

Planar graphs without 4-, 5-, 8-cycles and special 6-cycles are near-bipartite

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ABSTRACT

A graph is near-bipartite if its vertex set can be partitioned into two parts such that one part is an independent set and the other induces a forest. It is clear that near-bipartite graphs are 3-colorable. It was proved that planar graphs without 4-, 5- and 8-cycles are 3-colorable [Discuss. Math. Graph Theory, 31(2011): 775-789]. It is asked by Kang et al that whether it is true that the planar graphs without 4-, 5- and 8-cycles are near-bipartite [Discuss. Math. Graph Theory 45 (2025) 129-145]. A 6-cycle is called a special 6-cycle if this 6-cycle shares an edge with a triangle of G . In this paper, we prove that planar graphs without 4-, 5-, 8-cycles and special 6-cycles are near-bipartite which is a step toward the problem.

Keywords: 3-colorable, planar graph, near-bipartite

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

A graph G is k -degenerate if every subgraph H of G contains a vertex of degree at most k in H . Clearly, every k -degenerate graph is $(k + 1)$ -colorable. Borodin [1] in 1976 suggested to partition the vertex set of a graph into two sets such that the induced subgraphs have good degeneracy properties. A graph G is (a, b) -partitionable if its vertices can be partitioned into sets A and B such that the subgraph induced by A is a -degenerate and the subgraph induced by B is b -degenerate. Borodin [1] conjectured that every planar graph, which is 5-degenerate, is $(1, 2)$ - and $(0, 3)$ -partitionable. Thomassen [7] [8] confirmed these conjectures.

Clearly a graph is *bipartite* if and only if it is $(0, 0)$ -partitionable. A graph is called

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near-bipartite if it is $(0, 1)$ -partitionable. Borodin and Glebov [2] showed that every planar graph of girth at least 5 is near-bipartite. Dross, Montassier, and Pinlou [3] asked whether every triangle-free planar graph is near-bipartite. Liu and Yu [5] showed that planar graphs without cycles of lengths in $\{4, 6, 8\}$ are near-bipartite, which extend the result of Wang and Chen [9] that they are 3-colorable. Mondal [6] showed that planar graphs without 4-, 5-, 8-cycles are 3-colorable. Kang et al. [4] proved that every planar graph with neither 4- and 6-cycles nor special 9-cycles are near-bipartite and asked that whether it is true that the planar graphs without 4-, 5- and 8-cycles are near-bipartite. A 6-cycle is called a *special 6-cycle* if this 6-cycle shares an edge with a triangle of G . In this paper, in order to ask the question of Kang, we show that planar graphs without 4-, 5-, 8-cycles and special 6-cycles are near-bipartite as the following theorem.

Theorem 1.1. *Every planar graph without 4-, 5-, 8-cycles and special 6-cycles are near-bipartite.*

An IF -coloring of a graph is a partition of its vertices into two parts such that one part colored I induces an independent set and the other part colored F induces a forest. A path or cycle on only F -vertices is called an F -path or F -cycle, respectively. Given a graph G and a cycle C in G , an IF -coloring ϕ_c of $G[V(C)]$ super-extends to G if there exists an IF -coloring ϕ_G of G that extends ϕ_c with the property that there is no F -path with at least three vertices which has two end-vertices on C and all other vertices not on C . We say that C is super-extendable to G if every IF -coloring ϕ_c of $G[V(C)]$ can super-extend to G . A 12-cycle $[v_1v_2 \cdots v_{12}v_1]$ of a plane graph is called bad if its interior or exterior contains a common neighbor x of v_1, v_5 and v_9 (see Figure 1).

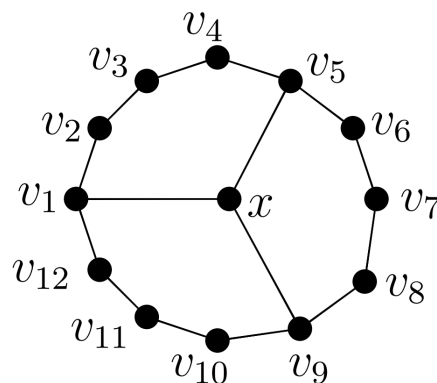


Fig. 1. The bad cycle

In Figure 1, we call x the *center vertex* of the bad cycle C . Instead of Theorem 1.1, we actually prove the following stronger result.

