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Ergodic results in operator algebras

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ABSTRACT

We provide a hierarchy of "nonconventional ergodic theorems" in quantum setting involving operators and unitaries acting on the Hilbert space of the Gelfand-Naimark-Segal covariant representation associated to a reference C^*-dynamical system. The first two levels correspond to the Mean Ergodic Theorem by J. von Neumann involving a unitary, and the ergodic theorem by Kovács and Szúcs relative to unitarily implemented automorphisms of von Neumann algebras, respectively. As a consequence, we provide multiple correlation results concerning the ergodic behaviour of "three-operator" expectations.

Keywords: noncommutative dynamical systems, ergodic theory, nonconventional ergodic theorems, multiple correlations

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1. Introduction

The aim of the present paper is to establish a hierarchy of ergodic theorems in operator algebras. They provide the generalisation, first of the celebrated Mean Ergodic Theorem by J. von Neumann (e.g. [23]) involving a unitary acting on a Hilbert space, and then that by Kovács and Szúcs (cf. [13]) relative to unitarily implemented groups of automorphisms of von Neumann algebras. Such theorems are milestones in Quantum Ergodic Theory, and have several applications to the general theory of dynamical systems, including those arising from Quantum Physics and Quantum Probability. The reader is referred to Section 4.3.1 of [3] and the reference cited therein for further details.

Such results can also be viewed as particular classes of "nonconventional ergodic theorems". A first example falling in this class has been established by H. Furstenberg (cf. [9]) in connection with a theorem by E. Szemerédi concerning arithmetic progressions.

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Successively, such results have been generalised in the quantum case in [6, 8] under some natural assumptions such as the ergodicity of the reference states (perhaps assumed also in [9]), or separability of the involved algebras in order to perform the ergodic disintegration (e.g. [19, 3]).

Ergodic theory is an important branch of Functional Analysis. It has relevant applications to several other branches of Mathematics including Probability, and Physics such as classical and quantum mechanics and statistical physics. Relatively to the classical case, Ergodic Theory is very well developed and, recently, its study is rapidly growing also in the quantum case. Even if the list is highly not exhaustive, we refer the reader to [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 19, 23] and the reference cited therein, for such recent investigations.

To be more precise, we fix a Hilbert space $\mathcal{H}$, denoting by $\mathcal{B}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the algebra of all linear bounded operators and the group of all unitaries acting on $\mathcal{H}$, respectively. A *Nonconventional Ergodic Theorem* is a result involving the convergence, in the strong or, merely, in the weak operator topology, of multiple Cesàro averages of the form

$$\frac{1}{n^k}\sum_{n_1,\dots,n_k=0}^{n-1} V_1^{n_{\alpha(1)}}A_1V_2^{n_{\alpha(2)}}\cdots V_{m-1}^{n_{\alpha(m-1)}}A_{m-1}V_n^{n_{\alpha(m)}}\,.$$

Here, $\{V_1,\dots,V_m\}$ are linear contractions (in many cases of interests, $\{V_1,\dots,V_m\}\subset\mathcal{U}(\mathcal{H})$), and $\{A_1,\dots,A_{m-1}\}\subset\mathcal{B}(\mathcal{H})$. For $m\geq k$, $\alpha:\{1,\dots,m\}\to\{1,\dots,k\}$ indeed describes a partition of the set $\{1,\dots,m\}$ in k parts. Recently, many other generalisations following this line have also been considered. Apart from several applications in other branches of mathematics and physics as mentioned above, such a class of ergodic theorems also finds application in Quantum Probability in order to provide a quantum version of the Central Limit Theorem, see e.g. [1].

In order to focus our starting point, we consider a reference C^*-dynamical system $(\mathfrak{A},\alpha,\omega)$ made of a C^*-algebra $\mathfrak{A}$ (supposed to be unital for the sake of simplicity), a $*$-automorphism α of $\mathfrak{A}$, and a state ω invariant under α. The classical case corresponds to $(C(K),T,\mu)$, $C(K)$ being the algebra of all continuous functions on the compact space K, $T:K\to K$ a homeomorphism whose pull-over generates a $*$-automorphism of $C(T)$, and μ an invariant Radon probability measure (i.e. an invariant state on $C(T)$ by the Riesz-Markov Theorem). This case is mostly well known and widely studied. We also assume that the support $s(\omega)$ of the reference state ω in the bidual $\mathfrak{A}^{**}$ is in the centre $Z(\mathfrak{A}^{**})$. Notice that the last property is trivially satisfied in the classical case.

Let $(\mathcal{H}_\omega,\pi_\omega,V_\omega,\xi_\omega)$ be the associated Gelfand-Naimark-Segal (GNS for short) covariant representation. With $M:=\pi_\omega(\mathfrak{A})''$ and $M':=\pi_\omega(\mathfrak{A})'$, the C^*-algebra $\mathfrak{M}:=M\otimes_{\max}M'$ acts in a natural way on both $\mathcal{H}_\omega\otimes\mathcal{H}_\omega$, and on $\mathcal{H}_\omega$. Then the vector state

$$M\otimes_{\max}M'\ni a\otimes b'\mapsto\langle ab'\xi_\omega,\xi_\omega\rangle,$$

is the quantum “diagonal measure" corresponding to the product state

$$M\otimes_{\max}M'\ni a\otimes b'\mapsto\langle a\xi_\omega,\xi_\omega\rangle\langle b'\xi_\omega,\xi_\omega\rangle\,.$$

In addition, on $\mathfrak{M}$ are naturally acting the $*$-automorphisms

$$\beta_{k_1,k_2} := \mathrm{Ad}_{V_\omega^{k_1}} \otimes \mathrm{Ad}_{V_\omega^{k_2}}\,,$$

where $k_1, k_2 \in \mathbb{Z}$ are any pair of fixed integers. The product state is automatically invariant under the action of β_{k_1,k_2}, whereas the diagonal measure is, in general, neither normal with respect to the product measure, nor invariant with respect to such actions.

The nonconventional ergodic theorems proved in the present paper go beyond the previous investigated cases corresponding to the product state (assuming ergodicity for $V_\omega^{k_2-k_1}$) in [9, 6], and the separability condition for $\mathfrak{A}$ in [8]. This is done by assuming an additional natural condition, perhaps difficult to check on concrete models, we are going to describe below.

As in the previous papers, such nonconventional ergodic theorems concern the study of the convergence in the strong operator topology of the sequence of the Cesáro means

$$\frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1} X V_\omega^{l(k_2-k_1)}\,, \tag{1}$$

when X belongs to the norm-closed linear subspace generated by M and M' in $\mathcal{B}(\mathcal{H}_\omega)$ (indeed a closed operator space), and k_1, k_2 are integers without any further restrictions on $\mathfrak{A}$, and on the pair of integers (k_1, k_2).

In the previous mentioned cases, the main ingredient to obtain the result was the property of *genericity* introduced in [9]. In order to get new ergodic results, the genericity, as originally defined in that paper, must be complemented by other natural conditions. These conditions are (almost) automatically satisfied in the cases involving the product and the diagonal "measures".

The framework of the more general cases we are going to treat, described in Secs. 4, 5, can be schematised as follows. We consider a class of states $\varphi_{k_2-k_1}$, denoted by φ, each of which is invariant w.r.t. one of the above $*$-automorphisms β_{k_1,k_2} denoted by β, together with the state ρ which is always the diagonal state φ_0 (i.e. the state "φ" corresponding to $k_1 = k_2$). Notice that, each of such objects, states and automorphisms, constructed by using the reference dynamical system $(\mathfrak{A}, \alpha, \omega)$, are given on the tensor product C^*-algebra $\mathfrak{M}$.

