

Exploring fibonacci cordiality in corona graphs

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ABSTRACT

A Fibonacci cordial (FC) labeling of a graph G is an injective function $f : V(G) \rightarrow \{F_0, F_1, \dots, F_n\}$, where F_i is the i^{th} Fibonacci number, such that the induced edge labeling $f^*(uv) = (f(u) + f(v)) \pmod{2}$ satisfies $|e_f(0) - e_f(1)| \leq 1$. A graph admitting such a labeling is called a Fibonacci cordial. First introduced by Rokad and Ghodasara (2016), FC labeling has been studied for several graph families. Mitra and Bhoumik (2020) extended this to complete graphs, cycles, and their corona (C_n and K_p for $p \leq 3$). Motivated to build upon their work, we investigate $C_n \odot K_p$ for $p \geq 4$. Additionally, we examine whether the aforementioned family of corona graphs retains Fibonacci cordiality when alterations are made to the order of the corona, as observed in the family $K_n \odot C_m$. Moreover, we investigate the conditions under which two additional corona graph families, namely $K_m \odot K_m$ and $K_{n,n} \odot K_p$, exhibit Fibonacci cordial labeling.

Keywords: Fibonacci numbers, Fibonacci cordial labeling, complete graphs, cycles, complete bipartite graphs, corona of graphs

2020 Mathematics Subject Classification: 05C78.

1. Introduction

Graph labeling, an area of increasing interest within graph theory research, has witnessed significant growth and diversification over the past few decades. This field encompasses a range of techniques aimed at assigning labels to the elements of a graph, thereby facilitating the analysis and understanding of its structural properties and relationships. Graph labeling can broadly be categorized into three main types: vertex labeling, edge labeling, and total labeling, which incorporates both vertex and edge labeling approaches. Throughout our analysis, we employ standard graph-theoretic terminology and notations

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Received 03 Jun 2025; Revised 20 Dec 2025; Accepted 13 Jan 2026; Published Online 30 Mar 2026.

DOI: [10.61091/um126-14](https://doi.org/10.61091/um126-14)

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as outlined in Bondy and Murty [1].

Definition 1.1. A function $f : V(G) \rightarrow \{0, 1\}$ is said to be Cordial Labeling if the induced function $f^* : E(G) \rightarrow \{0, 1\}$ defined by

$$f^*(uv) = |f(u) - f(v)|,$$

satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$, as well as $|e_f(0) - e_f(1)| \leq 1$, where, $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled i , respectively, for $i = 0, 1$.

This concept of cordial labeling, first introduced by Cahit [2], represents a seminal contribution to graph labeling theory. Initially explored through various vertex labeling strategies, cordial labeling has since been extended and adapted to encompass a diverse array of labeling schemes. These include divisor cordial labeling, product cordial labeling, total product cordial labeling, prime cordial labeling, and numerous others. Researchers seeking a comprehensive overview of the field often turn to Gallian's extensive and dynamic survey [3].

Fibonacci cordial labeling emerges as a distinctive vertex labeling technique, leveraging the mathematical properties of the Fibonacci sequence to assign labels to the vertices of a graph. By integrating the Fibonacci sequence's number-theoretic properties with the underlying structural characteristics of the graph family under investigation, this labeling approach aims to achieve cordiality within the labeled graph. This labeling is basically an extension of Cordial labeling, where we label the vertices with Fibonacci numbers instead of 0 and 1. This version of cordial labeling is challenging compared to the original, as the number of odd and even vertex labels are not equal, whereas the induced edges of both parity must differ by at most 1.

Definition 1.2. The sequence $\{F_n\}$ of Fibonacci numbers is defined as the linear homogeneous recurrence relation of degree 2:

$$F_n = F_{n-1} + F_{n-2}; \quad F_0 = 0, F_1 = F_2 = 1,$$

Lemma 1.3. *In the set of Fibonacci numbers $\{F_0, F_1, \dots, F_N\}$, the number of even labels available is $v_f(0) = \lfloor N/3 \rfloor + 1$.*

Proof. We prove by (strong form of) induction that F_n is even if and only if $n \equiv 0 \pmod{3}$. For the base cases, $F_0 = 0$, $F_1 = 1$, and $F_2 = 1$ satisfy the hypothesis. Now for the inductive step, we assume the hypothesis holds for all $m \leq k$. For $F_{k+1} = F_k + F_{k-1}$:

- If $k + 1 = 3j$, then $k = 3j - 1$ and $k - 1 = 3j - 2$. Both are odd by hypothesis, so F_{k+1} is even.
- If $k + 1 = 3j + 1$, then F_k is even and F_{k-1} is odd, so F_{k+1} is odd.
- If $k + 1 = 3j + 2$, then F_k is odd and F_{k-1} is even, so F_{k+1} is odd.

Thus, exactly one out of every three labels is even, yielding $\lfloor N/3 \rfloor + 1$ even labels. \square

Definition 1.4. An injective function $f : V(G) \rightarrow \{F_0, F_1, \dots, F_n\}$ is said to be Fibonacci cordial labeling if the induced function $f^* : E(G) \rightarrow \{0, 1\}$ defined by

$$f^*(uv) = (f(u) + f(v)) \pmod{2},$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Note that, unlike standard cordial labeling, the vertex parities $v_f(0)$ and $v_f(1)$ are determined by the properties of the Fibonacci sequence and are not required to be balanced, but rather should satisfy the Lemma 1.3.

Fibonacci cordial labeling was first introduced by Rokad & Ghodasara [9]. In their work, they provide the results for the Petersen graph, Wheel graph, Shell graph, Bistar, and some product families of graphs (corona, etc.). Later in 2017, it was explored for more families of graphs by Rokad [8]. Continuous efforts were made by different researchers over the past few years to explore Fibonacci Cordiality among various families of graphs. In their recent paper [4], Mitra & Bhoumik investigated the Fibonacci cordial labeling for the fundamental graphs, viz., complete graphs, cycles, and corona products of C_n and K_m , for $m \leq 3$. In that paper they proposed the following theorem.

Theorem 1.5 ([4], Theorem 2.4). *The complete list of Fibonacci cordial complete graphs are $K_1, K_2, K_3, K_4, K_6, K_7, K_9, K_{11}, K_{18}$, and K_{22} .*

Continuous efforts were made by different researchers over the past decade to explore Fibonacci Cordiality among various families of graph [7, 5, 6, 10]. In this paper, we are extending their work for the generalized family of $C_n \odot K_m$ where $m \geq 4$. Further, we investigate another family, namely the corona of complete bipartite graphs with complete graphs i.e., $K_{n,n} \odot K_p$.

The rest of the paper is organized as follows: We provide a few results in Section 2 which will be referred to in the following sections. We discuss the results obtained on the corona families $C_n \odot K_m$ and $K_{n,n} \odot K_p$ in Section 3 and Section 4 respectively, followed by concluding remarks and future scope in Section 5.

2. Preliminaries

Before presenting the main results, we establish a few preliminary lemmas and propositions that will be instrumental in the subsequent analysis.

Lemma 2.1. *For any cordial labeling f of a cyclic graph C_n , the number of odd edges (edges labeled with 1) is always even, and satisfies the inequality*

$$e_f(1) \leq 2 \min\{v_f(0), v_f(1)\},$$

where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled i , respectively, for $i = 0, 1$.