Theorem 1.2. *For every plane graph G without 4-, 5-, 8-cycles and special 6-cycles, given any cycle C in G of length at most 12 except bad cycles, (G, C) is super-extendable.*

By [2], every planar graph of girth 5 has an IF -coloring. Since G contains no 4-cycles,

G must have a triangle which is a good cycle. Thus Theorem 1.1 follows from Theorem 1.2 easily.

The graphs considered in this paper are finite and simple. A vertex is classified as an *external* vertex if it lies on the boundary of the unbounded face; otherwise, it is considered to be *internal*. For a cycle C , let $int(C)$ and $ext(C)$ denote the set of vertices in the interior and exterior of C , respectively. A cycle C is *separating* if both $int(C)$ and $ext(C)$ are nonempty. Let $d(v)$ be the degree of v . A k -vertex (k^+ -vertex, k^- -vertex) is a vertex of degree k (at least k , at most k). A k -face (k^+ -face, k^- -face) is a face of degree k (at least k , at most k). A k -cycle (k^+ -cycle, k^- -cycle) is a cycle of degree k (at least k , at most k).

2. The properties of a minimum counterexample to Theorem 1.2

Let (G, C_0) be a counterexample to Theorem 1.2 with the minimum value about $|V(G)| + |E(G)|$, where C_0 is a cycle of length at most 12 in G and not a bad cycle that is precolored for $G[V(C)]$. If C_0 is a separating cycle, then C_0 is super-extendable to both $G - ext(C_0)$ and $G - int(C_0)$. Thus, C_0 is super-extendable to G , contrary to the choice of C_0 . So we may assume that C_0 is the boundary of the outer face of G in the rest of this paper. Call a vertex v internal if $v \notin C_0$ and a face f internal if $f \neq C_0$. We have the following lemmas about the counterexample G .

Lemma 2.1. *Every internal vertex of G has degree at least 3.*

Proof. Suppose to the contrary that G has an internal vertex v such that $d(v) \leq 2$. By the minimality of (G, C_0) , the IF -coloring of C_0 can super-extend to $G - v$. If v has a neighbor colored I , then color v with F ; otherwise, color v with I . In all cases, C_0 can super-extend to G , a contradiction. □

Lemma 2.2. *Except for the bad cycles, the graph G has no separating cycle of length at most 12.*

Proof. Suppose otherwise that C is a separating cycle of length at most 12 but not the bad cycle and $C \neq C_0$. Let $H_1 = G - int(C)$ and $H_2 = G[C \cup int(C)]$. By the minimality of G , (H_1, C_0) is super-extendable, and after that, $G[V(C)]$ is colored. By the minimality of (G, C_0) again, (H_2, C) is super-extendable. Further, we get that (G, C_0) is super-extendable, a contradiction. □

Lemma 2.3. *C_0 has no chords.*

Proof. If C_0 has a chord, then by Lemma 2.2, $V(C_0) = V(G)$. By our assumption, the precoloring of C_0 is already a IF -coloring of G . □

A vertex (or an edge) is *triangular* if it is incident with a triangle. We say that a vertex is *bad* if it is an internal triangular 3-vertex; otherwise, it is a good vertex. A *tetrad* in a

plane graph is a path $v_1v_2v_3v_4$ of four internal 3-vertices contained in the boundary of a face, so that both v_1v_2 and v_3v_4 are edges of triangles (see Figure 2).

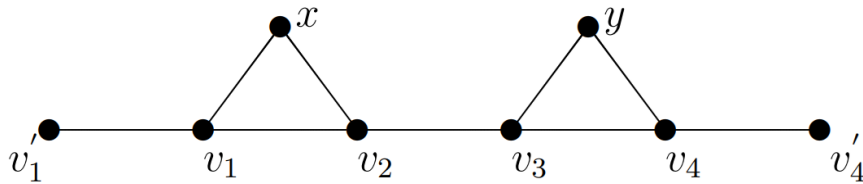


Fig. 2. A tetrad

Lemma 2.4. G contains no tetrad.