It is easily shown that ρ is generic for the states φ w.r.t. the $*$-subalgebra $\mathfrak{M}_o := \pi_\omega(\mathfrak{A})'' \otimes \pi_\omega(\mathfrak{A})'$, relatively to the automorphisms β (cf. (6) where $\mathfrak{A}$ and $\mathfrak{A}_o$ are $\mathfrak{M}$ and $\mathfrak{M}_o$, respectively). In order to apply genericity, we need that the invariant elements $\pi_\varphi(\mathfrak{M}_o^\beta)\xi_\varphi$ must generate the closed subspace of $\mathcal{H}_\varphi$ made of the invariant vectors $E_1^{V_\varphi}\mathcal{H}_\varphi$ (cf. (7)). Due to the ergodicity of the state ω, the last condition is satisfied for the previous cases treated in [9]. The result is extended to the situation in [6] by using the Ergodic Disintegration, available when the reference algebra $\mathfrak{A}$ is separable. Concerning the generalisations in the present paper, one easily gets

$$\pi_\varphi(\mathfrak{M}_o^\beta)\xi_\varphi \subset \pi_\varphi(\mathfrak{M}_o)^{\mathrm{ad}_{V_\varphi}}\xi_\varphi\,. \tag{2}$$

Now, on one hand, $\overline{\pi_\varphi(\mathfrak{M}_o)^{\mathrm{ad}_{V_\varphi}}\xi_\varphi} = E_1^{V_\varphi}\mathcal{H}_\varphi$ (cf. Proposition 3.1) but, on the other hand, the above inclusion might be strict. In order to avoid this natural obstruction and manage the more general cases considered in the present paper, the genericity is now complemented by

$$x \in \mathfrak{M}_o \ \& \ x - \beta(x) \in \mathrm{Ker}(\pi_\varphi) \Rightarrow \rho(x^*x) = \rho(\beta(x^*x)) \,, \quad \text{(cf. (8))}.$$

It should be noted that, if π_φ is faithful, (8) is automatically satisfied and, in addition, we directly have also the equality in (2). By Proposition 6.3 and (i) in Theorem 4.1, another condition which would assure (7) is the centrality of the support in the bidual of the involved state $\varphi \equiv \varphi_{k_2-k_1}$. Unfortunately, both properties mentioned above relative to such states "φ" (i.e. the faithfulness of π_φ and/or the centrality of the support $s(\varphi)$) are very difficult to check, and it should be done for specific models. However, we point out that many new examples can be managed by considering states ω on the full matrix algebra $\mathbb{M}_n(\mathbb{C})$ $(n > 1)$ with central support, that is those corresponding to the density-matrices T with strictly positive eigenvalues. In this case, all kernels of $\pi_{\varphi_{k_2-k_1}}$ are trivial since

$$\mathbb{M}_n(\mathbb{C}) \otimes_{\max} \mathbb{M}_n(\mathbb{C}) = \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \sim \mathbb{M}_{n^2}(\mathbb{C}) \,,$$

and the full matrix algebra $\mathbb{M}_{n^2}(\mathbb{C})$ is a simple C^*-algebra.

Summarising, under the conditions listed above we prove (cf. Sec. 5) that the sequence of the Cesàro averages in (1) converges in the strong operator topology for X in the norm-closed operator space spanned by M and M'. In particular, after denoting by

$$\mathfrak{M} \ni x \mapsto x_\psi \in \mathcal{H}_\psi =: L^2(\mathfrak{M}, \psi),$$

the L^2-embedding for the state $\psi \in \mathcal{S}(\mathfrak{M})$, we provide the following explicit formula

$$\lim_{n\to+\infty} \left\{ \begin{array}{c} \frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1} a V_\omega^{l(k_2-k_1)} b'\xi_\omega \\ \frac{1}{n}\sum_{l=0}^{n-1} V_{\omega,\alpha}^{lk_1} b' V_\omega^{l(k_2-k_1)} a\xi_\omega \end{array} \right\} = V(a \otimes b')_{\varphi_{k_2-k_1}} \,, \tag{3}$$

$a \in M$, $b' \in M'$, where V is the partial isometry canonically associated to the dynamical system $(\mathfrak{A}, \alpha, \omega)$ given in (9).

Since ξ_ω is a standard vector for $M = \pi_\omega(\mathfrak{A})''$, the limit in (3) can be viewed in two equivalent ways. The first one is as a limit in the strong operator topology of $\mathcal{B}(\mathcal{H}_\omega)$ for the sequences on the r.h.s. considering a (b') as a fixed operator and allowing $b'\xi_\omega$ $(a\xi_\omega)$ to run into the dense set $M'\xi_\omega$ $(M\xi_\omega)$ in $\mathcal{H}_\omega$. The second way is to fix both $a \in M$ and $b' \in M'$, and look at the above limit directly inside $\mathcal{H}_\omega$.

It should also be noted that $k_1 = k_2$ provides the von Neumann Mean Ergodic Theorem (e.g. [23]), whereas $k_2 = 0$ leads to the Kovács and Szúcs Ergodic Theorem (cf. [13]).

The result concerning the behaviour of the Cesàro Means (1) allows us to investigate the limit for the three-point correlations associated to sequences of the form

$$\left(\frac{1}{n}\sum_{l=0}^{n-1} \omega\left(a_0 \alpha^{lk_1}(a_1)\alpha^{lk_2}(a_2)\right)\right),$$

for $a_0, a_1, a_2 \in \mathfrak{A}$, which might have relevant applications in Quantum Physics and Quantum Probability. This is done in Section 6.

The paper is complemented with an appendix devoted to the study of the invariant part of the subspace $\mathfrak{M}_o$ which would allow to extend our result directly (i.e. without assuming (8)) to the case when the state $\varphi_{k_2-k_1}$ has central support in the bidual.

2. Preliminaries

2.1. *Basic facts*

For the basic notions of Linear Algebra, Functional Analysis, Operator Algebras and other standard definitions and results, we refer the reader to the corresponding textbooks. All linear spaces X considered here are supposed to be nontrivial (i.e. $X \neq \{0\}$), and on the complex scalar field $\mathbb{C}$. In particular, for details about the theory of Operator Algebras, and its representations such as the Gelfand-Naimark-Segal (GNS for short) representation associated to a positive linear functional φ (typically a state) and denoted by $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$, the reader is referred to [19, 21]. The unit circle $\{z \in \mathbb{C}\,;\, |z|= 1\}$ is denoted by $\mathbb{T}$.

Let $\mathcal{H}$ be a Hilbert space, its inner product, linear w.r.t. the 1st variable, is denoted by $\langle\ ,\ \rangle$. For $S \subset \mathcal{H}$ being a subset of $\mathcal{H}$, with $[S] := \overline{\mathrm{span}(S)}$ we denote its closed linear span in $\mathcal{H}$. The space (indeed a concrete W^*-algebra, see below) of all linear bounded operators and the group of all unitary operators acting on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$, respectively. A *von Neumann algebra* M acting on the Hilbert space $\mathcal{H}$ (often denoted by the pair $(M, \mathcal{H})$) is a selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ coinciding with its double commutant: $M = (M')' =: M''$. For the von Neumann algebra $(M, \mathcal{H})$, we always have $I_\mathcal{H} \in M$ and thus M is unital with $\mathbf{1}_M = I_\mathcal{H}$.

In the present note, we deal with C^*-algebras which, to simplify, we suppose to be unital without any further mention, even if all presented results can be straightforwardly extended to the non unital case. We also recall that a W^*-algebra $\mathfrak{M}$ is a C^*-algebra which has a, necessarily unique, predual $\mathfrak{M}_*$ (i.e. $(\mathfrak{M}_*)^* \sim \mathfrak{M}$ as Banach spaces). By Sakai Theorem (cf. [19]), this is equivalent to require that $\mathfrak{M}$ is $*$-isomorphic to a von Neumann algebra: $\mathfrak{M} \sim \pi(\mathfrak{M})$ for some representation π such that $\pi(\mathfrak{M}) = \pi(\mathfrak{M})''$.

For the positive linear functional φ on the C^*-algebra $\mathfrak{A}$, we consider the closed left ideal and the kernel of the GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$

$$\mathfrak{n}_\varphi := \{a \in \mathfrak{A}\,;\, \varphi(a^*a) = 0\}\,, \quad \mathrm{Ker}(\pi_\varphi) = \{a \in \mathfrak{A}\,;\, \pi_\varphi(a) = 0\}\,,$$

respectively. One easily has $\mathrm{Ker}(\pi_\varphi) \subset \mathfrak{n}_\varphi$, but the inclusion might be strict as is shown when φ is a vector state ω_ξ on $\mathcal{B}(\mathcal{H})$, $\xi \in \mathcal{H}$ being a unit vector and $\dim(\mathcal{H}) > 1$.