Proof. An edge is odd exactly when its endpoints receive different labels. Consider a walk around the cyclic graph C_n . Each odd edge flips the current label type. Since the walk returns to its starting point, the total number of flips must be even. Therefore, the number of odd edges is even.

For the upper bound, every vertex of the smaller parity class (say there are $s = \min\{v_f(0), v_f(1)\}$ such vertices) is incident to at most two edges of the cycle, and each odd edge is incident to exactly one vertex of that smaller class. Hence, each of these s vertices can contribute at most two odd edges, giving

$$e_f(1) \leq 2s = 2 \min\{v_f(0), v_f(1)\},$$

as required. \square

Lemma 2.2. *Let K_n be a complete graph with n odd, say $n = 2m + 1$. For any labeling f that assigns parities to vertices, the number of odd edges $e_f(1)$ is even.*

Proof. Let p be the number of odd-labeled vertices (so $n - p$ even-labeled vertices). An edge is odd exactly when it joins one odd-labeled and one even-labeled vertex, hence

$$e_f(1) = p(n - p).$$

Since n is odd, p and $n - p$ have opposite parity, so their product is even. Therefore $e_f(1)$ is even. \square

Lemma 2.3. *Let G and H be two complete graphs of order m , and let $f_0 : V(G \cup H) \rightarrow \{F_0, F_1, \dots, F_{2m}\}$ be a labeling such that G and H contain g and h vertices labeled with odd Fibonacci numbers, respectively, with $g \geq h$. Define*

$$\tilde{v}_i = |v_G(i) - v_H(i)| \quad \text{and} \quad \tilde{e}_i = |e_G(i) - e_H(i)|, \quad \text{for } i = 0, 1.$$

Then, swapping x odd-labeled vertices from G with x even-labeled vertices from H increases (or decreases) \tilde{v}_1 and correspondingly increases (or decreases) \tilde{e}_0 .

Proof. Let f_1 be the labeling obtained from f_0 by swapping x odd vertices from G with x even vertices from H . After the swap, G has $g - x$ odd vertices and H has $h + x$ odd vertices. The number of edges labeled 1 under f_0 is:

$$e_{f_0}(1) = g(m - g) + h(m - h).$$

Under the labeling f_1 , we have:

$$e_{f_1}(1) = (g - x)(m - h - x) + (h + x)(m - g + x).$$

Clearly if $g \neq h$, as $0 < x \leq g$, $e_{f_1}(1) > e_{f_0}(1)$ when $x < g - h$, and $e_{f_1}(1) < e_{f_0}(1)$ if $x > g - h$. If $x = g - h$, then the swap will result in the same composition of blocks, so we do not consider this case. If $g = h$, then any value of $x > 0$ will result in $e_{f_1}(1)$ being less than $e_{f_0}(1)$. \square

Theorem 2.4. *Let S be any collection of n complete graphs of order m , each labeled with a Fibonacci cordial labeling such that the total number of vertices labeled 0 across all graphs is $v_f(0) = p$. Let S_0 be the specific collection of n graphs isomorphic to K_m , also with $v_f(0) = p$, where each graph (block) contains either $\lceil \frac{p}{nm} \rceil$ or $\lfloor \frac{p}{nm} \rfloor$ many even labeled vertices. Then, if $S \neq S_0$, the total number of odd-labeled edges in S_0 exceeds that in S .*

Proof. Let B be any block in S . Define $r_B = v_B(0) - \lfloor \frac{p}{nm} \rfloor$ and $q_B = \lfloor \frac{p}{nm} \rfloor - v_B(0)$, representing the surplus or deficit of even-labeled vertices relative to the target distribution in S_0 . Let $\{X_1, X_2, \dots, X_k\}$ be the blocks in S with fewer than $\lfloor \frac{p}{nm} \rfloor$ even-labeled vertices (i.e., $q_B > 0$), and after reindexing, assume $q_{X_1} \leq q_{X_2} \leq \dots \leq q_{X_k}$. Similarly, let $\{Y_1, Y_2, \dots, Y_j\}$ be the blocks in S with more than $\lfloor \frac{p}{nm} \rfloor$ even-labeled vertices (i.e., $r_B > 0$), and assume $r_{Y_1} \geq r_{Y_2} \geq \dots \geq r_{Y_j}$.

Now, for each X_i , perform a series of vertex swaps to exchange q_{X_i} odd-labeled vertices from X_i with an equal number of even-labeled vertices from blocks in $\{Y_1, \dots, Y_j\}$, starting with the largest r_{Y_j} available. For example, swap as many vertices as possible between X_1 and Y_1 , then Y_2 , and so on, until q_{X_1} is reduced to 0 or 1. Repeat this process for X_2, X_3, \dots, X_k , ensuring after each step that $q_{X_i} \in \{0, 1\}$ and $r_{Y_i} = 0$.

This redistribution transforms S into S_0 , the unique configuration in which each block contains either $\lfloor \frac{p}{nm} \rfloor$ or $\lceil \frac{p}{nm} \rceil$ even-labeled vertices.

Finally, by Lemma 2.3, each such swap strictly increases the number of odd-labeled edges. Therefore, S_0 contains more odd edges than S , completing the proof. \square

The following Corollary can be derived from the above Theorem immediately.

Corollary 2.5. *Let S be any set of n many K_m graph blocks and $F = \{f_1, f_2, \dots, f_t\}$ be the set of all possible unique injective vertex labeling (up to isomorphism) on S such that $e_{f_i}(0) = p$ for all $1 \leq i \leq t$. Then there exists some f_M such that $e_{f_i}(1) \leq e_{f_M}(1)$ for all $1 \leq i \leq t$ where $i \neq M$. Moreover, the resulting blocks of f_M will each contain either $\lceil \frac{p}{nm} \rceil$ or $\lfloor \frac{p}{nm} \rfloor$ many even nodes.*

Lemma 2.6. *Let S be any set of n many K_m graph blocks, where $nm \equiv 0 \pmod{3}$. The maximum total number of odd edges is achieved when the total number of even-labeled vertices q is maximized, specifically at $q = \frac{nm}{3} + 1$. This maximum occurs when $n - 1$ blocks contain $\frac{m}{3}$ even vertices and one block contains $\frac{m}{3} + 1$ even vertices. Moreover, the total number of odd edges in S cannot exceed*

$$\frac{2nm^2}{9} + \frac{m}{3} - 1.$$

Proof. By Theorem 2.4, for any fixed total number q of even-labeled vertices in S , the number of odd edges is maximized when each block contains either $\lceil q/n \rceil$ or $\lfloor q/n \rfloor$ even vertices. Since $nm \equiv 0 \pmod{3}$, the number of available even Fibonacci labels among the nm distinct vertex labels equals $\frac{nm}{3}$ or $\frac{nm}{3} + 1$, depending on whether the final Fibonacci index is even. The larger value,

$$q_{\max} = \frac{nm}{3} + 1,$$

is attainable under the Fibonacci cordial labeling constraints.

Because each block contributes $x(m-x)$ odd edges when it contains x even vertices, and because the function $x(m-x)$ is strictly increasing on $0 \leq x \leq m/2$, the total number of odd edges increases with q as long as $q \leq nm/2$. Since $q_{\max} = \frac{nm}{3} + 1 < nm/2$, the maximum total number of odd edges occurs at $q = q_{\max}$.