Proof. Let $v_1v_2v_3v_4$ be a tetrad in G (see Figure 2), $N(v_1)=\{x, v_2, v'_1\}$ and $N(v_4)=\{y, v_3, v'_4\}$. Let G' be the graph obtained by identifying y and v'_1 of $G-\{v_1, v_2, v_3, v_4\}$. Let v^* denote the new vertex obtained by identifying y and v'_1 .

Note that the identification does not create a chord of C_0 or identify two vertices of C_0 . Suppose the identification does create a chord, then either y or v'_1 must be on C_0 . Without the loss of generality, assume that $v'_1 \in C_0$ and $y' \in C_0$ which is a neighbor of y different from v_3 and v_4 . There exists a path $P=y'yv_3v_2v_1v'_1$ that divides C_0 into two parts P_1 and P_2 . Obviously, x is not on C_0 since G has neither 4-, 5-, 8-cycles nor special 6-cycles. Since $|C_0| \leq 12$, $|P_1| + 5 + |P_2| + 5 \leq 22$. Then either $P \cup P_1$ or $P \cup P_2$ is a cycle of length at most 11 that separates x and v_4 , a contradiction to Lemma 2.2.

In $G - \{v_1, v_2, v_3, v_4\}$, we will prove that there is no path Q of length at most 8 between y and v'_1 . For otherwise, assume that $|Q| \leq 8$. Then $Q \cup yv_3v_2v_1v'_1$ would have a cycle of length at most 12 of G . Obviously, this cycle separates x and v_4 since x is not on Q by the assumption that G has no 4-, 5-, 8-cycles and special 6-cycles. In addition, we claim that the separation cycle cannot be a bad cycle as follows, thus we get a contradiction to Lemma 2.2. If $C = Q \cup yv_3v_2v_1v'_1$ is a bad 12-cycle and x is inside C , it is impossible that x is the center vertex of C . So there is another vertex x' inside C which is the center vertex of C . Note that $v_1v_2v_3v_4$ is a tetrad of G . Both v'_1 and y are adjacent to x' . Then $v'_1x'yv_3v_2v_1v'_1$ separates x and v_4 , a contradiction. If $C = Q \cup yv_3v_2v_1v'_1$ is a bad 12-cycle and x is outside C , it is impossible that v_4 is the center vertex of of the bad cycle C . So there is another vertex t inside C which is the center vertex of of the bad cycle C . Note that $v_1v_2v_3v_4$ is a tetrad of G . Both v'_1 and y are adjacent to t . Then $v'_1tyv_3v_2v_1v'_1$ separates x and v_4 , still a contradiction. So there is no path Q of length at most 8 between y and v'_1 in $G - \{v_1, v_2, v_3, v_4\}$ and no new 8⁻-cycles are created in G' . This implies that G' contains no 4-, 5-, 8-cycles and special 6-cycles. Since $|V(G')| < |V(G)|$, the precoloring of C_0 can be super-extended to a IF -coloring of G' . Now we will super-extend it to an IF -coloring of G .

First, we color v'_1 and y with the color of the identified vertex. If v'_1, y are colored I , then color v_1, v_3, v_4 with F and color v_2 with a color different from the color of x . If v'_1, y are colored F and x is colored I , then both v_1 and v_2 are colored F . Next we discuss the

color of v'_4 . If v'_4 is colored I , then v_4 is colored F and v_3 is colored I , we are done. If v'_4 is colored F , then v_4 is colored I and v_3 is colored F . Notice that $yv_3v_2v_1v'_1$ might be one part of an F -path, but this is impossible. Suppose not, G' would also have an F -path, a contradiction. If v'_1, y are colored F and x is also colored F . Next we discuss the color of v'_4 . If v'_4 is colored I , then both v_4 and v_2 are colored F , both v_1 and v_3 are colored I , we are done. If v'_4 is colored F , there are the following two cases.