An *Operator Space* $\mathfrak{X}$ is a linear subspace of some C^*-algebra. Any Operator Space $\mathfrak{X}$ can be abstractly characterised through the assignment of a sequence of norms $\{\|\ \|_n\}$ on matrix-spaces $\{\mathbb{M}_n(\mathfrak{X})\}$, $n = 1, 2, \cdots$, satisfying appropriate compatibility properties, see [17].

Suppose that $(M, \mathcal{H})$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is said to be cyclic (for M) if $[M\xi] = \mathcal{H}$. It is possible to see that this property is equivalent to the separation property for the commutant M': $M' \ni b'$ such that $b'\xi = 0 \Rightarrow b' = 0$. Therefore, a *standard vector* $\xi \in \mathcal{H}$ is a vector which is cyclic and separating for M or, equivalently, for M'.

We now recall that, if $\varphi \in \mathcal{S}(\mathfrak{A})$ is a state on the C^*-algebra $\mathfrak{A}$ with GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ and support $s(\varphi)$ in the bidual $\mathfrak{A}^{**}$, it possible to show that

$$s(\varphi) \in Z(\mathfrak{A}^{**}) \iff \xi_\varphi \text{ is standard}\,.$$

For the proof of such an assertion, we refer the reader to [15].

2.2. *Some celebrated ergodic theorems*

For the convenience of the reader, we report the celebrated ergodic theorems by von Neumann (e.g. [23]), and Kovács and Szúcs (cf. [13]). Indeed, let $U \in \mathcal{U}(\mathcal{H})$ be a unitary acting on the Hilbert space $\mathcal{H}$. For $z \in \mathbb{T}$, with E^U_z we denote the selfadjoint projection onto the eigenspace of U corresponding to the eigenvalue z. Then $z \in \sigma_{\rm pp}(U) \iff E^U_z \neq 0$. In particular, $E^U_1\mathcal{H}$ corresponds to the subspace of the invariant vectors under U.

Theorem 2.1 (von Neumann mean Ergodic theorem)**.** *For* $U \in \mathcal{U}(\mathcal{H})$*,*

$$\operatorname{sot-lim}_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} U^k = E^U_1\,,$$

(“$\operatorname{sot-lim}$*” stands for the limit in the strong operator topology of* $\mathcal{B}(\mathcal{H})$*).*

Let now $(M, \mathcal{H})$ and $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ be a von Neumann algebra and a unitary representation of the group G, respectively, acting on the same Hilbert space $\mathcal{H}$, such that $\mathrm{Ad}_{U_g}(M) = M$, $g \in G$. As before, denote by $E := E^U_1 \in \mathcal{B}(\mathcal{H})$ the selfadjoint projection onto the invariant vectors under the unitary representation U_g, and by $\Gamma(x)$ the weak operator closed convex hull of the orbit $\{U_g x U_g^*\,;\, g \in G\}$ of x under the action of G. Assume further that the set of invariant vectors $E\mathcal{H}$ separates M:

$$x \in M \;\&\; x\xi = 0 \;\forall\, \xi \in E\mathcal{H} \Rightarrow x = 0\,.$$

Theorem 2.2 (Kovács and Szúcs)**.** *Under the above conditions, there exists a map*

$$M \ni x \mapsto \Phi(x) \in M,$$

characterised in either of the following equivalent way:

(a) $\Phi(x)$ *is the only element of* $\Gamma(x) \bigcap U'_G$*;*

(b) $\Phi(x)E = ExE$*;*

(c) $\Phi(x)$ *is the only weak operator continuous* G*-invariant linear map, reducing to the identity on the fixed-point subalgebra* M^G*.*

The map Φ *is a faithful conditional expectation and, further, for each net of finite convex combinations* $\left(\sum_n c_n U^*_{g_n}\right)_n$ *converging in the strong (weak) operator topology to* E*,*

$$\sum_n c_n U_{g_n} x U^*_{g_n} \to \Phi(x), \tag{4}$$

in the strong (weak) operator topology, for all $x \in M$*.*

2.3. Automorphisms and expectations in W^*-algebras

Let $\mathfrak{A}$ be a C^*-algebra and α a $*$-automorphism. In such a situation,

$$\mathfrak{A}^\alpha := \{a \in \mathfrak{A}\,;\, \alpha(a) = a\}\,, \quad \mathcal{S}(\mathfrak{A})^\alpha := \{\varphi \in \mathcal{S}(\mathfrak{A})\,;\, \varphi \circ \alpha = \varphi\},$$

denote the C^*-subalgebra of $\mathfrak{A}$ of the invariant elements and the $*$-weakly compact convex subset of the invariant states of $\mathfrak{A}$, respectively. For $\varphi \in \mathcal{S}(\mathfrak{A})^\alpha$, we consider the *GNS covariant representation*, unique up to unitary equivalence, which provides the unitary implementation of α (and its powers). It is made of a quadruple $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi, \xi_\varphi)$, where $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is the GNS representation of φ, and V_φ is determined by

$$V_\varphi \pi_\varphi(a)\xi_\varphi = \pi_\varphi(\alpha(a))\xi_\varphi\,, \quad a \in \mathfrak{A}\,,$$

satisfying, for each $a \in \mathfrak{A}$, $V_\varphi \pi_\varphi(a) V_\varphi^* = \pi_\varphi(\alpha(a))$.

We say (with an abuse of notation) that $\varphi \in \mathcal{S}(\mathfrak{A})^\alpha$ is *ergodic* if $\mathcal{H}_\varphi$ contains only one unit vector, up to a phase, which is invariant under V_φ. This simply means that the range of $E_1^{V_\varphi}$ is one dimensional. It is possible to show that an invariant state φ is extremal invariant if it is ergodic. The reverse implication holds true if φ is $\mathbb{Z}$-abelian, see Theorem 2.2 in [8] and the references cited therein.

For the W^*-algebra $\mathfrak{M}$, let $\mathcal{B}(\mathfrak{M}) := \mathcal{B}(\mathfrak{M}, \mathfrak{M})$ denote the bounded linear self-maps on $\mathfrak{M}$. Since $\mathfrak{M} = (\mathfrak{M}_*)^*$, the remarkable result in Lemma 7.1 of [16] asserts that $\mathcal{B}(\mathfrak{M})$ is a dual space: $\mathcal{B}(\mathfrak{M})$ is isometrically isomorphic with the dual of some Banach space Z. Therefore, its unit ball $\mathcal{B}(\mathfrak{M})_1$ is compact w.r.t. the weak topology inherited by the weak*-topology of Z^*. Such a weak topology is named by *Bounded Weak* (BW for short) topology, and is generated by the separating family of seminorms

$$\{|f(T(x))|\,;\, f \in \mathfrak{M}_*,\, x \in \mathfrak{M}\}\,. \tag{5}$$

Now we state a result, crucial for our analysis, which is known to the experts. We sketch its proof for the convenience of the reader.

Theorem 2.3. *Let α be a $*$-automorphism of a W^*-algebra $\mathfrak{M}$. Then the sequence $\left(\frac{1}{n}\sum\limits_{k=0}^{n-1}\alpha^k\right)_n \subset \mathcal{B}(\mathfrak{M})_1$ admits cluster points w.r.t. the BW topology. Each of such cluster points ε is a, not necessarily normal, conditional expectation onto the fixed-point subalgebra $\mathfrak{M}^\alpha$ satisfying $\alpha \circ \varepsilon = \varepsilon = \varepsilon \circ \alpha$.*

Proof. We already noticed that the unit ball $\mathcal{B}(\mathfrak{M})_1$ is compact in the BW topology, and thus there is a subnet $(n(\iota))_\iota$ of $\mathbb{N}$ such that $\lim_\iota \left(\frac{1}{n(\iota)}\sum\limits_{k=0}^{n(\iota)-1}\alpha^k\right)$ exists as an element of $\mathcal{B}(\mathfrak{M})_1$. Let us denote $\varepsilon := \lim_\iota \left(\frac{1}{n(\iota)}\sum\limits_{k=0}^{n(\iota)-1}\alpha^k\right)$ for the above limit.