For this value of q , we have

$$\frac{q}{n} = \frac{m}{3} + \frac{1}{n},$$

so the distribution required by Theorem 2.4 is that $(n-1)$ blocks contain $\frac{m}{3}$ even vertices and one block contains $\frac{m}{3} + 1$ even vertices. We now compute the resulting number of odd edges:

$$e_i(1) = \frac{m}{3} \cdot \frac{2m}{3} = \frac{2m^2}{9} \quad \text{for each of the } n-1 \text{ identical blocks,}$$

and for the exceptional block,

$$e_j(1) = \left(\frac{m}{3} + 1\right) \left(\frac{2m}{3} - 1\right) = \frac{2m^2}{9} + \frac{m}{3} - 1.$$

Therefore,

$$e(1) = (n-1) \cdot \frac{2m^2}{9} + \left(\frac{2m^2}{9} + \frac{m}{3} - 1\right) = \frac{2nm^2}{9} + \frac{m}{3} - 1.$$

This is the claimed upper bound. □

The following Proposition provides a critical refinement to the existing literature regarding complete graphs (see Theorem 1.5), and proves that K_{22} is not Fibonacci cordial.

Proposition 2.7. *K_m is not Fibonacci cordial for $m > 18$.*

Proof. Let $E_v = \lfloor m/3 \rfloor + 1$ be the maximum number of even Fibonacci numbers among F_0, \dots, F_{m-1} , and $O_v = m - E_v$ the number of odd-labeled vertices. Now in K_m , the number of odd edges is $e_f(1) = E_v O_v$, and even edges is $e_f(0) = \binom{m}{2} - E_v O_v$. Thus, for Fibonacci cordiality, we require $|e_f(1) - e_f(0)| \leq 1$, which simplifies to:

$$\left| \binom{m}{2} - 2E_v O_v \right| \leq 1.$$

Substituting E_v and O_v :

$$\left| \frac{m(m-1)}{2} - 2 \left(\lfloor \frac{m}{3} \rfloor + 1 \right) \left(m - \lfloor \frac{m}{3} \rfloor - 1 \right) \right| \leq 1.$$

Direct computation shows this inequality fails for $m \geq 21$. For $m = 19, 20$, exhaustive verification confirms the condition is also violated. Thus K_m is not Fibonacci cordial for $m > 18$. □

3. Corona of cycles and complete graphs $(C_n \odot K_m)$

The *corona* $G_1 \odot G_2$ of two graphs is the graph obtained by taking one copy of G_1 , and p_1 copies of G_2 (where $|V(G_1)|=p_1$), and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 . The number of vertices of $C_n \odot K_m$ is $n(m + 1)$, and the number of edges is $n\binom{m}{2} + nm + n = n(m(\frac{m+1}{2}) + 1)$. In the graph $G = C_n \odot K_m$, let $V = \{v_i^t | i \in \mathbb{N}\}$, where $t = 0$ denotes the vertices of C_n (we refer to them as inner vertices), and $t \in \mathbb{Z}_m^+$ represent the vertices that do not lie on the cycle. We refer to them as outer vertices. We can identify the $G = C_n \odot K_m$ as n many K_{m+1} graphs where all the complete graphs are like pendants, and connected by a closed string (C_n). Throughout the paper, we will follow these notations:

- We call each complete subgraph K_{m+1} of G a *block* of G .
- We denote the i^{th} block by K_{m+1}^i .
- For any induced subgraph H of G , the number of even edges is denoted by $e_H(0)$, analogously $e_H(1)$ denote the number of odd degrees.
- $e_H(1) - e_H(0)$ will be denoted as \tilde{e}_H .

Theorem 3.1. $C_n \odot K_4$ is not Fibonacci cordial when $n \equiv 2 \pmod{4}$.

Proof. The graph $C_n \odot K_4$ consists of an inner cycle C_n where a K_5 block is attached at each cycle node. The total number of edges in this corona graph is $|E|= 11n$. For the graph to be Fibonacci cordial, the number of odd edges $e_f(1)$ must satisfy the condition $|e_f(0) - e_f(1)| \leq 1$. Since $|E|= 11n$ is even when $n \equiv 2 \pmod{4}$, cordiality requires $e_f(1) = \frac{11n}{2}$. Because $n = 4k + 2$ for some integer k , we have:

$$e_f(1) = \frac{11(4k + 2)}{2} = 22k + 11.$$

This target value for $e_f(1)$ is an odd number.

We now analyze the structural parity of $e_f(1)$. Let $e_{K_5^i}(1)$ denote the total number of odd edges within each of the n blocks $\{K_5^1, K_5^2, \dots, K_5^n\}$. By Lemma 2.2, for any complete graph K_p where p is odd, the number of odd edges $e_{K_p}(1)$ is always even. Since each block in this graph is a K_5 , every block must contribute an even number of odd edges. Consequently, the total number of odd edges across all n blocks, $\sum_{i=1}^n e_{K_5^i}(1)$, is a sum of even numbers and is therefore even.

Furthermore, by Lemma 2.1, the number of odd edges in the inner cycle C_n , denoted $e_{C_n}(1)$, is always even. The total number of odd edges in the entire graph is the sum of these two components:

$$e_f(1) = \sum_{i=1}^n e_{K_5^i}(1) + e_{C_n}(1).$$

Since each $e_{K_5^i}(1)$ and $e_{C_n}(1)$ are even, their sum $e_f(1)$ must be even. This contradicts the requirement that $e_f(1)$ must be an odd number $(22k + 11)$. Thus, no Fibonacci cordial labeling exists for $n \equiv 2 \pmod{4}$. □

Theorem 3.2. $C_n \odot K_4$ is Fibonacci cordial when $n \not\equiv 2 \pmod{4}$.

Proof. There are $5n$ many vertices and $11n$ many edges in the corona graph $C_n \odot K_4$. The graph consists of a cycle C_n inside, and at each cycle node, a K_4 is attached, which we consider as a block. All possible parity assignments for a K_4 block yield the following imbalances:

$(v(0), v(1))$	$e(1)$	$e(0)$	\tilde{e}
$(0, 5), (5, 0)$	0	10	10
$(1, 4), (4, 1)$	4	6	2
$(2, 3), (3, 2)$	6	4	2

Thus the minimum achievable imbalance in K_4 is 2, attained whenever $v(0) \in \{1, 2, 3, 4\}$. Among these, the assignments using the fewest even labels are $v(0) = 1$ and $v(0) = 2$, corresponding exactly to the Type II and Type I blocks in Figure 1. Using only these two block types therefore minimizes even-label usage while achieving the optimal imbalance.

