

Partial hyperbolicity and average pseudo-orbit tracing property

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ABSTRACT

In this paper, given a homeomorphism f of a compact metric space X , we show that the set of all asymptotic average shadowable points of f is an open and invariant set and f has the asymptotic average shadowing property if and only if the set of all asymptotic average shadowable points of f is X if and only if any Borel probability measure μ of X has the asymptotic average shadowing property.

Keywords: shadowing property, shadowable point, asymptotic average shadowing property, asymptotic average shadowable point, Borel probability measure

2020 Mathematics Subject Classification: 37C50, 37D05, 37D30.

1. Introduction

The pseudo-orbit tracing properties are very useful notion to investigate of the stability theory. Sakai [13] and Robinson [12] proved that a diffeomorphism f of a compact smooth manifold M has the robustly pseudo-orbit tracing property if and only if it is structurally stable. Later, a type of the pseudo-orbit tracing property was introduced in [1] which is called the average pseudo-orbit tracing property. The average pseudo-orbit tracing property has been studied by many people (see. [4, 6, 5, 7, 9, 10]). For example, Sakai proved in [14] that a diffeomorphism f of a two dimensional manifold M has the robustly average pseudo-orbit tracing property then it is Anosov. However, it is still open if the dimension of M is greater than 3. So we consider a weakly hyperbolic (partially hyperbolic) dynamical system. Regarding this system, Bonatti, Díaz and Turcat [2] proved that if a diffeomorphism f of the three dimensional manifold M is partially hyperbolic then it does not have the pseudo-orbit tracing property. Lee and Ahn [8] proved that if a

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Received 26 Dec 2025; Revised 10 Jan 2026; Accepted 28 Jan 2026; Published Online 07 Mar 2026.

DOI: [10.61091/um126-09](https://doi.org/10.61091/um126-09)

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diffeomorphism f of any dimensional manifold M is partially hyperbolic then it does not have the pseudo-orbit tracing property which is a generalization of the result of [2]. For the previous results of the pseudo-orbit tracing property([2, 8]), we deal with the average pseudo-orbit tracing property.

2. Basic notions and main Theorem

In this paper, we assume that M is a compact smooth Riemannian manifold. We denote $\text{Diff}(M)$ as the set of all C^1 -diffeomorphisms of M with the C^1 -topology. Let d be a metric on M induced by a Riemannian metric $\|\cdot\|$ on the tangent bundle TM .

For any $f \in \text{Diff}(M)$, we set $\text{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$ and it is said to be the orbit of x . For any $\delta > 0$, a sequence $\{x_i : i \in \mathbb{Z}\}$ is a δ pseudo-orbit of f if for all $i \in \mathbb{Z}$, we have

$$d(f(x_i), x_{i+1}) < \delta.$$

A diffeomorphism $f : M \rightarrow M$, f has the *pseudo-orbit tracing property* if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ pseudo-orbit $\{x_i : i \in \mathbb{Z}\}$ of f can be ϵ pseudo-orbit traced by some point of f , that is,

$$d(f^i(z), x_i) < \epsilon, \forall i \in \mathbb{Z}.$$

For any $\delta > 0$, a sequence $\{x_i : i \in \mathbb{Z}\}$ is a δ average pseudo-orbit of f if for any $i \in \mathbb{Z}$, there is a positive integer N such that for any $n \geq N$ and $i \in \mathbb{Z}$, we have

$$\frac{1}{n} \sum_{k=1}^n d(f(x_{k+i}), x_{k+i+1}) < \delta.$$

Note that we can see that if a sequence $\{x_i : i \in \mathbb{Z}\}$ is a δ pseudo-orbit of f then the sequence is a δ average pseudo-orbit of f .

For any $\epsilon > 0$, a δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ is *positively ϵ -traced in average* by some point $z \in \Lambda$ if

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Analogously, we define *negatively average pseudo-orbit tracing property*.