Since for $f \in \mathfrak{M}_*$ and $a \in \mathfrak{M}$,

$$f\left(\alpha\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k(a)\right)\right) = f\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k(\alpha(a))\right)$$

$$
\begin{aligned}
&= f\left(\frac{1}{n(\iota)}\sum_{k=1}^{n(\iota)}\alpha^k(a)\right)\\
&= f\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k(a)\right)+\frac{f\left(\alpha^{n(\iota)}(a)\right)}{n(\iota)}-\frac{f(a)}{n(\iota)},
\end{aligned}
$$

and, taking into account that $\lim_\iota n(\iota)=+\infty$, we deduce

$$
\alpha\left(\lim_\iota\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k\right)\right)=\lim_\iota\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k\right)=\lim_\iota\left(\frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1}\alpha^k\right)\alpha,
$$

in the BW topology. This means that ε is both equivariant and invariant w.r.t. α: $\alpha\varepsilon=\varepsilon\alpha=\varepsilon$, which implies that ε is a conditional expectation onto the fixed-point subalgebra. □

An analogous situation is reported in Section 2.5 of [11], where the limit along a subnet of the integers is denoted as E_ω, $\omega\in\beta(\mathbb{N})\smallsetminus\mathbb{N}$ by using free ultrafilters in view of the identification with elements of the Stone-Čech compactification $\beta(\mathbb{N})$, where $\mathbb{N}$ corresponds to principal ultrafilters and $\beta(\mathbb{N})\smallsetminus\mathbb{N}$ to the free ones.

2.4. *Genericity for C^*-dynamical systems*

Let $\mathfrak{A}$ be a C^*-algebra, β a $*$-automorphism, and $\rho\in\mathcal{S}(\mathfrak{A})$ $\varphi\in\mathcal{S}(\mathfrak{A})^\beta$ be states, with the latter invariant under β. We report for the reader's convenience the original definition of *genericity* established in pag. 211 of [9] for ergodic abelian dynamical systems.

Indeed, with the above notations, we say that ρ is *generic* for φ w.r.t. the $*$-subalgebra $\mathfrak{A}_o\subset\mathfrak{A}$ if

$$
\lim_n\frac{1}{n}\sum_{k=0}^{n-1}\rho(\beta^k(x))=\varphi(x)\,,\quad x\in\mathfrak{A}_o\,. \tag{6}
$$

Genericity is a necessary condition to provide many ergodic theorems. It is usually complemented with the following natural property

$$
\overline{\pi_\varphi(\mathfrak{A}_o^\beta)\xi_\varphi}=E_1^{V_\varphi}\mathcal{H}_\varphi\,, \tag{7}
$$

in order to establish the existence of the partial isometry in (9) needed to obtain the desired ergodic theorems.

The last condition is not so natural to be satisfied. In fact, on one hand Proposition 3.1 asserts that $\overline{\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi}=E_1^{V_\varphi}\mathcal{H}_\varphi$ for every norm-dense $*$-subalgebra $\mathfrak{A}_o$. On the other hand, $\pi_\varphi(\mathfrak{A}_o^\beta)\subset\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}$ where, perhaps, the inclusion might be strict. We show in the next section that this difficulty can be overcome with the following condition: generity should be satisfied w.r.t. a norm-dense $*$-subalgebra $\mathfrak{A}_o$ for which

$$
x-\beta(x)\in\mathrm{Ker}(\pi_\varphi)\Rightarrow\rho(x^*x)=\rho(\beta(x^*x))\,. \tag{8}
$$

Notice that, if π_φ is faithful, (8) is automatically satisfied. If, instead, φ has central support, Proposition 6.3 in the appendix together with (i) in Theorem 4.1 would allow (7).

In the cases previously treated, (7) is reached by using ergodicity in [6], and by using the ergodic disintegration in[8] available only when the involved algebra $\mathfrak{A}$ is separable.

Since all these conditions (faithfulness of the GNS representations and centrality of the support of the involved states) or, directly, conditions (7) and (8), are very delicate ones to be checked in quite general settings. For situations different from those previously described, such conditions should be investigated directly for the specific situations of interests.

3. On C^*-dynamical systems

Let β be a $*$-automorphism of a C^*-algebra $\mathfrak{A}$. By defining $\beta_n := \beta^n$, $n \in \mathbb{Z}$, we have an action $\mathbb{Z} \overset{\beta_n}{\curvearrowright} \mathfrak{A}$ of the group of the integers $\mathbb{Z}$. As usual, we encode in the definition of *dynamical system* also a fixed invariant state, being the last crucial in the forthcoming analysis.

Indeed, for a (discrete) C^*-dynamical system we then mean a triplet $(\mathfrak{A}, \beta, \varphi)$ made of a unital C^*-algebra, a $*$-automorphism $\beta \in \mathrm{Aut}(\mathfrak{A})$, and an invariant state $\varphi \in \mathcal{S}(\mathfrak{A})^\beta$. The quadruple $(\pi_\varphi, \mathcal{H}_\varphi, V_\varphi, \xi_\varphi)$ will denote the covariant GNS representation of φ as explained above. The unitary V_φ is also called the *Koopman operator* (in analogy with the classical situation) associated to the state φ, invariant under the action of the automorphism β.

As a preliminary step, we provide the proof of the following useful result.

Proposition 3.1. *For the C^*-dynamical system $(\mathfrak{A}, \beta, \varphi)$ and a norm-dense involutive subalgebra $\mathfrak{A}_o \subset \mathfrak{A}$, we have $\overline{\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi} = E_1^{V_\varphi}\mathcal{H}_\varphi$.*

Proof. With $M = \pi_\varphi(\mathfrak{A})''$ and $E := E_1^{V_\varphi}$, we first note that $\mathcal{H}_\varphi = \overline{\pi_\varphi(\mathfrak{A})\xi_\varphi} = \overline{M\xi_\varphi}$, and thus $E\mathcal{H}_\varphi = \overline{EM\xi_\varphi}$. Let now $n(\iota)$ be the subnet of $\mathbb{N}$ and ε the Ad_{V_φ}-invariant conditional expectation onto $M^{\mathrm{Ad}_{V_\varphi}}$ given in Theorem 2.3 applied with $\mathfrak{M} = M$ and $\alpha = \mathrm{Ad}_{V_\varphi}$. By the Mean Ergodic Theorem and the above mentioned theorem, for $X \in M$ and $\xi \in \mathcal{H}_\varphi$, we get

$$\begin{aligned}\langle EX\xi_\varphi, \xi\rangle &= \lim_{n\to+\infty}\left\langle \frac{1}{n}\sum_{k=0}^{n-1} V_\varphi^k X\xi_\varphi, \xi\right\rangle = \lim_\iota \left\langle \frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1} V_\varphi^k X\xi_\varphi, \xi\right\rangle \\ &= \lim_\iota \left\langle \frac{1}{n(\iota)}\sum_{k=0}^{n(\iota)-1} \mathrm{Ad}_{V_\varphi}^k(X)\xi_\varphi, \xi\right\rangle = \langle \varepsilon(X)\xi_\varphi, \xi\rangle\,,\end{aligned}$$

and thus $EM\xi_\varphi = M^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi$. Now,

$$E\mathcal{H}_\varphi = E\overline{M\xi_\varphi} = \overline{EM\xi_\varphi} = \overline{\varepsilon(M)\xi_\varphi} = \overline{M^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi}\,.$$

We now use Kaplansky Density Theorem (e.g. [21], Theorem II.4.7), applied to bounded parts

$$M_n := \{X \in M\,;\, \|X\| \leq n\}$$

of M. Indeed, by the previous part and using the above mentioned density result in the

4th equality, we get

$$\begin{aligned}E\mathcal{H}_\varphi &=\overline{M^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi} = \overline{\bigcup_{n=1}^{+\infty}(M_n)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi} = \overline{\bigcup_{n=1}^{+\infty}\left(M^{\mathrm{Ad}_{V_\varphi}}\right)_n\xi_\varphi}\\ &=\overline{\bigcup_{n=1}^{+\infty}\left(\pi_\varphi(\mathfrak{A})^{\mathrm{Ad}_{V_\varphi}}\right)_n\xi_\varphi} = \overline{\pi_\varphi(\mathfrak{A})^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi} = \overline{\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi}\,.\end{aligned}$$

□

Consider now another state $\rho\in\mathcal{S}(\mathfrak{A})$. Notice that ρ is supposed in general to be neither invariant under the action of β, nor normal (i.e. not of the form $\rho=\mathrm{Tr}(\pi_\varphi(\ )T)$ for some positive, normalised trace class operator T acting on $\mathcal{H}_\varphi$) w.r.t. φ.