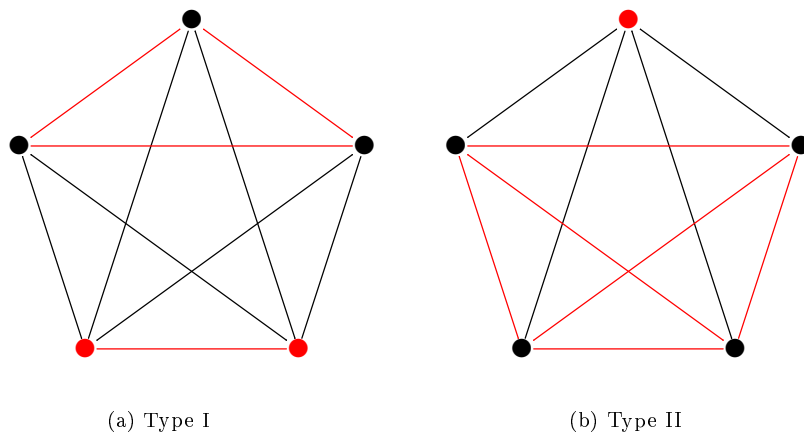


Fig. 1. Two different assignments of the odd and even Fibonacci numbers to the blocks

We will now approach this analysis through cases. It's notable that any node within a $C_n \odot K_4$ block can be selected to be part of the inner cycle. As both Type I and Type II blocks comprise at least one odd and one even node, the inner cycle of any $C_n \odot K_4$ graph exclusively composed of Type I or Type II blocks can accommodate any combination of odd and even nodes. By Lemma 2.1, a cycle can feature any even number of odd edges if there are no constraints on the quantity of odd and even nodes within the cycle.

Table 1 lists for each case of $n \pmod{12}$ the distribution of Type I and Type II blocks, alongside cycle edge parities required for Fibonacci cordiality. The n blocks are partitioned into Type I (2 even nodes, 3 odd nodes) and Type II (1 even node, 4 odd nodes) to manage the consumption of even Fibonacci labels. According to Lemma 1.3, the number of even labels available in the set $\{F_0, F_1, \dots, F_N\}$ is exactly $\lfloor N/3 \rfloor + 1$.

Consider the case $n \equiv 0 \pmod{12}$, where $n = 12q$ for some $q \in \mathbb{N}$. The graph order is $|V| = 60q$, and the labels used are $\{F_0, \dots, F_{60q}\}$. By the inductive result in Lemma 1.3, the available labels consist of $\lfloor (60q)/3 \rfloor + 1 = 20q + 1$ even labels and $(60q + 1) - (20q + 1) = 40q$ odd labels. Assigning $8q$ Type I blocks and $4q$ Type II blocks

requires exactly $8q(2) + 4q(1) = 20q$ even labels and $8q(3) + 4q(4) = 40q$ odd labels. As these requirements are within the available parity counts, injectivity is preserved. Regarding edge balance, this configuration yields $8q(6) + 4q(4) = 64q$ odd edges and $8q(4) + 4q(6) = 56q$ even edges within the blocks. With $10q$ even and $2q$ odd cycle edges (permitted by Lemma 2.1 as $2q$ is even), the total edge counts are $e_f(1) = 64q + 2q = 66q$ and $e_f(0) = 56q + 10q = 66q$. Thus, $|e_f(0) - e_f(1)| = 0$, satisfying the cordiality condition. The remaining cases in Table 1 follow a similar verification of label availability and edge balance. □

Table 1. Assignment of available Fibonacci numbers to the vertices

Cases	n	Type I	Type II	Cycle Even	Cycle Odd
$n \equiv 0 \pmod{12}$	$12q$	$8q$	$4q$	$10q$	$2q$
$n \equiv 4 \pmod{12}$	$12q + 4$	$8q + 3$	$4q + 1$	$10q + 4$	$2q$
$n \equiv 8 \pmod{12}$	$12q + 8$	$8q + 6$	$4q + 2$	$10q + 8$	$2q$
$n \equiv 9 \pmod{12}$	$12q + 9$	$8q + 6$	$4q + 3$	$10q + 7$	$2q + 2$
$n \equiv 1 \pmod{12}$	$12q + 1$	$8q + 1$	$4q$	$10q + 1$	$2q$
$n \equiv 5 \pmod{12}$	$12q + 5$	$8q + 4$	$4q + 1$	$10q + 5$	$2q$
$n \equiv 3 \pmod{12}$	$12q + 3$	$8q + 2$	$4q + 1$	$10q + 3$	$2q$
$n \equiv 7 \pmod{12}$	$12q + 7$	$8q + 5$	$4q + 2$	$10q + 7$	$2q$
$n \equiv 11 \pmod{12}$	$12q + 11$	$8q + 8$	$4q + 3$	$10q + 11$	$2q$

Theorem 3.3. $C_n \odot K_5$ is Fibonacci cordial for all n .

Proof. There are $6n$ vertices and $16n$ edges in the graph, i.e. $|V(C_n \odot K_5)| = 6n$ and $|E(C_n \odot K_5)| = 16n$. The set of Fibonacci numbers used are $\{F_0, F_1, \dots, F_{6n}\}$, which implies there are $2n + 1$ many even Fibonacci labels (by Lemma 1.3) and $4n$ many odd Fibonacci labels available to be assigned to the vertices.

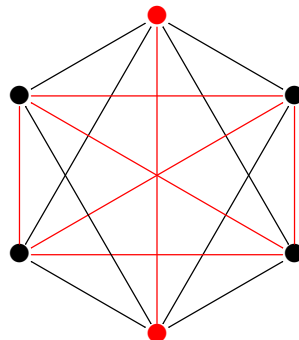


Fig. 2. Assignment of odd and even labels in each block of $C_n \odot K_5$

In each block K_6 , we assign two even and four odd labels (as shown in Figure 2 in such a manner that all the vertices in the cycle will be assigned labels of same parity (all odd labels or all even labels). Hence, during the assignment, total $2n$ even Fibonacci numbers

and $4n$ odd Fibonacci numbers are used. As a result, total $8n$ odd edges (from the blocks) and $8n$ even edges ($7n$ from the blocks and n from the cycle). \square

Theorem 3.4. $C_n \odot K_6$ is not Fibonacci cordial when n is an odd number.

Proof. $|V(C_n \odot K_6)| = 7n$ and $|E(C_n \odot K_6)| = 22n$, which implies for any value of n , total number of edges in the corona graph is $22n$, an even number. If $C_n \odot K_6$ is Fibonacci Cordial, then the induced edge labeling must generate $11n$ many edges of each parity. If n is odd, then we need a total $11n$ i.e., an odd number of odd edges which is impossible to achieve due to the results described in Lemma 2.1. This completes the proof. \square

The above theorem indicates that if $C_n \odot K_6$ is Fibonacci Cordial, then n must be even. Therefore, we investigate the following three cases viz. $n \equiv 0 \pmod{6}$, $n \equiv 2 \pmod{6}$, and $n \equiv 4 \pmod{6}$. We consider three different patterns of assignment of the odd and even labels to the nodes of the blocks (i.e., K_7) as shown in Figure 3. In Type I, II, and III one, two, and three vertices are labeled with even labels, and hence six, five, and four vertices are assigned odd labels respectively. As a result, the respective induced functions have 6 odd edges and 15 even edges in Type I, 10 odd edges and 11 even edges in Type II, and 12 odd and 9 even edges in Type III blocks.