Definition 2.1. Let Λ be a closed f -invariant set. We say that a diffeomorphism f of M has the average pseudo-orbit tracing property in Λ if for any $\epsilon > 0$, there is $\delta > 0$ such that any δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ can be positively and negatively ϵ -traced in average by some point $z \in \Lambda$ (z is called an average pseudo-orbit tracing point), that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Moreover, if $\Lambda = M$ then we say that f has the *average pseudo-orbit tracing property*. Note that if the average pseudo-orbit tracing point contained in M , then f has the average

pseudo-orbit tracing property on Λ . That is, any $\epsilon > 0$, a δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ is ϵ -traced in average by some point $z \in M$ if

$$\limsup_{n \rightarrow \pm\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.$$

For a closed f -invariant set $\Lambda \subset M$, Λ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a continuous Df -invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n,$$

for all $x \in \Lambda$ and $n \geq 0$. Moreover, if $\Lambda = M$ then we say that f is *Anosov*.

A closed f -invariant set Λ has a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df -invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n,$$

for all $x \in \Lambda$ and $n \geq 0$. It is clear that if a closed f -invariant set Λ is hyperbolic then Λ has a dominated splitting.

If either E is uniformly contracting or F is uniformly expanding then Λ is called *partially hyperbolic* for f . It is clear that if Λ is hyperbolic then it has a dominated splitting, also, it is partially hyperbolic. If $\Lambda = M$ then we say that f is a *partially hyperbolic*. A point $x \in M$ is said to be *periodic* if there is $n \in \mathbb{N}$ such that $f^n(x) = x$. Denote the set of all periodic points of f as $P(f)$.

In [14], Sakai showed that the relation with the average pseudo-orbit tracing property on the two-dimensional manifold M and hyperbolicity, and for any dimensional manifold M , Lee and Wen [10] proved that if a diffeomorphism f has the robustly average pseudo-orbit tracing property then it has a dominated splitting.

For a hyperbolic periodic point p of f with period $\pi(p)$, the stable manifold of p , $W^s(p) = \{x \in M : d(f^{\pi(p)i}(x), p) \rightarrow 0 \text{ as } i \rightarrow \infty\}$ and the unstable manifold of p , $W^u(p) = \{x \in M : d(f^{\pi(p)i}(x), p) \rightarrow 0 \text{ as } i \rightarrow -\infty\}$. Let $H(p)$ be the set of closures of the non-empty transverse intersection of $W^s(p)$ and $W^u(p)$.

In [6], Lee proved that if the homoclinic class $H(p)$ has a dominated splitting, every periodic point in $H(p)$ is hyperbolic, and a diffeomorphism f has the average pseudo-orbit tracing property on $H(p)$, then $H(p)$ is hyperbolic.

A subset \mathcal{R} is *residual* if it contains a dense G_δ subset of $\text{Diff}(M)$. A dynamic property of diffeomorphisms is called C^1 *generic* if it holds on a residual set in $\text{Diff}(M)$.

Lee proved in [7] that there is a residual subset \mathcal{R} of the set of two dimensional manifold M such that a diffeomorphism $f \in \mathcal{R}$ has the average pseudo-orbit tracing property then it is Anosov. It is a generalization of result of Sakai [14]. Lee and Park proved in [9] that there is a residual subset \mathcal{R} of $\text{Diff}(M)$ such that if a diffeomorphism $f \in \mathcal{R}$ has the average pseudo-orbit tracing property and every periodic points p and q are homoclinically related then it is Anosov, where the periodic points p and q are *homoclinically related* if $W^s(p)$ and $W^u(q)$ are non-empty transversal intersections and $W^u(p)$ and $W^s(q)$ are too.

In the paper, we consider the relationship with a type of the average pseudo-orbit tracing property and a homoclinic tangent associated with a hyperbolic periodic point p .

Theorem 2.2 (Theorem A). *For any C^1 generic $f \in \text{Diff}(M)$, if f have the average pseudo-orbit tracing property then f is partially hyperbolic.*

3. Proof of Theorem 2.2

Let M be as before, and let $f : M \rightarrow M$ be a diffeomorphism.