Concerning the present paper, the application of genericity is summarised in the following

Theorem 3.2. *Suppose that there exists a norm-dense $*$-subalgebra $\mathfrak{A}_o\subset\mathfrak{A}$ such that, for each $x\in\mathfrak{A}_o$, (8), and (6) (e.g. genericity according to the definition in [9]), are satisfied. Then the map*

$$\mathcal{H}_\varphi\supset\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi\ni\pi_\varphi(x)\xi_\varphi\mapsto\pi_\rho(x)\xi_\rho\in\mathcal{H}_\rho, \tag{9}$$

uniquely defines a partial isometry $V:\mathcal{H}_\varphi\mapsto\mathcal{H}_\rho$ with initial range $V^*V=E_1^{V_\varphi}\mathcal{H}_\varphi$ and, in addition,

$$\lim_n\frac{1}{n}\sum_{k=0}^{n-1}\pi_\rho(\beta^k(x))\xi_\rho=V(\pi_\varphi(x)\xi_\varphi)\,,\quad x\in\mathfrak{A}_o\,.$$

Proof. Suppose that (8) and (6) hold true for the norm dense involutive subalgebra $\mathfrak{A}_o$, and pick $x\in\mathfrak{A}_o$ with $\pi_\varphi(x)\in\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}$. We first note that

$$\pi_\varphi(x)=V_\varphi\pi_\varphi(x)V_\varphi^*=\pi_\varphi(\beta(x))\,,$$

and thus $x-\beta(x)\in\mathrm{Ker}(\pi_\varphi)$. Since, for the invariant state φ, $\beta\left(\mathrm{Ker}(\pi_\varphi)\right)=\mathrm{Ker}(\pi_\varphi)$, for such elements we also get $\beta^k(x)-\beta^{k+1}(x)\in\mathrm{Ker}(\pi_\varphi)$, k being any natural number. Therefore, we can compute

$$\begin{aligned}\|\pi_\rho(x^*x)\xi_\rho\|^2 &=\rho(x^*x)\overset{(8)}{=}\rho\left(\beta^k(x^*x)\right)=\frac{1}{n}\sum_{k=0}^{n-1}\rho\left(\beta^k(x^*x)\right)\\ &=\lim_n\left(\frac{1}{n}\sum_{k=0}^{n-1}\rho\left(\beta^k(x^*x)\right)\right)\overset{(6)}{=}\varphi(x^*x)=\|\pi_\varphi(x^*x)\xi_\rho\|^2\,.\end{aligned}$$

Summarising, we have shown that the map V given by $\pi_\varphi(x)\xi_\varphi\mapsto\pi_\rho(x)\xi_\rho$ is isometric, and thus well defined, on $\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi$. It is then enough to put V identically zero on $\left(\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi\right)^\perp$ and extend it to $\overline{\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}\xi_\varphi}$.

In order to prove the last property, we reason as in Theorems 2.2 in [9] and 3.5 in [6]. Indeed, for $\varepsilon>0$ and $x\in\mathfrak{A}_o$, choose $x_\varepsilon\in\pi_\varphi(\mathfrak{A}_o)^{\mathrm{Ad}_{V_\varphi}}$ such that $\|(E\pi_\varphi(x)-\pi_\varphi(x_\varepsilon))\xi_\varphi\|<\varepsilon$.

By the mean ergodic theorem, $\varphi\left(\left|\frac{1}{m}\sum_{l=0}^{m-1}\beta^k(x-x_\varepsilon)\right|^2\right) < \varepsilon^2$ for all sufficiently large integers m. Put $\Gamma\equiv\Gamma_m := \frac{1}{m}\sum_{l=0}^{m-1}\beta^k(x-x_\varepsilon)$.

Since $\mathfrak{A}_o$ is a $*$-algebra which is invariant under β, $\Gamma\in\mathfrak{A}_o$ as well, and thus, by the first part, $\frac{1}{n}\sum_{k=0}^{n-1}\rho(\beta^k(\Gamma^*\Gamma))\to\varphi(\Gamma^*\Gamma)$. By applying Lemma 3.3 in [6], we have for a fixed m sufficiently large and each n sufficiently large, $\rho\left(\left|\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{m}\sum_{l=0}^{m-1}\beta^{k+l}(x-x_\varepsilon)\right|^2\right) < \varepsilon^2$ which, by Lemma 3.4 in [6], becomes $\left\|\frac{1}{n}\sum_{k=0}^{n-1}\pi_\rho(\beta^k(x)-x_\varepsilon)\xi_\rho\right\| < \varepsilon$.

The proof ends as

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=0}^{n-1}\pi_\rho(\beta^k(x))\xi_\rho - VE\pi_\varphi(x)\xi_\varphi\right\| \leq& \left\|\frac{1}{n}\sum_{k=0}^{n-1}\pi_\rho(\beta^k(x)-x_\varepsilon)\xi_\rho\right\|\\ &+\|V(\pi_\varphi(x_\varepsilon)-E\pi_\varphi(x))\xi_\varphi\|< 2\varepsilon\,.\end{aligned}$$

□

4. Dynamical systems on the tensor product

Let us consider a reference C^*-dynamical system $(\mathfrak{A},\alpha,\omega)$ such that ω has central support in the bidual. As usual, $(\mathcal{H}_\omega,\pi_\omega,V_\omega,\xi_\omega)$ will be the associated covariant GNS representation. Therefore, (the closure of) Tomita's involution S_ω is meaningful. Its polar decomposition is $S_\omega=J_\omega\Delta_\omega^{1/2}$, J_ω,Δ_ω being the conjugation and modular operator, respectively. Put $M:=\pi_\omega(\mathfrak{A})''$, $M'=\pi_\omega(\mathfrak{A})'$, and

$$\begin{aligned}M_z &:= \{a\in M\mid V_\omega a V_\omega^* = za\}\,,\\ M_z' &:= \{a'\in M'\mid V_\omega a' V_\omega^* = za'\}\,.\end{aligned}$$

For the convenience of the reader, we collect the main properties (e.g. [6], Proposition 2.1 and 2.2) enjoyed by these subspaces.

Theorem 4.1. *Let $(\mathfrak{A},\alpha,\omega)$ be a C^*-dynamical system such that $s(\omega)\in Z(\mathfrak{A}^{**})$. Then, with the previous notations,*

(i) $\overline{M_z\xi_\omega}=E_z^{V_\omega}\mathcal{H}_\omega=\overline{M_z'\xi_\omega}$;

(ii) *the subspace $E_1^{V_\omega}\mathcal{H}_\omega$ of the invariant vectors separates $\pi_\omega(\mathfrak{A})''$ and $\pi_\omega(\mathfrak{A})'$;*

(iii) $\sigma_{\rm pp}(V_\omega)=\sigma_{\rm pp}(V_\omega)^{-1}$;

(iv) $J_\omega E_z^{V_\omega}=E_{\bar z}^{V_\omega}J_\omega$.

Suppose further that ω is ergodic, then

(v) $E_z^{V_\omega}\mathcal{H}=\mathbb{C}V_z\xi_\omega=\mathbb{C}W_z\xi_\omega$, *$v\in\sigma_{\rm pp}(U_\omega)$ for a unitary $V_z\in M$ ($W_z\in M'$), uniquely determined up to a phase.*

As a consequence of (v),

(vi) *$\sigma_{\rm pp}(V_\omega)$ is a subgroup of $\mathbb{T}$.*

Proof. (i) By (3.2) in [15], we check that $M_z\xi_\omega$ is dense in $E_z^{V_\omega}\mathcal{H}_\omega$. By exchanging the role between M and M', we obtain the analogous result $\overline{M'_z\xi_\omega}=E_z^{V_\omega}\mathcal{H}_\omega$.

(ii) Since, in such a situation,

$$\mathcal{H}_\omega=[\pi_\omega(\mathfrak{A})'\xi_\omega]=[\pi_\omega(\mathfrak{A})'E_1^{V_\omega}\xi_\omega]\subset[\pi_\omega(\mathfrak{A})'E_1^{V_\omega}\mathcal{H}_\omega]\subset\mathcal{H}_\omega\,,$$

$E_1^{V_\omega}\mathcal{H}_\omega$ is cyclic for M' and thus it separates M. By reversing M' with M, we get that $E_1^{V_\omega,\alpha}\mathcal{H}_\omega$ separates M' as well.