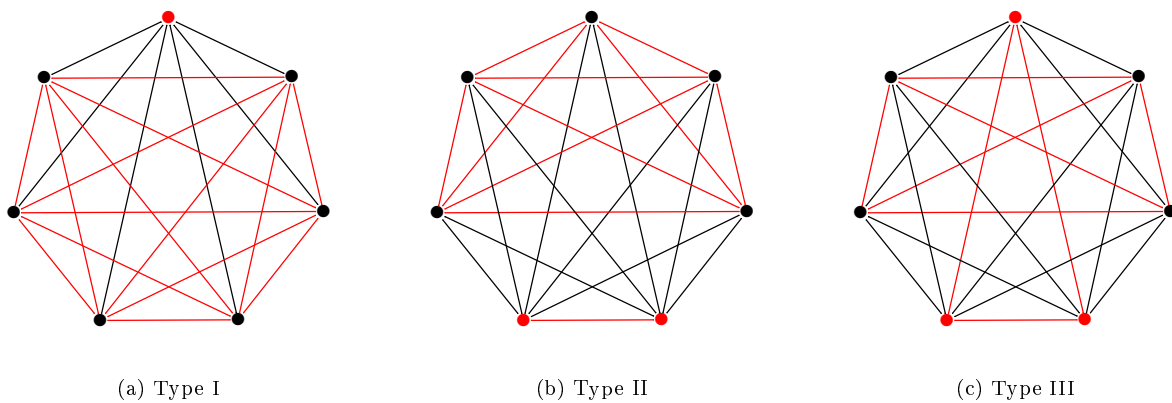


Fig. 3. Three different assignments of the odd and even Fibonacci numbers to the blocks

Theorem 3.5. $C_n \odot K_6$ is Fibonacci Cordial if $n \equiv 0 \pmod{2}$

Proof. Fibonacci Cordial graph labeling when $n \equiv 0, 2$ or $4 \pmod{6}$, i.e. $n = 6p, 6p + 2$, or $6p + 4$ for some $p \in \mathbb{N}$, will be induced using combinations of three types of K_7 graph blocks. Type I, II, and III blocks will contain one, two, and three even nodes respectively.

When $n \equiv 0 \pmod{6}$, we will assign $4p$ many blocks as Type II and $2p$ many as Type III. When $n \equiv 2 \pmod{6}$, we will assign p many blocks as Type I, $2p + 1$ many as Type II, and $3p + 1$ many as Type III. And when $n \equiv 4 \pmod{6}$, we will assign p many blocks as Type I, $2p + 2$ many as Type II, and $3p + 2$ many as Type III.

As an example, for the case of $n \equiv 0 \pmod{6}$, $|V| = 42p$, implying that $28p$ and $14p + 1$ many odd and even node labels are available (Lemma 1.3). We will assign $4p$ many blocks

as Type II and $2p$ many as Type III. The induced edge labeling results in $64p$ many odd and $62p$ many even edges in the blocks. Therefore, if $4p$ cycle edges are even and $2p$ cycle edges are odd, allowed by Lemma, total odd and even edges in the graph will both equal $66q$. Given the assignments previously provided, the remaining two cases can similarly be shown to result in Fibonacci Cordial Labeling. \square

We conclude this section by examining the Fibonacci cordiality of the corona product $C_n \odot K_p$ for large values of p . In particular, we identify specific conditions under which this family fails to admit a Fibonacci cordial labeling.

Theorem 3.6. *The corona graph $C_n \odot K_p$ is not Fibonacci cordial for $p \geq 13$.*

Proof. The total number of edges in $C_n \odot K_p$ is $|E| = \frac{n(p^2+p+2)}{2}$. For Fibonacci cordiality, the number of odd edges $e_f(1)$ must satisfy $e_f(1) \in \{\lfloor |E|/2 \rfloor, \lceil |E|/2 \rceil\}$. Thus, $e_f(1) \approx \frac{n(p^2+p+2)}{4}$.

The maximum number of odd edges available is the sum of the maximum odd edges from n blocks of order $p + 1$ and the cycle C_n . By Lemma 2.6, the maximum odd edges in a block K_{p+1} is given by $M(p) = \frac{2(p+1)^2}{9} + \frac{p+1}{3} - 1$. Since the cycle contributes at most n odd edges, Fibonacci cordiality is only possible if:

$$\frac{n(p^2 + p + 2)}{4} \leq n \left(\frac{2(p + 1)^2}{9} + \frac{p + 1}{3} - 1 \right) + n.$$

Dividing by n and simplifying the quadratic inequality in terms of p :

$$\begin{aligned} \frac{p^2 + p + 2}{4} &\leq \frac{2p^2 + 7p + 5}{9} \\ 9p^2 + 9p + 18 &\leq 8p^2 + 28p + 20 \\ p^2 - 19p - 2 &\leq 0. \end{aligned}$$

This inequality fails for all $p \geq 20$. For $13 \leq p \leq 19$, direct verification of the exact $e_f(1)$ requirement and available Fibonacci parities confirms that no cordial labeling exists. \square

Theorem 3.7. *The corona graph $C_n \odot K_p$ is not Fibonacci cordial in each of the following cases:*

- (a) *When n is odd and $p \equiv 6 \pmod{8}$.*
- (b) *When $p \geq 13$.*

Proof.

Case 1. Let $p = 8k + 6$ for some integer k , and suppose n is odd. Then the total number of edges in $C_n \odot K_p$ is:

$$|E| = n \cdot \left\{ \frac{(8k + 6)(8k + 7)}{2} + 1 \right\} = n(32k^2 + 52k + 22).$$

In order to $C_n \odot K_{8k+6}$ be Fibonacci cordial, the edge labels with 0 and 1 must occur equally, implying the number of edges labeled 1 must be:

$$e_f(1) = \frac{|E|}{2} = n(16k^2 + 26k + 11).$$

This expression is odd when n is odd, contradicting Lemma 2.1, which states that the number of odd edges in a cycle (and thus in C_n) must be even. Hence, no such labeling is possible.

Case 2. The total number of edges for the graph $C_n \odot K_p$ is $\frac{n(p^2+p+2)}{2}$. For Fibonacci Cordiality, the required number of odd edges is approximately

$$e(1) \approx \frac{|E(G)|}{2} = \frac{n(p^2 + p + 2)}{4}.$$

The maximum available odd edges is the sum of the maximum odd edges from n blocks K_{p+1} and the cycle C_n . Using the maximization principle from Lemma 2.6 for the blocks (K_{p+1}) and noting $e_{C_n}(1) \leq n$, the Fibonacci labeling is only possible when,

$$\frac{n(p^2 + p + 2)}{4} \leq \left(\frac{2n(p + 1)^2}{9} + \frac{p + 1}{3} - 1 \right) + n.$$

This inequality holds for $n \geq 3$ and $1 \leq p \leq 12$, thus establishing the result. □

Example 3.8. We now provide an explicit example of $C_3 \odot K_4$ to demonstrate the Fibonacci cordial labeling construction. The graph order is $|V| = 5n = 15$, and we assign labels from the set $\{F_0, F_1, \dots, F_{15}\}$. According to Lemma 1.3, the number of available even Fibonacci labels in this set is $v_f(0) = \lfloor 15/3 \rfloor + 1 = 6$, and the number of odd labels is $v_f(1) = 16 - 6 = 10$.

Per Table 1, for the case $n = 3$, we utilize two Type I blocks and one Type II block. The vertex parity distribution is assigned as follows:

- Type I blocks (K_5^1, K_5^2): 2 even vertices and 3 odd vertices each.
- Type II block (K_5^3): 1 even vertex and 4 odd vertices.

The total number of even vertices required is $2(2) + 1(1) = 5$, which is within the available limit of 6 even labels. We define the specific assignment of even Fibonacci labels to vertices v_j^i (where i is the block index and j is the vertex index within the block) as follows:

$$f(v_0^1) = F_0, \quad f(v_1^1) = F_3, \quad f(v_0^2) = F_6, \quad f(v_1^2) = F_9, \quad f(v_0^3) = F_{12}.$$

The remaining 10 vertices are assigned the available odd Fibonacci labels from the set $\{F_1, F_2, F_4, F_5, F_7, F_8, F_{10}, F_{11}, F_{13}, F_{14}\}$ injectively. Regarding the induced edge labeling:

- Each Type I block generates 6 odd edges and 4 even edges.
- The Type II block generates 4 odd edges and 6 even edges.

