$operation	4
$G_1 \diamond G_2$	Edge corona	$n(G_1)$
$C_n \star G_2$	Neighborhood Corona	n
$P_n \star G_2$	Neighborhood Corona	$n + 2\gamma(G_2)$
$W_n \star G_2$	Neighborhood Corona	$\lceil \frac{n}{2} \rceil + 1$
$K_n \star G_2$	Neighborhood Corona	3
$K_{m,n} \star G_2$	Neighborhood Corona	4
$K_{1,n} \star G_2$	Neighborhood Corona	$(n+1) + l.\gamma(G_2)$
$S_{r,t} \star G_2$	Neighborhood Corona	$(r+t) + l.\gamma(G_2)$
$G_1^{(S)} \odot G_2$	Subdivision-vertex corona	$n_1(1 + \gamma(G_2))$
$G_1^{(S)} \ominus G_2$	Subdivision-edge corona	$m_1(1 + \gamma(G_2)) + l(G_1); \delta(G_1) = 1; m_1(1 + \gamma(G_2))$; otherwise
$C_n^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$m(C_n) + \lceil \frac{n}{2} \rceil$
$P_n^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$m(P_n) + \lceil \frac{n}{2} \rceil + 2\gamma(G_2)$
$W_n^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$\frac{m(W_n)}{2} + \lceil \frac{n(W_n)-1}{2} \rceil + \lfloor \frac{n(W_n)-1}{2} \rfloor + 1$
$K_n^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$2n$
$K_{m,n}^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$mn + min\{m,n\}$
$K_{1,n}^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$m(K_{1,2}) + l(1 + \gamma(G_2))$
$S_{r,t}^{(S)} \boxdot G_2$	Subdivision-vertex Neighborhood Corona	$m(S_{r,t}) + l(1 + \gamma(G_2))$
$G_1^{(S)} \boxminus G_2$	Subdivision-edge Neighborhood Corona	$n_1 + \lceil \frac{n_1}{2} \rceil$

$K_{1,n}^{(S)} \boxminus G_2$	Subdivision-edge Neighborhood Corona	$(n+1) + l(K_{1,n})$
$S_{r,t}^{(S)} \boxminus G_2$	Subdivision-edge Neighborhood Corona	$(r+t) + l(S_{r,t})$
$(G_1^{(R)} \odot G_2)$	R-vertex corona	$n_1(1 + \gamma(G_2))$
$(G_1^{(R)} \ominus G_2)$	R-edge corona	$m_1(1 + \gamma(G_2)) + s_1, \delta(G_1) = 1; m_1(1 + \gamma(G_2));$ otherwise
$(G_1^{(R)} \boxdot G_2)$	R-vertex neighborhood corona	$n(G_1)$
$(G_1^{(R)} \boxminus G_2)$	R-vertex neighborhood corona	$n(G_1)$

7. Conclusion

This work studies the impact of various unary and binary operations on graphs, including inflation and cubic inflation and various corona operations on double domination number. We also defined two new unary operations and explored the impact of these operations on the double domination number. The inflation and cubic inflation operations increase the order and size of the graph, and thus the double domination number increases. The two new operations defined in this paper also increase the order and size of the graph and thus creates an impact to the double domination number of graphs . Further, we studied the double domination number of various corona operations of two graphs. These findings contribute to the better understanding of double domination in various graph structures.

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Conflicts of Interest

The authors declare no conflicts of interest.

Authors Contributon

The authors have equally contributed to this work.

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