(iii) By taking into account that the adjoint action of V_ω on M (or equivalently on M') is a $*$-automorphism, if $A\in M_z$ is non-null, A^* is a non-null element of $M_{\bar z}$, that is $z\in\sigma_{\rm pp}(V_\omega)$ implies $\bar z\in\sigma_{\rm pp}(V_\omega)$.

(iv) It easily follows as J_ω commutes with V_ω, see Proposition 3.3 (c) of [15].

(v) Suppose ω ergodic, fix $z\in\sigma_{\rm pp}(V_\omega)$ and choose non-null elements $A,B\in M_z$ which always exist by Proposition 3.2 of [15]. Then $A^*B=\alpha I$ and $BA^*=\beta I$ for some non-null numbers α,β. Indeed, $A^*B\xi_\omega$ is invariant under V_ω. Thus by ergodicity, $A^*B\xi_\omega=\alpha\xi_\omega$, and by the fact that ξ_ω is separating, $A^*B=\alpha I$. In addition, suppose that $\alpha=0$. As AA^* is a non-null multiple, say c, of the identity, we have $AA^*B=0$, which means $B=0$, a contradiction. At the same way, we verify $BA^*\neq0$. Now, $\alpha^{-1}A^*$ and $\beta^{-1}A^*$ are left and right inverses of B. This means that B is invertible and $B^{-1}=\alpha^{-1}A^*$. At the same way, A is invertible too. Moreover, $AB^{-1}=\alpha^{-1}AA^*=\alpha^{-1}cI$. This means $A=\alpha^{-1}cB$, that is A is a multiple of B. In addition, in this situation $AA^*=cI$ means that A is a multiple of the unitary $A/\sqrt{c}$.

(vi) follows from (v). Indeed, if $z,w\in\sigma_{\rm pp}(U)$, let $V_z\in M_z$, $V_w\in M_w$ be the corresponding unitaries (uniquely determined, up to a phase).

Then $V_zV_w\in M_{zw}$ is non-null, that is $zw\in\sigma_{\rm pp}(U)$. □

Let $\mathfrak{M}:=M\otimes_{\rm max}M'$ be the completion of the algebraic tensor product $\mathfrak{M}_o:=M\otimes M'$ w.r.t. the maximal C^*-norm (cf. [21], Section IV.4). On the C^*-algebra $\mathfrak{M}$, we define a collection of states as we are going to explain.

Indeed, we consider the von Neumann algebras M, M' and the representation of $\mathbb{Z}$ generated by V_ω acting on $\mathcal{H}_\omega$. Since the adjoint action of the powers of V_ω leaves M and M' globally stable, we can apply the Kovács-Szúcs Theorem 2.2 with $G=\mathbb{Z}$ and $U_n=(V_\omega^k)^n=V_\omega^{kn}$, for the von Neumann algebras $(M,\mathcal{H}_\omega)$, $(M',\mathcal{H}_\omega)$ and all choices of integers $k\in\mathbb{Z}$. For such k, we denote by

$$E_k:M\to M^{{\rm Ad}_{V_\omega^k}}\,,\quad E'_k:M'\to(M')^{{\rm Ad}_{V_\omega^k}}\,,$$

the corresponding faithful conditional expectations onto the fixed-point subalgebras. Such conditional expectations exist as a consequence of Kovács-Szúcs Theorem which can be applied by (ii) of Theorem 4.1. Notice that $E_k=E_{-k}$ and $E'_k=E'_{-k}$.

On the generators $a\otimes b'$, $b\in M$ and $b'\in M'$, of $\mathfrak{M}_o$, define

$$\varphi_k(a\otimes b'):=\langle aE'_k(b')\xi_\omega,\xi_\omega\rangle\,,\quad a\in M,\ b'\in M'\,. \tag{10}$$

Proposition 4.2. *For each $k\in\mathbb{Z}$, the functional φ_k is well defined and positive on $\mathfrak{M}_o$. In addition, it extends to a state on the whole $\mathfrak{M}$.*

Proof. We consider the maps defined on $\mathfrak{M}_o$ given by $\mathrm{id}_M \otimes E'_k$, and the multiplication map $\mathrm{m} : \mathfrak{M}_o \to \mathcal{B}(\mathcal{H}_\omega)$, given on elementary tensors by $\mathrm{m}(a \otimes b') = ab'$. By the results in [21], Section IV.4, both extend to bounded, indeed completely positive, maps, again denoted by $\mathrm{id}_M \otimes E'_k$ and m respectively. Denoting by ω_{ξ_ω} the vector state on $\mathcal{B}(\mathcal{H}_\omega)$ corresponding to ξ_ω, it is immediate to show that

$$\varphi_k = \omega_{\xi_\omega} \circ \mathrm{m} \circ (\mathrm{id}_M \otimes E'_k)\,,$$

and the assertion follows. □

The proof of Proposition 4.2 tells us that the extension to the whole $\mathfrak{M}$ of the functionals φ_k in (10), denoted again by the same symbol with an abuse of notation, is given by

$$\varphi_k(x) = \omega_{\xi_\omega}\left(\mathrm{m}(\mathrm{id}_M \otimes E'_k)\right)(x)\,, \quad x \in \mathfrak{M}\,,$$

where $\omega_\xi := \langle \quad \xi, \xi\rangle$ is the vector functional associated to the vector ξ.

Since $E_k = E_{-k}$, again by the Kovács-Szúcs Theorem, we get

$$\varphi_k(a \otimes b') = \langle aE'_k(b')\xi_\omega, \xi_\omega\rangle = \langle E_{-k}(a)b'\xi_\omega, \xi_\omega\rangle = \langle E_k(a)b'\xi_\omega, \xi_\omega\rangle\,.$$

By following the notations of Section 4 of [6] (see also [8]), φ_0 is nothing else than the diagonal state corresponding to the product state $(\omega_{\xi_\omega}\lceil_M) \times (\omega_{\xi_\omega}\lceil_{M'})$. If the unitary V_ω^k is ergodic, that is $E_1^{V_\omega^k}\mathcal{H}_\omega$ is one-dimensional, then φ_k is just the product state mentioned above. Notice that, φ_0 does not have, in general, central support in the bidual. Conversely, the product state $(\omega_{\xi_\omega}\lceil_M) \times (\omega_{\xi_\omega}\lceil_{M'})$, coinciding with φ_k in the ergodic case, always has central support. As explained above, it would be also of interest to provide conditions under which the φ_k have central support.

Fix now $k_1, k_2 \in \mathbb{Z}$, and define

$$\beta_{k_1,k_2}(a \otimes b') := \mathrm{Ad}_{V_\omega^{k_1}}(a) \otimes \mathrm{Ad}_{V_\omega^{k_2}}(b')\,, \quad a \in M,\ b' \in M'\,,$$

which easily extends to a $*$-automorphism on all of $\mathfrak{M}$.

Proposition 4.3. *The state $\varphi_{k_2-k_1}$ is invariant under β_{k_1,k_2}.*

Proof. It is enough to check the invariance on the total set of generators $a \otimes b'$ for $a \in M$ and $b' \in M'$. By (4), we have

$$\begin{aligned}
\varphi_{k_2-k_1}\left(\beta_{k_1,k_2}(a \otimes b')\right) =& \langle V_\omega^{k_1} a V_\omega^{-k_1} E'_{k_2-k_1}\left(V_\omega^{k_2} b' V_\omega^{-k_2}\right)\xi_\omega, \xi_\omega\rangle \\
=& \lim_{n \to +\infty} \frac{1}{n} \sum_{l=0}^{n-1} \langle aV_\omega^{-k_1} V_{\omega,}^{l(k_2-k_1)} V_\omega^{k_2} b'\xi_\omega, \xi_\omega\rangle \\
=& \lim_{n \to +\infty} \frac{1}{n} \sum_{l=1}^{n} \langle aV_\omega^{l(k_2-k_1)} b'\xi_\omega, \xi_\omega\rangle \\
=& \langle aE'_{k_2-k_1}(b')\xi_\omega, \xi_\omega\rangle = \varphi_{k_2-k_1}(a \otimes b')\,.
\end{aligned}$$