Lemma 3.1. *Let $\Lambda \subset M$ be a closed f -invariant set. If f has the average pseudo-orbit tracing property, then f has the average pseudo-orbit tracing property in Λ .*

Proof. Since Λ is a closed set, a closed set $A \subset \Lambda$, there is $r > 0$ such that $B(A, r) \subset \Lambda$, where $B(A, r)$ is a r -neighborhood of A . For $\epsilon = r$, let $0 < \delta \leq \epsilon$ be the number of the average pseudo-orbit tracing property for f . Let $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ be a δ average pseudo-orbit of f such that $B(\overline{\{x_i : i \in \mathbb{Z}\}}, \epsilon) \subset \Lambda$. Assume that the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ can be ϵ traced in average by a point $z \in M$ such that $\text{Orb}_f(z) \cap \Lambda = \emptyset$. Then we see that $d(f^i(z), x_i) \geq \epsilon$ for all $i \in \mathbb{Z}$. This is a contradiction by the average pseudo-orbit tracing property for f . Thus the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$ can be ϵ traced in average by a point $z \in M$ such that $\text{Orb}_f(z) \cap \Lambda \neq \emptyset$. Indeed, since Λ is f -invariant set, if there is $j \in \mathbb{Z}$ such that $f^j(z) \in \Lambda$ then $f^i(f^j(z)) \in \Lambda$ for all $i \in \mathbb{Z}$. Thus, z is a average pseudo-orbit tracing point in Λ , and so f has the average pseudo-orbit tracing property in Λ . \square

Note that if f has the average pseudo-orbit tracing property, then f^i has the average pseudo-orbit tracing property, for all $i \in \mathbb{Z} \setminus \{0\}$ (see [11, Theorem 3.2]).

For a hyperbolic periodic point p of f , we assume that $x \in W^s(p) \cap W^u(p) \setminus \{p\}$ a non-transversal point of f . Let $\Gamma = \text{Orb}(x) \cup \{\text{Orb}(p)\}$.

Lemma 3.2. *If f have the average pseudo-orbit tracing property in Γ , then the average pseudo-orbit tracing point z belongs to neither $W^s(p)$ nor $W^u(p)$.*

Proof. For simplicity, assume that p is a fixed point, i.e., $f(p) = p$. Since $x \in W^s(p) \cap W^u(p) \setminus \{p\}$, for any sufficiently small $r > 0$, there exists an integer $l > 0$ such that $f^l(x) \in B_{r/4}(p)$, $f^{l-1}(x) \notin B_{r/4}(p)$, $f^{-l}(x) \in B_{r/4}(p)$ and $f^{-l+1}(x) \notin B_{r/4}(p)$. Then we have a sequence $\xi_1 = \{x_i : i = 0, 1, \dots, 2l + 1\}$ such that

$$\begin{aligned} \{x_0 = p, x_1 = f^{-l}(x), x_2 = f^{-l+1}(x), \dots, \\ x_l = f^{-1}(x), x_{l+1} = x, x_{l+2} = f(x), \dots, \\ x_{2l} = f^{l-1}(x), x_{2l+1} = p\}. \end{aligned}$$

The sequence $\xi_1 = \{x_i : i = 0, 1, \dots, 2l + 1\}$ constitutes an $(r/4l)$ -average pseudo-orbit

of f . Indeed, we have:

$$\begin{aligned} \frac{1}{2l+1} \sum_{i=0}^{2l} d(f(x_i), x_{i+1}) &= \frac{1}{2l+1} (d(f(x_0), x_1) + d(f(x_{2l}), x_{2l+1})) \\ &= \frac{1}{2l+1} (d(p, x_1) + d(f(f^{l-1}(x)), p)) \\ &< \frac{1}{2l+1} \left(\frac{r}{4} + \frac{r}{4}\right) < \frac{r}{4l}. \end{aligned}$$

Next, we construct a sequence $\xi_2 = \{x_i : i = 0, 1, \dots, 4l + 1\}$ by extending ξ_1 with a sequence of p :

$$\underbrace{\{p(= x_0), f^{-l}(x), \dots, f^{-1}(x), x, f(x), \dots, f^{l-1}(x)(= x_{2l}), p(= x_{2l+1}), p, \dots, p, p(= x_{4l}), f^{-l}(x)(= x_{4l+1})\}}_{2l}.$$