- By assigning the inner cycle C_3 vertices such that they are all even (e.g., vertices v_0^1, v_0^2, v_0^3), the cycle contributes 0 odd edges and 3 even edges.

The resulting total edge counts are $e_f(1) = 6+6+4+0 = 16$ and $e_f(0) = 4+4+6+3 = 17$. Thus, $|e_f(0) - e_f(1)| = 1$, which satisfies the condition for Fibonacci cordiality.

4. Corona product of complete bipartite graphs and complete graphs

Theorem 4.1. K_{m_1, m_2} is Fibonacci cordial if either m_i is even and $m_i \leq 2m_j$ or $m_1 + m_2 \leq 12$ where both m_i is odd.

Proof. Let us denote the vertices of K_{m_1, m_2} by $\{v_{11}, v_{12}, \dots, v_{1m}, v_{21}, v_{22}, \dots, v_{2n}\}$. In order to make K_{m_1, m_2} Fibonacci cordial, let us label p_i many vertices of $\{v_{ij}\}$ by even Fibonacci labels, and the rest by odd.

$e_f(1) = p_1(m_2 - p_2) + p_2(m_1 - p_1)$ and $e_f(0) = p_1p_2 + (m_1 - p_1)(m_2 - p_2)$, we get

$$e_f(1) - e_f(0) = (2p_1 - m_1)(m_2 - 2p_2), \tag{1}$$

upon simplification. Thus $|e_f(1) - e_f(0)| \leq 1$ if either $m_i = 2p_i$ or $|m_1 - 2p_1| = 1 = |m_2 - 2p_2|$.

First, we consider the former case. Without loss of generality, we consider the case where m_1 is even, and $m_1/2$ of them are labeled with even Fibonacci labels. As there are $\lfloor (m_1 + m_2)/3 \rfloor + 1$, many even labels, $m_1/2 \leq \lfloor (m_1 + m_2)/3 \rfloor + 1 \leq (m_1 + m_2)/3 + 1$, which implies that $m_1 \leq 2m_2 + 6$.

Now for the later case, as $m_i = 2p_i \pm 1$, $p_1 + p_2 \leq \lfloor (m_1 + m_2)/3 \rfloor + 1 \leq (2k_1 + 2k_2 + 2)/3 + 1$, which simplifies to $p_1 + p_2 \leq 5$, and consequently $m_1 + m_2 \leq 12$. □

The corona graph $K_{n,n} \odot K_p$ consists of an inner complete bipartite graph $K_{n,n}$ where each of the $2n$ vertices is identified with one vertex of a K_{p+1} block. This structure can be viewed as comprising two columns, each containing n copies of the graph K_{p+1} . We refer to each such copy of K_{p+1} as a block, and to the two groups of n blocks as the left and right columns, corresponding to the two partite sets of $K_{n,n}$. The total number of vertices and edges is given by: $|V(G)| = 2n(p+1)$, $|E(G)| = 2n\binom{p+1}{2} + n^2 = np(p+1) + n^2$. Following the structural notation established in the previous section, we define the vertex set of the corona graph $G = K_{n,n} \odot K_p$ as:

$$V = \{v_{i,j,k} \mid 1 \leq i \leq 2, 1 \leq j \leq n, 0 \leq k \leq p\},$$

where $k = 0$ denotes the *root vertices* belonging to the base bipartite graph $K_{n,n}$, and $k \in \{1, \dots, p\}$ represents the outer vertices of the p -order complete graphs attached to each root. The index i partitions the graph into two columns, with $i = 1$ and $i = 2$ representing the left and right partite sets, respectively. Finally, j denotes the specific level or row within each partite set.

Theorem 4.2. $K_{n,n} \odot K_1$ is Fibonacci Cordial for all $n \in \mathbb{N}$.

Proof. This graph was examined in six cases based on the value of n . For each case, we have provided an assignment of nodes and edges that produces a Fibonacci Cordial graph. For the first case, as an example, we have shared the calculations that prove its Fibonacci Cordiality. The proofs of the other five cases can be calculated in the same manner.

Case 1. $n \equiv 0 \pmod{6}$, i.e. $n = 6x$ for some $x \in \mathbb{N}$. This implies there are $24x$ vertices and $36x^2 + 12x$ edges. Therefore, $8x + 1$ many even Fibonacci numbers (by Lemma 1.3) and $16x$ many odd Fibonacci numbers to be used as vertex labels. In each column of the bipartite graph, we will assign $3x$ many nodes as odd and $3x$ many nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: x many will be even leaf nodes connected to even nodes in the left column, x many will be even leaf nodes connected to odd nodes in the left column, $2x$ will be odd leaf nodes connected to even nodes in the left column, and $2x$ will be odd leaf nodes connected to odd nodes in the left column. Each leaf node connected to a node in the right column of the bipartite graph will be labeled odd. The induced node labeling generates: $v_f(1) = 16x$ and $v_f(0) = 8x$. While these vertex parities are not balanced, they are within the available limits of the Fibonacci sequence indices as established in Lemma 1.3. Because the induced edge labeling results in $e_f(1) = e_f(0) = 18x^2 + 6x$, the edge-balance condition $|e_f(0) - e_f(1)| \leq 1$ for Fibonacci cordiality is strictly satisfied.

Case 2. $n \equiv 1 \pmod{6}$ i.e. $n = 6x + 1$ for some $x \in \mathbb{N}$. In each column of the bipartite graph, we will assign $3x + 1$ nodes as odd and q nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: x many will be even leaf nodes connected to even nodes in the left column, $x - 1$ many will be even leaf nodes connected to odd nodes in the left column, $2x$ will be odd leaf nodes connected to even nodes in the left column, and $2x + 2$ will be odd leaf nodes connected to odd nodes in the left column. Finally, each leaf node connected to a node in the right column of the bipartite graph will be labeled odd.

Case 3. $n \equiv 2 \pmod{6}$ i.e. $n = 6x + 2$ for some $x \in \mathbb{N}$. In the left column of the bipartite graph, we will assign $3x + 2$ many nodes as odd and $3x$ many nodes as even. In the right column of the bipartite graph, we will assign $3x + 1$ many nodes as odd and $3x + 1$ many nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: x many will be even leaf nodes connected to even nodes in the left column, $x + 1$ many will be even leaf nodes connected to odd nodes in the left column, $2x$ many will be odd leaf nodes connected to even nodes in the left column, and $2x + 1$ many will be odd leaf nodes connected to odd nodes in the left column. Finally, each leaf node connected to a node in the right column of the bipartite graph will be labeled odd.

Case 4. $n \equiv 3 \pmod{6}$ i.e. $n = 6x + 3$ for some $x \in \mathbb{N}$. In each column of the bipartite graph, we will assign $3x + 2$ many nodes as odd and $3x + 1$ many nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: $x + 1$ many will be even leaf nodes connected to even nodes in the left column, x many will be even leaf nodes connected to odd nodes in the left column, $2x$ many will be odd leaf nodes connected to even nodes in the left column, and $2x + 2$ many will be odd

leaf nodes connected to odd nodes in the left column. Finally, each leaf node connected to a node in the right column of the bipartite graph will be labeled odd.