Case 1. Assume that there is an F -path P between x and y in the IF -coloring of G' . Then both v_1 and v_3 are colored F , both v_2 and v_4 are colored I . We are done as follow. Notice that $P_1v'_1v_1xP_2$ or $P_1v'_1v_1xPP_3$ may form an F -path which has two end-vertices on C_0 and all other vertices are not on C_0 (see Figure 3). In Figure 3, C_0 is the outer cycle. But this is impossible. Otherwise, the coloring of G' would also form an F -path $P_1v^*PxP_2$ or $P_1v^*P_3$ which has two end-vertices on C_0 and all other vertices are not on C_0 , a contradiction. In addition, $P_4v'_1v_1x$ may form an F -cycle. Then this is also impossible. Suppose not, the coloring of G' would also form an F -cycle xP_4v^*Px , a contradiction.

Case 2. Assume that there is no F -path P between x and y in the IF -coloring of G' . Then both v_2 and v_3 are colored F , both v_1 and v_4 are colored I . We are done as follow. Notice that $P_2xv_2v_3yP_3$ may form an F -path which has two end-vertices on C_0 and all other vertices are not on C_0 (see Figure 3). In Figure 3, C_0 is the outer cycle. In this case, we re-color them: both v_1 and v_3 are colored F , and both v_2 and v_4 are colored I . After re-coloring, we still need to note that $P_1v'_1v_1xP_2$ may form an F -path which has two end-vertices on C_0 and all other vertices are not on C_0 (see Figure 3). But this is impossible. Otherwise, the coloring of G' would also form an F -path $P_1v^*P_3$ which has two end-vertices on C_0 and all other vertices are not on C_0 , a contradiction. In addition, $P_4v'_1v_1x$ may form an F -cycle. Then this is also impossible. Suppose not, the coloring of G' would also form an F -path $P_2xP_4v^*P_3$ which has two end-vertices on C_0 and all other vertices are not on C_0 , a contradiction. □

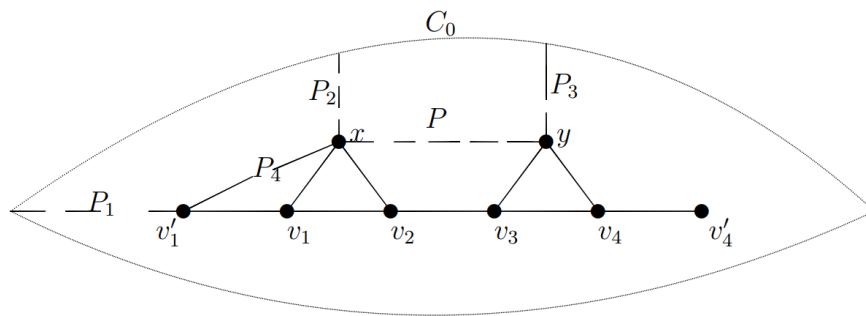


Fig. 3. The case when v'_1, y, x and v'_4 are colored F

3. The proof of Theorem 1.2

We are now ready to present a discharging procedure that will complete the proof of Theorem 1.2. Let $x \in V(G) \cup (F(G) - C_0)$ have an initial charge of $ch(x) = d(x) - 4$,

and $ch(C_0) = |C_0| + 4$. By Euler's Formula, $\sum_{x \in V \cup F} ch(x) = 0$. Let $ch^*(f)$ be the charge of $x \in V \cup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $ch^*(x) \geq 0$ for all $x \in V(G) \cup (F(G) - C_0)$ and $ch^*(C_0) > 0$.

Here are the discharging rules:

(R1). Each 3-face gets $\frac{1}{3}$ from each incident vertex.

(R2). (a) Let v be an external 2-vertex. v gets $\frac{2}{3}$ from the internal face containing v .

(b) Let v be an internal 3-vertex. If v is a bad vertex, then v gets $\frac{2}{3}$ from each 9^+ -face containing v . Otherwise, v gets $\frac{1}{3}$ from each 6^+ -face containing v .