□

5. A hierarchy of ergodic theorems

The present section is devoted to a hierarchy of ergodic theorems on von Neumann algebras which can be viewed as generalisations of von Neumann Mean Ergodic theorem and the Kovács and Szúcs Theorem. We focus the attention on the C^*-dynamical system $(\mathfrak{M}, \beta_{k_1,k_2}, \varphi_{k_2-k_1})$ made of $\mathfrak{M} = M \otimes_{\max} M'$, $\beta_{k_1,k_2} = \mathrm{Ad}_{V_\omega^{k_1}} \otimes \mathrm{Ad}_{V_\omega^{k_1}}$, and $\varphi_{k_2-k_1}(a \otimes b') = \langle aE_{k_2-k_1}(b')\xi_\omega, \xi_\omega\rangle$ where $E_{k_2-k_1}$ is the map Φ in Theorem 2.2 corresponding to $U = V_\omega^{k_2-k_1}$. Finally, $\mathfrak{X} := \overline{M + M'}^{\|\ \|_{\mathcal{B}(\mathcal{H}_\omega)}}$ is the closed operator space generated by M and M'. For a state ψ on a C^*-algebra $\mathfrak{B}$ and $x \in \mathfrak{B}$, denote by x_ψ its embedding in the GNS Hilbert space $\mathcal{H}_\psi$:

$$\mathfrak{B} \ni x \mapsto x_\psi := \pi_\psi(x)\xi_\psi \in \mathcal{H}_\psi\,.$$

Lemma 5.1. *For each $k_1, k_2 \in \mathbb{Z}$, the state φ_0 is generic for $\varphi_{k_2-k_1}$ w.r.t. the norm-dense subalgebra $\pi_\omega(\mathfrak{A})'' \otimes \pi_\omega(\mathfrak{A})'$.*

Proof. We already showed (cf. Proposition 4.3) that $\varphi_{k_2-k_1}$ is invariant w.r.t. the $*$-automorphism β_{k_1,k_2}. Now, by linearity it is enough to check (6) for the generators $x = a \otimes b'$ of $M \otimes M' = \pi_\omega(\mathfrak{A})'' \otimes \pi_\omega(\mathfrak{A})'$. Indeed, by von Neumann Mean Ergodic Theorem,

$$\begin{aligned}\frac{1}{n}\sum_{l=0}^{n-1}\varphi_0\left((\beta^l_{k_1,k_2}(x)\right) =&\frac{1}{n}\sum_{l=0}^{n-1}\langle aV_\omega^{l(k_2-k_1)}b'\xi_\omega, \xi_\omega\rangle \to \langle aE_1^{V_\omega^{k_2-k_1}}b'\xi_\omega, \xi_\omega\rangle\\ \to&\langle aE'_{k_2-k_1}(b')\xi_\omega, \xi_\omega\rangle = \varphi_{k_2-k_1}(x)\,.\end{aligned}$$

□

Here, there is the main result of the present paper.

Theorem 5.2. *Let $(\mathfrak{A}, \alpha, \omega)$ be a C^*-dynamical system such that $s(\omega) \in Z(\mathfrak{A}^{**})$. Suppose that, for $x \in \pi_\omega(\mathfrak{A})'' \otimes \pi_\omega(\mathfrak{A})'$, (8) is satisfied with $\beta = \beta_{k_1,k_2}$, $\varphi = \varphi_{k_2-k_1}$ and $\rho = \varphi_0$. Then we get:*

(i) *for each $X \in \mathfrak{X}$, the sequence $\left(\frac{1}{n}\sum_{l=0}^{n-1} V_{\omega,}^{lk_1}XV_{\omega,}^{l(k_2-k_1)}\right)_n$ converges in the strong operator topology;*

(ii) *the isometry in (9) is meaningful and, in particular, for $a \in M$ and $b' \in M'$,*

$$\left.\begin{array}{l}\frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1}aV_\omega^{l(k_2-k_1)}b'\xi_\omega\\ \frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1}b'V_\omega^{l(k_2-k_1)}a\xi_\omega\end{array}\right\}\xrightarrow[n\to+\infty]{} V(a\otimes b')_{\varphi_{k_2-k_1}}\,. \tag{11}$$

Proof. We start by noticing that, by Lemma 5.1, the isometry in (9) is meaningful under the notations listed above. We also note that, by a standard approximation argument, and taking int account of the symmetric role played by M and M', (i) is proven for $\mathfrak{X}$ once it holds true for elements of the form $X = a \otimes I$ with $a \in M$.

We start by noticing that $(\mathcal{H}_{\varphi_0}, \pi_{\varphi_0}, \xi_{\varphi_0})$ coincides, up to unitary equivalence, with $(\mathcal{H}_\omega, \mathrm{m}, \xi_\omega)$, where $\mathrm{m}(a\otimes b') = ab'$ on generators $a\otimes b'$ of $\mathfrak{M}$. Now, the proof is the direct consequence of Theorem 3.2.

Indeed by the above consideration, for $a \in M$, $b \in M'$ such that $x = a\otimes b'$ and $\eta = \pi_{\varphi_0}(I\otimes b')\xi_{\varphi_0} = b'\xi_\omega$, we compute

$$\begin{aligned}\frac{1}{n}\sum_{l=0}^{n-1}\pi_{\varphi_0}\left(\beta^{k_1,k_2}(x)\right)\xi_{\varphi_0} &= \frac{1}{n}\sum_{l=0}^{n-1}\left(V^{lk_1}aV^{-lk_1}\right)\left(V^{lk_2}b'V^{-lk_2}\right)\xi_\omega\\ &= \frac{1}{n}\sum_{l=0}^{n-1}V^{lk_1}aV^{l(k_2-k_1)}b'\xi_\omega\\ &\overset{\text{Theorem 3.2}}{\longrightarrow} V\pi_{\varphi_{k_2-k_1}}(x)\xi_{\varphi_{k_2-k_1}}\\ &= V(a\otimes b')_{\varphi_{k_2-k_1}}\,,\end{aligned}$$

where V is the partial isometry in (9). □

Remark 5.3. Since $[M\xi_\omega] = \mathcal{H}_\omega = [M'\xi_\omega]$, Theorem 5.2 immediately gets:

(i) for $k_1 = k_2$ and $a \in M$, $b' \in M'$,

$$\lim_n \frac{1}{n}\sum_{l=0}^{n-1}\left(V_\omega^{k_1}\right)^l ab'\xi_\omega = E_1^{V_\omega^{k_1}}ab'\xi_\omega\,,$$

which is nothing else than the von Neumann Mean Ergodic Theorem relative to the unitary $V_\omega^{k_1}$;

(ii) for $k_2 = 0$ and $a \in M$, $b' \in M'$,

$$\begin{aligned}\lim_n \frac{1}{n}\sum_{l=0}^{n-1}\left(V_\omega^{k_1}\right)^l a\left(V_\omega^{k_1}\right)^{-l}b'\xi_\omega &= E_{k_1}(a)b'\xi_\omega\\ =&E'_{k_1}(b')a\xi_\omega = \lim_n\frac{1}{n}\sum_{l=0}^{n-1}\left(V_\omega^{k_1}\right)^l b'\left(V_\omega^{k_1}\right)^{-l}a\xi_\omega\,,\end{aligned}$$

which is the Kovács and Szúcs Theorem relative to M and M' equipped with the adjoint action of $V_\omega^{k_1}$.

Concerning the von Neumann ergodic theorem, condition (i) in Theorem 3.2 is automatically satisfied, and thus we again recover such a result for the Koopman unitary $V_\omega^{k_1}$.

Concerning the Kovács and Szúcs Theorem, we are recovering a particular case of it due to the additional condition (i) in Theorem 3.2. It should be noted that, under such a condition, we are able to provide a quite explicit formula (11) for the limit $\Phi(a)b'\xi_\omega$ (or, equivalently, for $\Phi(b')a\xi_\omega$) which is not provided by Theorem 2.2.

6. Multiple correlations

The aim of the present section is to provide a three-point correlation associated to the C^*-dynamical system $(\mathfrak{A},\alpha,\omega)$ such that $s(\omega)\in Z(\mathfrak{A}^{**})$ as for the cases in [6, 8]. We start with the following

Lemma 6.1. *Let $a\in M$ and $b',c'\in M'$. Then*
(i) $\left\|\frac{1}{n}\sum_{l=0}^{n-1}V^{lk_1}_{\omega,\alpha}aV^{l(k_2-k_1)}_{\omega,\alpha}(b'\xi_\omega-c'\xi_\omega)\right\|\leq\|a\|\|b'\xi_\omega-c'\xi_\omega\|$;
(ii) $\left\|V(a\otimes(b'-c'))_{\varphi_{k_2-k_1}}\right\|\leq\|a\|\|b'\xi_\omega-c'\xi_\omega\|$.