Then we see that

$$\frac{1}{4l+1} \sum_{i=0}^{4l} d(f(x_i), x_{i+1}) < \frac{1}{4l+1} \cdot \frac{3r}{4} < \frac{r}{4l}.$$

Thus, ξ_2 is also an $(r/4l)$ -average pseudo-orbit. By concatenating these sequences, we obtain an infinite sequence ξ_3 such that for any $k \in \mathbb{N}$:

$$\frac{1}{2kl+1} \sum_{i=0}^{2kl} d(f(x_i), x_{i+1}) < \frac{1}{2kl+1} \cdot \frac{(2k+1)r}{4} < \frac{r}{4l}.$$

Let $\xi = \{x_i : i \in \mathbb{Z}\}$ be a bi-infinite sequence defined by $\xi_4 = \{x_i = p : i \leq 0\}$ and $\xi = \xi_4 \cup \xi_3$. One can easily verify that ξ is an $(r/4l)$ -average pseudo-orbit of f .

Take $\epsilon \in (0, r/4)$ and let $0 < \delta < \epsilon$ be the constant of the average pseudo-orbit tracing property such that $\delta = r/4l$.

We assume that the average pseudo-orbit tracing point $z \in W^s(p)$ (the case $z \in W^u(p)$ is similar). Since $z \in W^s(p)$, there is $j \geq l$ such that $f^j(z) \in B_{r/4}(p) \cap W^s(p)$ and $f^{j-1}(z) \notin B_{r/4}(p) \cap W^s(p)$. Then we see that $f^{j+i}(z) \in B_{r/4}(p) \cap W^s(p)$ for all $i \geq 0$, and $d(f^j(z), f^{-l}(x)) > r/4$.

Since f has the average pseudo-orbit tracing property in Γ , the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ can be negatively and positively ϵ pseudo-orbit traced in average by the point $z \in W^s(p)$.

By contradiction, suppose that f has the average pseudo-orbit tracing property in Γ . Then there exists a point $z \in \Gamma$ that ϵ -traces ξ in average.

It is enough to show that the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ is positively ϵ pseudo-orbit traced in average by the point z , that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, consider $2kl \leq n \leq 2(k+1)l$. We see that if $n \rightarrow \infty$ then $k \rightarrow \infty$. Since $d(f^{j+s}(z), f^{-l+i}(x)) > r/4$ for all $s \geq 0$ and $i = 1, \dots, 2l$, we have that

$$\begin{aligned} \frac{1}{n+1} \sum_{i=0}^n d(f^i(z), x_i) &\geq \frac{1}{2(k+1)l+1} \sum_{i=0}^{2kl} d(f^i(z), x_i) \\ &> \frac{1}{2(k+1)l+1} \cdot \frac{r}{4} \cdot 2(k-2)l = \frac{(k-2)lr}{4((k+1)l+1)}. \end{aligned}$$

Since $n \rightarrow \infty$ then $k \rightarrow \infty$, we see that

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n d(f^i(z), x_i) > \lim_{k \rightarrow \infty} \frac{(k-2)lr}{4((k+1)l+1)} = \frac{r}{4} > \epsilon.$$

This contradicts the assumption that z ϵ -traces ξ in average. Therefore, f does not have the average pseudo-orbit tracing property in $\Gamma = \text{Orb}(x) \cup \{\text{Orb}(p)\}$. \square

Lemma 3.3. *If f have the average pseudo-orbit tracing property in Γ , then the average pseudo-orbit tracing point z not belong to $\text{Orb}(p)$.*

Proof. Take $\epsilon \in (0, r/4)$ and let $0 < \delta < \epsilon$ be the constant of the average pseudo-orbit tracing property such that $\delta = r/4l$. Then as in the proof of Lemma 3.2, we make a δ average pseudo-orbit

$$\{x_i : i \in \mathbb{Z}\} = \{\dots, p, f^{-l}(x), \dots, f^{-1}(x), x, f(x), \dots, f^{l-1}(x), p, \dots, p, f^{-l}(x), \dots\} \subset \Gamma.$$