Case 5. $n \equiv 4 \pmod{6}$ i.e. $n = 6x + 4$ for some $x \in \mathbb{N}$. In each column of the bipartite graph, we will assign $3x + 2$ many nodes as odd and $3x + 2$ many nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: $x + 1$ many will be even leaf nodes connected to even nodes in the left column, $x + 1$ many will be even leaf nodes connected to odd nodes in the left column, $2x + 1$ many will be odd leaf nodes connected to even nodes in the left column, and $2x + 1$ many will be odd leaf nodes connected to odd nodes in the left column. Finally, each leaf node connected to a node in the right column of the bipartite graph will be labeled odd.

Case 6. $n \equiv 5 \pmod{6}$ i.e. $n = 6x + 5$ for some $x \in \mathbb{N}$. In each column of the bipartite graph, we will assign $3x + 3$ many nodes as odd and $3x + 2$ many nodes as even. Labels for the leaf nodes connected to each node in the left column of the bipartite graph are as follows: $x + 1$ many will be even leaf nodes connected to even nodes in the left column, x many will be even leaf nodes connected to odd nodes in the left column, $2x + 1$ will be odd leaf nodes connected to even nodes in the left column, and $2x + 3$ will be odd leaf nodes connected to odd nodes in the left column. Finally, each leaf node connected to a node in the right column of the bipartite graph will be labeled odd.

□

Theorem 4.3. $K_{n,n} \odot K_2$ is Fibonacci Cordial if $n \equiv 0 \pmod{2}$

Proof. Let us assume that $n \equiv 0 \pmod{2}$, i.e., $n = 2q$ for some $q \in \mathbb{N}$. This implies there are $12q$ vertices and $4q^2 + 12q$ edges. Therefore, $4q + 1$ many even Fibonacci numbers (by Lemma 1.3) and $8q$ many odd Fibonacci numbers to be used as vertex labels. Let each block in the left column be Type I and let each block in the right column be Type II, as shown in Figure 4.



Fig. 4. Two different assignments of the odd and even Fibonacci numbers to the blocks

Type I consists of only odd nodes, so each block in the left column will have an odd node included in the bipartite graph. Type II consists of one odd and two even nodes, so each Type II block can have either an odd or an even node included in the bipartite graph. Let $\frac{n-2}{2}$ many blocks in the right column have an odd node included in the bipartite graph, and $n - \frac{n-2}{2}$ many have an even node included in the bipartite graph. The induced node labeling generates: $v_f(1) := 4n$ and $v_f(1) := 2n$. These values are within the available limits. The induced edge labeling generates: total odd edges in the blocks as $0 + 2n = 2n$,

total even edges in the blocks as $3n + n = 4n$, total odd edges in the bipartite graph as $n(n - \frac{n(n-2)}{2})$, and total even edges in the bipartite graph as $\frac{n(n-2)}{2}$, resulting in $\frac{n^2+6n}{2}$ total odd edges and $\frac{n^2+6n}{2}$ total even edges. This ensures that the defined labeling satisfies all the conditions of Fibonacci cordial labeling. \square

Theorem 4.4. $K_{n,n} \odot K_3$ is Fibonacci Cordial for all n .

Proof. In this graph, the total number of vertices and edges are given by $|V(K_{n,n} \odot K_3)| = 8n$ and $|E(K_{n,n} \odot K_3)| = n^2 + 12n$ respectively. In this case, each block forms K_4 . The assignment of the vertex labels to the $2n$ many blocks is a combination of two types I and II as shown in Figure 5. There are three other possible arrangements of vertex labels to the blocks but we consider these two as they give rise to 3 edges of each parity and hence it is already balanced. To achieve Fibonacci cordial-ness it is sufficient to balance the induced edges in the bipartite graph. If n is even then the left column has $\frac{n}{2}$ even nodes and $\frac{n}{2}$ odd nodes and the right column assignment is insignificant. If n is odd then the left column has $\frac{n+1}{2}$ even nodes and $\frac{n-1}{2}$ odd nodes and the right column has $\frac{n-1}{2}$ odd nodes and $\frac{n+1}{2}$ even nodes. The blocks are complete graphs, so their orientation can be determined to achieve the desired number of odd vertices and even vertices on the columns of the bipartite graph. We discuss the following six cases.

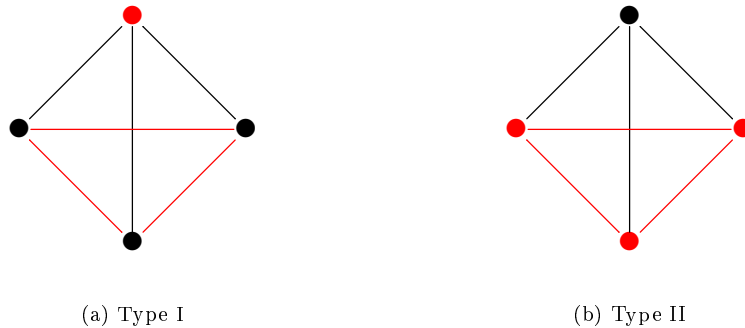


Fig. 5. Two different assignments of the odd and even Fibonacci numbers to the blocks

Case 1. Let $n \equiv 0 \pmod{6}$ then $n = 6q$ for some $q \in \mathbb{N}$.

Consequently, $|V(K_{n,n} \odot K_3)| = 48q$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 72q$. Thus, there are $16q + 1$ many even (proceeds from Lemma 1.3) and $32q$ many odd Fibonacci numbers available to be used for vertex labeling. Among the $12q$ blocks, $10q$ are of Type I, and $2q$ are of Type II. This results in $v_f(0) = 16q$ and $v_f(1) = 32q$, both falling within the intended limits.

There are $3q$ even and $3q$ odd nodes on both the columns of the bipartite graph. Consequently, we have $e_f(0) = e_f(1) = 18q^2 + 36q$ which satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ as required.

Case 2. Let $n \equiv 1 \pmod{6}$ then $n = 6q + 1$ for some $q \in \mathbb{N}$.

This implies $|V(K_{n,n} \odot K_3)| = 48q + 8$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 84q + 13$. So, there are $16q + 3$ many even and $32q + 6$ many odd Fibonacci numbers available to be used for

vertex labeling. Out of a total $12q + 2$ blocks, $10q + 2$ are of Type I, and $2q$ are of Type II. This leads to $v_f(0) = 16q + 2$ and $v_f(1) = 32q + 6$. Observe that both these values are within the intended limits.

As per the description of the assignment, there are $3q + 1$ even and $3q$ odd nodes on the left column and $3q + 1$ odd and $3q$ even nodes on the right column of the bipartite graph. As a result, we have $e_f(0) = e_f(1) = 18q^2 + 42q + 6$ which fulfills the condition $|e_f(0) - e_f(1)| \leq 1$ as stipulated.

Case 3. Let $n \equiv 2 \pmod{6}$ then $n = 6q + 2$ for some $q \in \mathbb{N}$.