(c) Let v be an internal 4-vertex. If v is incident with only one 3-face, then v gets $\frac{1}{3}$ from the unique face f which contains v and is not adjacent to this 3-face; if v is incident with two 3-faces, then v gets $\frac{1}{3}$ from each 9^+ -face containing v .

(R3). Each 3-vertex on C_0 which is not in a triangle gives $\frac{1}{6}$ to each incident internal 6-face. Each 4⁺-vertex on C_0 gives $\frac{1}{6}$ to each incident internal 6-face.

(R4). The outer face C_0 gives $\frac{4}{3}$ to each incident 2-vertex or 3-vertex, and 1 to each other incident vertex.

After the discharging of rules from (R1) to (R4) is completed, the discharging of rule (R5) is performed.

(R5). Each internal face adjacent to C_0 gives the surplus charge to C_0 .

Lemma 3.1. *Every vertex v in G has nonnegative final charge.*

Proof. Let $d(v) = 2$. Then $v \in C_0$ by Lemma 2.1. By (R4), v gets $\frac{4}{3}$ from C_0 . By (R2), v gets $\frac{2}{3}$ from the other incident face. So $ch^*(v) \geq 2 - 4 + \frac{4}{3} + \frac{2}{3} = 0$.

Let $d(v) = 3$. If $v \in C_0$ and C_0 is a 3-face, there are two 9^+ -faces containing v . By (R1), v gives $\frac{1}{3}$ to C_0 . By (R4), v also gets $\frac{4}{3}$ from C_0 . So $ch^*(v) \geq 3 - 4 + \frac{4}{3} - \frac{1}{3} = 0$. If $v \in C_0$ and C_0 is a 6-face or 7-face, there are exactly two internal 6^+ -faces containing v since G has no 8-cycles and special 6-cycles. By (R4), v also gets $\frac{4}{3}$ from C_0 . By (R3), v gives at most $\frac{1}{6} \times 2$ to its incident internal 6-faces. So $ch^*(v) \geq 3 - 4 + \frac{4}{3} - \frac{1}{6} \times 2 = 0$. If $v \in C_0$ and C_0 is a 9^+ -face, the set of degrees of the internal faces containing v is one of the following: $\{3, 9^+\}$, $\{6^+, 6^+\}$ since G has no special 6-cycles and without 4-, 5-, 8-cycles. By (R4), v gets $\frac{4}{3}$ from C_0 . For the case $\{3, 9^+\}$, v gives $\frac{1}{3}$ to the inner 3-face by (R1), so we have that $ch^*(v) \geq 3 - 4 + \frac{4}{3} - \frac{1}{3} \geq 0$. For the case $\{6^+, 6^+\}$, by (R3), v gives at most $\frac{1}{6} \times 2$ to its incident internal 6-faces. Then $ch^*(v) \geq 3 - 4 + \frac{4}{3} - \frac{1}{6} \times 2 = 0$. If $v \notin C_0$, the set of degrees of the faces containing v is one of the following: $\{3, 9^+, 9^+\}$, $\{6^+, 6^+, 6^+\}$. For the case $\{3, 9^+, 9^+\}$, v is a bad 3-vertex. By (R2), v gets $\frac{2}{3} \times 2$ from the two incident 9^+ -faces. By (R1), v gives $\frac{1}{3}$ to its incident 3-face. For the case $\{6^+, 6^+, 6^+\}$, v is not a bad 3-vertex. By (R2), v gets $\frac{1}{3} \times 3$ from the three incident faces. So we get that $ch^*(v) \geq 3 - 4 + \min\{\frac{2}{3} \times 2 - \frac{1}{3}, \frac{1}{3} \times 3\} = 0$.