By contradiction, suppose that the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ is positively ϵ pseudo-orbit traced in average by the point $z = p$, that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Since $d(p, f^i(x)) > r/4$ for all $i = -l, -l+1, \dots, l-2$, we see that for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, $2kl \leq n \leq 2(k+1)l$. For the sequence ξ_1 in Lemma 3.2, we see that the number of elements of the set $\{i : d(p, x_i) > r/4, i = 1, \dots, 2l\}$ is $2l$. Then as in the proof of Lemma 3.2, we know that

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{i=0}^n d(f^i(z), x_i) > \lim_{k \rightarrow \infty} \frac{(k-2)lr}{4((k+1)l+1)} = \frac{r}{4} > \epsilon.$$

This means that the δ pseudo-orbit $\{x_i : i \in \mathbb{Z}\}$ is not positively ϵ pseudo-orbit traced in average by the $z = p$, whis is a contradiction \square

Lemma 3.4. *Let $\epsilon > 0$ and $\delta > 0$ be as in definition 2.1. Then f does not satisfy the average pseudo-orbit tracing property in Γ .*

Proof. By contradiction, suppose that f has the average pseudo-orbit tracing property in $\Gamma = Orb(x) \cup \{Orb(p)\}$. As in the proof of Lemma 3.2 and Lemma 3.3, we can make a δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$. Since f has the average pseudo-orbit tracing property in Γ , we can find the point $z \in Orb(x)$ or $z = p$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.$$

However, it is a contradiction by Lemma 3.2 and Lemma 3.3. □

Proposition 3.5. *Let $\epsilon > 0$ and $\delta > 0$ be as in Definition 2.1. If f has the average pseudo-orbit tracing property and a δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ as in Lemma 3.2, then an average pseudo-orbit tracing point is in Γ .*

Proof. As in Lemma 3.2, we consider the sequence $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ which is a δ average pseudo-orbit of f . Since f has the average pseudo-orbit tracing property, there is $z \in M$ such that

$$\limsup_{n \rightarrow \pm\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Assume that the point $z \in M \setminus \Gamma$.

Case 1. Consider that if $z \in M \setminus W^s(p) \cup W^u(p) \cup Orb(p)$ then $f^i(z)$ can not converges to p as $i \pm \infty$ and by λ -lemma $f^i(W^s(p)) \rightarrow W^s(p)$ as $i \rightarrow -\infty$ and $f^i(W^u(p)) \rightarrow W^u(p)$ as $i \rightarrow \infty$, and thus there is $j \in \mathbb{Z}$ such that $d(f^{j+k}(z), x_{j+k}) \geq \epsilon$ for all $k \in \mathbb{Z}$. Then we see that the point z is not average pseudo-orbit tracing point of f .

Case 2. Consider that if $z \in W^s(p) \cup W^u(p) \cup Orb(p)$ then by Lemma 3.2 and Lemma 3.3 the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ can not be ϵ pseudo-orbit traced in average by z .

By Case 1 and Case 2, the δ average pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Gamma$ can be ϵ pseudo-orbit traced in average by $z \in \Gamma$. □

We say that f has a *homoclinic tangency* associated with a periodic point p if $W^s(p)$ and $W^u(p)$ are non-transversal intersections. Denote by \mathcal{HT} the set of all diffeomorphisms having the homoclinic tangency associated with periodic points. The following was proved in [3].

Lemma 3.6. *There is a residual subset \mathcal{G} in $\text{Diff}(M)$ such that for given $f \in \mathcal{G}$, either f has a homoclinic tangency associated with a periodic point p or f is partially hyperbolic.*

Proof of Theorem 2.2. Let $f \in \mathcal{G} \subset \text{Diff}(M)$ and f have the average pseudo-orbit tracing property. For simplicity, we assume that $f(p) = p$. For any point $x \in W^s(p) \cap W^u(p) \setminus \{p\}$, by Proposition 3.5 we assume that f the average pseudo-orbit tracing property in Γ . According to Lemma 3.3 and Lemma 3.4, f does not satisfy the average pseudo-orbit tracing property in Γ . Thus if a diffeomorphism f has the average

pseudo-orbit tracing property in Γ , then we have that $f \notin \mathcal{HT}$. By Lemma 3.6, if $f \in \mathcal{G}$ has the average pseudo-orbit tracing property in Γ , then f is partially hyperbolic. \square

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