Then $|V(K_{n,n} \odot K_3)| = 48q + 16$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 96q + 28$. So, there are $16q + 6$ many even and $32q + 11$ many odd Fibonacci numbers available to be used for vertex labeling. Out of a total of $12q + 4$ blocks, $10q + 3$ are of Type I, and $2q + 1$ are of Type II. This leads to $v_f(0) = 16q + 6$ and $v_f(1) = 32q + 10$, both satisfying the intended limits. As per the assignment, $3q + 1$ even and $3q + 1$ odd nodes on both the columns of the bipartite graph. As a result, we have $e_f(0) = e_f(1) = 18q^2 + 48q + 14$ which satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ as required.

Case 4. Let $n \equiv 3 \pmod{6}$ then $n = 6q + 3$ for some $q \in \mathbb{N}$.

Then $|V(K_{n,n} \odot K_3)| = 48q + 24$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 108q + 45$. Consequently, there are $16q + 9$ many even and $32q + 16$ many odd Fibonacci numbers available to be used for vertex labeling. Out of a total of $12q + 6$ blocks, $10q + 5$ are of Type I, and $2q + 1$ are of Type II. This leads to $v_f(0) = 16q + 8$ and $v_f(1) = 32q + 16$. Observe that both these values are within the intended limits.

There are $3q + 2$ even and $3q + 1$ odd nodes on the left column and $3q + 2$ odd and $3q + 1$ even nodes on the right column of the bipartite graph. As a result, we have $e_f(0) = 18q^2 + 54q + 22$ and $e_f(1) = 18q^2 + 54q + 23$ which fulfills the condition $|e_f(0) - e_f(1)| \leq 1$ as stipulated.

Case 5. Let $n \equiv 4 \pmod{6}$ then $n = 6q + 4$ for some $q \in \mathbb{N}$.

This implies $|V(K_{n,n} \odot K_3)| = 48q + 32$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 96q + 28$. So, there are $16q + 11$ many even and $32q + 22$ many odd Fibonacci numbers available to be used for vertex labeling. Out of a total of $12q + 8$ blocks, $10q + 7$ are of Type I, and $2q + 1$ are of Type II. This leads to $v_f(0) = 16q + 10$ and $v_f(1) = 32q + 22$, both satisfying the intended limits. As per the assignment, $3q + 2$ even and $3q + 2$ odd nodes on both the columns of the bipartite graph. As a result, we have $e_f(0) = e_f(1) = 18q^2 + 60q + 32$ which satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ as required.

Case 6. Let $n \equiv 5 \pmod{6}$ then $n = 6q + 5$ for some $q \in \mathbb{N}$.

Then $|V(K_{n,n} \odot K_3)| = 48q + 40$ and $|E(K_{n,n} \odot K_3)| = 36q^2 + 132q + 85$. So, there are $16q + 14$ many even and $32q + 27$ many odd Fibonacci numbers available to be used for vertex labeling. Out of a total of $12q + 10$ blocks, $10q + 8$ are of Type I, and $2q + 2$ are of Type II. This leads to $v_f(0) = 16q + 14$ and $v_f(1) = 32q + 26$. Observe that both these values are within the intended limits.

There are $3q + 3$ even and $3q + 2$ odd nodes on the left column and $3q + 3$ odd and $3q + 2$ even nodes on the right column of the bipartite graph. As a result, we have $e_f(0) = 18q^2 + 66q + 42$ and $e_f(1) = 18q^2 + 66q + 43$ which fulfills the condition $|e_f(0) - e_f(1)| \leq 1$ as stipulated.

□

Example 4.5. We provide an explicit Fibonacci cordial labeling for $K_{2,2} \odot K_3$ using the label set $\{F_0, \dots, F_{16}\}$. By Lemma 1.3, we utilize the 6 available even labels $\{F_0, F_3, F_6, F_9, F_{12}, F_{15}\}$. The 16 vertices are labeled using the triple-index notation $v_{i,j,k}$ as follows:

- *Roots (Inner $K_{2,2}$, $k = 0$):* $f(v_{1,1,0}) = F_0$, $f(v_{1,2,0}) = F_4$, $f(v_{2,1,0}) = F_{12}$, $f(v_{2,2,0}) = F_{10}$.
- *Outer nodes for $i = 1$ ($k \in \{1, 2, 3\}$):* Level $j = 1$: F_3, F_1, F_2 . Level $j = 2$: F_6, F_9, F_5 .
- *Outer nodes for $i = 2$ ($k \in \{1, 2, 3\}$):* Level $j = 1$: F_{15}, F_7, F_8 . Level $j = 2$: F_{11}, F_{13}, F_{14} .

(The labels F_{16} remain unassigned.) The internal block edges generate $\sum e_{K_4}(1) = 4 + 4 + 4 + 0 = 12$ and $\sum e_{K_4}(0) = 2 + 2 + 2 + 6 = 12$. The $K_{2,2}$, with root parities $\{E, O\}$ for both partite sets ($i = 1$ and $i = 2$), induces $e_{K_{2,2}}(1) = 2$ and $e_{K_{2,2}}(0) = 2$. Thus, the total counts are $e_f(1) = 14$ and $e_f(0) = 14$, yielding $|e_f(0) - e_f(1)| = 0$, which satisfies Fibonacci cordiality.

We close this section with the following result, which examines the corona product of $K_{n,n}$ and K_p for sufficiently large values of p . The theorem establishes a condition under which this graph fails to be Fibonacci cordial.

Theorem 4.6. *The graph $K_{n,n} \odot K_{p-1}$ is not Fibonacci cordial when*

$$p > \frac{3(3n + 2 + \sqrt{4n^3 + 9n^2 + 8n + 4})}{2n}.$$

Proof. By Lemma 2.6, if $K_{n,n} \odot K_{p-1}$ is Fibonacci cordial, then the total number of odd-labeled edges within its $2n$ block copies of K_p cannot exceed

$$\left(\frac{4np^2}{9} + \frac{p}{3} - 1 \right).$$

Additionally, the total number of odd-labeled edges within the inner bipartite graph $K_{n,n}$ is at most n^2 . On the other hand, the total number of edges in $K_{n,n} \odot K_{p-1}$ is

$$\frac{2n \cdot p(p-1)}{2} + n^2 = np(p-1) + n^2.$$

Therefore, it can not be Fibonacci Cordial if:

$$np(p-1) + n^2 - 2 \left(\frac{4np^2}{9} + \frac{p}{3} - 1 + n^2 \right) > 1.$$

Solving this inequality gives the bound on p :

$$p > \frac{3(3n + 2 + \sqrt{4n^3 + 9n^2 + 8n + 4})}{2n}.$$

Hence, under this condition, $K_{n,n} \odot K_{p-1}$ cannot be Fibonacci cordial. □

5. Conclusion

In this research, we investigated the Fibonacci cordiality of specific corona graph families, namely $C_n \odot K_p$ for $p \in \{4, 5, 6\}$. This work provides a foundational analysis of how Fibonacci cordiality behaves within corona structures. By extending our study to larger values of p , we established exact mathematical conditions under which the Fibonacci cordiality of $C_n \odot K_p$ fails as the order of the corona increases.

Furthermore, we explored the corona family of symmetric bipartite graphs and complete graphs, denoted as $K_{n,n} \odot K_p$ for $p \in \{1, 2, 3\}$. This investigation broadens the understanding of Fibonacci cordiality across diverse graph classes. By deriving specific bounds for cases where n and p are sufficiently large, we have highlighted the limitations of cordiality in increasingly complex graph structures. These findings provide a framework for future explorations into more generalized graph products and structural labeling constraints.

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