Proof. Since (i) is trivial, we restrict the matter to (ii). Indeed,

$$\begin{aligned}\|V(a\otimes(b'-c'))_{\varphi_{k_2-k_1}}\|^2\leq&\|(a\otimes(b'-c'))_{\varphi_{k_2-k_1}}\|^2\\=&\langle E_{k_2-k_1}(a^*a)(b'-c')^*(b'-c')\xi_\omega,\xi_\omega\rangle\\=&\langle E_{k_2-k_1}(a^*a)(b'-c')\xi_\omega,(b'-c')\xi_\omega\rangle\\\leq&\|a\|^2\|b'\xi_\omega-c'\xi_\omega\|^2\,.\end{aligned}$$

□

Theorem 6.2. *Let $(\mathfrak{A},\alpha,\omega)$ be a C^*-dynamical system with $s(\omega)\in Z(\mathfrak{A}^{**})$, $0<k_1<k_2$ integers and $a_0,a_1,a_2\in\mathfrak{A}$. Under the same conditions of Theorem 5.2 for the C^*-dynamical system $(\mathfrak{M},\beta_{k_1,k_2},\varphi_{k_2-k_1})$, the sequence $\left(\frac{1}{n}\sum_{l=0}^{n-1}\omega\left(a_0\alpha^{lk_1}(a_1)\alpha^{lk_2}(a_2)\right)\right)_n$ converges. Since $M'\xi_\omega$ is dense in $\mathcal{H}_\omega$ (by the centrality of the support of ω), one has*

$$\lim_n\frac{1}{n}\sum_{l=0}^{n-1}\omega\left(a_0\alpha^{lk_1}(a_1)\alpha^{lk_2}(a_2)\right)=\lim_{\xi\to\pi_\omega(a_2)\xi_\omega}\langle V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}},\pi_\omega(a_0)^*\xi_\omega\rangle\,,$$

whenever $\xi=b'\xi_\omega\in M'\xi_\omega$ converges to $\pi_\omega(a_2)\xi_\omega$.

Proof. Let $\varepsilon>0$, and choose $b'\in M'$ such that $\|b'\xi_\omega-\pi_\omega(a_2)\xi_\omega\|<\varepsilon$, which is possible since ω has central support in the bidual. By (ii) in Lemma 6.1, $V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}}$ converges to some vector $\eta\in\mathcal{H}_\omega$ as $b'\xi_\omega\to\pi_\omega(a_2)\xi_\omega$ and, in addition,

$$\|V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}}-\eta\|<\varepsilon\|\pi_\omega(a_1)\|\leq\varepsilon\|a_1\|\,.$$

We then compute

$$\begin{aligned}&\left|\frac{1}{n}\sum_{l=0}^{n-1}\omega\left(a_0\alpha^{lk_1}(a_1)\alpha^{lk_2}(a_2)\right)-\langle\eta,\pi_\omega(a_0)^*\xi_\omega\rangle\right|\\&\leq\left|\left\langle\frac{1}{n}\sum_{l=0}^{n-1}V^{lk_1}_\omega\pi_\omega(a_1)V^{l(k_2-k_1)}_\omega(\pi_\omega(a_2)-b')\xi_\omega,\pi_\omega(a_0)^*\xi_\omega\right\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\left\langle \frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1}\pi_\omega(a_1)V_\omega^{l(k_2-k_1)}b'\xi_\omega, \pi_\omega(a_0)^*\xi_\omega\right\rangle - \langle V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}}, \pi_\omega(a_0)^*\xi_\omega\rangle\right| \\
&+|\langle V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}} - \eta, \pi_\omega(a_0)^*\xi_\omega\rangle| \leq 2\|a_0\|\|a_1\|\varepsilon \\
&+\left|\left\langle \frac{1}{n}\sum_{l=0}^{n-1} V_\omega^{lk_1}\pi_\omega(a_1)V_\omega^{l(k_2-k_1)}b'\xi_\omega, \pi_\omega(a_0)^*\xi_\omega\right\rangle - \langle V(\pi_\omega(a_1)\otimes b')_{\varphi_{k_2-k_1}}, \pi_\omega(a_0)^*\xi_\omega\rangle\right| .
\end{aligned}
$$

Now, by (11), we get

$$
\limsup_{n\to+\infty}\left|\frac{1}{n}\sum_{l=0}^{n-1}\omega\left(a_0\alpha^{lk_1}(a_1)\alpha^{lk_2}(a_2)\right) - \langle\eta, \pi_\omega(a_0)^*\xi_\omega\rangle\right| \leq 2\|a_0\|\|a_1\|\varepsilon\,,
$$

which leads to the assertion as $\varepsilon > 0$ is arbitrary. □

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Appendix

The present appendix deals with a result of algebraic nature which is useful (by the application of (i) in Theorem 4.1) in checking directly the condition

$$\overline{\pi_{\varphi_{k_2-k_1}}\left((\pi_\omega(\mathfrak{A})''\otimes\pi_\omega(\mathfrak{A})')^{\beta_{k_1,k_2}}\right)\xi_{\varphi_{k_2-k_1}}}=E_1^{V_{\varphi_{k_2-k_1}}}\mathcal{H}_{\varphi_{k_2-k_1}},$$

insuring (11), provided $\varphi_{k_2-k_1}$ has central support in the bidual $(\pi_\omega(\mathfrak{A})''\otimes_{\max}\pi_\omega(\mathfrak{A})')^{**}$.

Proposition 6.3. *With the previous notations,*

$$\mathfrak{N}:=\left\{\sum r_i\otimes s'_i\mid r_i\in M_{z_i}, s'_i\in M'_{w_i}, z_i,w_i\in\mathbb{T},\ z_i^{k_1}w_i^{k_2}=1\right\}=\mathfrak{M}_o^{\beta_{k_1,k_2}}.$$

Proof. We first note that $\mathfrak{N}\subset\mathfrak{M}_o^{\beta_{k_1,k_2}}$. For the reverse inclusion, fix a non null eigenvector

$$\mathfrak{M}_o\ni x \text{ s.t. } \beta_{k_1,k_2}(x)=\lambda x\in\mathfrak{M}_o\,,\quad \lambda\in\mathbb{T}\,.$$

We now reason as in Lemma 4.17 of [10] (here, the unitarity of the involved operators plays no role since we are reasoning on finite dimensional subspaces) by applying Lemma 1.1 in [20], see also [22], Lemma 1: for some positive integer n, x has the form $\sum\limits_{j=1}^n a_j\otimes b'_j$, where the $a_j\in M$ and the $b'_j\in M'$ can be chosen to be linearly independent (e.g. Lemma 1.1 in [20]). Since x is a non null eigenvector of β_{k_1,k_2}, again by Lemma 1.1 in [20], this linear map induces an invertible map between the space of dimension $n\times n$ generated by the linearly independent vectors $\{a_j\otimes b'_k\}_{j,k=1}^n$ in $\mathfrak{M}_o$. We first recall the pure-point version of the spectral mapping theorem (e.g. [18], Theorems 10.28 and 10.33) $\sigma_{\rm pp}(T^m)=\sigma_{\rm pp}(T)^m$ applied to T coinciding with β restricted to M, M' and $m=k_1,k_2$, respectively. Then, after noticing that

$$\beta_{k_1,k_2}\left(\sum_{j=1}^n a_j\otimes b'_j\right)=\sum_{j=1}^n\beta^{k_1}(a_j)\otimes\beta^{k_2}(b'_j)=\lambda\sum_{j=1}^n a_j\otimes b'_j\,,$$

we apply Lemma 4.17 of [10] and conclude that x is a finite sum the form $\sum\limits_i r_i\otimes s'_i$, $r_i\in M$ and $s'_i\in M'$, of eigenvectors corresponding to eigenvalues z_i, w_i of β belonging to the unit circle and satisfying $z_i^{k_1}w_i^{k_2}=z$. The assertion now corresponds to the particular case $z=1$. □

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