Let $d(v) = 4$. If $v \in C_0$ and C_0 is a 3-face, the sets of degrees of the internal faces containing v are $\{9^+, 6^+, 9^+\}$, and $\{9^+, 3, 9^+\}$ since G has no special 6-cycles. For the case $\{9^+, 6^+, 9^+\}$, v gets 1 from C_0 by (R4), v gives $\frac{1}{3}$ to C_0 by (R1) and v gives at most $\frac{1}{6}$ to the incident internal faces by (R3). Then $ch^*(v) \geq 4 - 4 + 1 - \frac{1}{3} - \frac{1}{6} \geq 0$. For the case $\{9^+, 3, 9^+\}$, v gets 1 from C_0 by (R4) and v gives $\frac{1}{3} \times 2$ to the outer 3-face C_0 and

the internal 3-face by (R1). Then $ch^*(v) \geq 4 - 4 + 1 - \frac{1}{3} \times 2 \geq 0$. If $v \in C_0$ and C_0 is a 6-face or 7-face, the set of degrees of the internal faces containing v is one of the following: $\{6^+, 6^+, 6^+\}$, $\{9^+, 3, 9^+\}$. For the case $\{6^+, 6^+, 6^+\}$, v gets 1 from C_0 by (R4), and v gives at most $\frac{1}{6} \times 3$ to the incident internal 6-faces by (R3). Then $ch^*(v) \geq 4 - 4 + 1 - \frac{1}{6} \times 3 \geq 0$. For the cases $\{9^+, 3, 9^+\}$, v gets 1 from C_0 by (R4) and v gives $\frac{1}{3}$ to the internal 3-face by (R1). Then $ch^*(v) \geq 4 - 4 + 1 - \frac{1}{3} \geq 0$.

If $v \in C_0$ and C_0 is a 9^+ -face, the set of degrees of the internal faces containing v is one of the following: $\{3, 9^+, 3^+\}$, $\{9^+, 3, 9^+\}$, $\{9^+, 6, 9^+\}$, $\{6^+, 6^+, 6^+\}$. By (R4), v gets 1 from C_0 . For the case $\{3, 9^+, 3^+\}$, v gives the most $\frac{1}{3} \times 2$ to the incident two 3-faces by (R1). For the case $\{9^+, 3, 9^+\}$, v gives $\frac{1}{3}$ to the incident 3-face by (R1). For the case $\{9^+, 6, 9^+\}$, v gives $\frac{1}{6}$ to the incident 6-face by (R3). For the case $\{6^+, 6^+, 6^+\}$, v gives at most $\frac{1}{6} \times 3$ to the incident internal faces by (R3). So $ch^*(v) \geq 4 - 4 + \min\{1 - \frac{1}{3} \times 2, 1 - \frac{1}{3}, 1 - \frac{1}{6}, 1 - \frac{1}{6} \times 3\} \geq 0$.

If $v \notin C_0$, the set of degrees of the faces containing v in clock-wise order is one of the following: $\{3, 9^+, 3, 9^+\}$, $\{3, 9^+, 6^+, 9^+\}$, $\{6^+, 6^+, 6^+, 6^+\}$. For the case $\{3, 9^+, 3, 9^+\}$, by (R1) and R(2), v gives $\frac{1}{3}$ to each of the two incident 3-faces and gets $\frac{1}{3}$ from each of the two incident 9^+ -faces. For the cases $\{3, 9^+, 6^+, 9^+\}$, by (R1) and R(2), v gives $\frac{1}{3}$ to the incident 3-face and gets $\frac{1}{3}$ from the incident 6^+ -face which is not adjacent to the 3-face containing v . For the case $\{6^+, 6^+, 6^+, 6^+\}$, by (R2), do not charge value. So $ch^*(v) \geq 4 - 4 + \min\{\frac{1}{3} \times 2 - \frac{1}{3} \times 2, \frac{1}{3} - \frac{1}{3}\} \geq 0$.

Finally, let $d(v) \geq 5$. If $v \in C_0$ and C_0 is a 3-face, v is contained in at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces. By (R4), v gets 1 from C_0 . By (R1), v gives at most $\frac{1}{3} \times \lfloor \frac{d(v)}{2} \rfloor$ to all the incident 3-faces. So $ch^*(v) \geq d(v) - 4 + 1 - \lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{3} > 0$. If $v \in C_0$ and C_0 is a 6-face or 7-face, v is contained in at most $\lfloor \frac{d(v)-2}{2} \rfloor$ 3-faces since G has no 8-cycles and special 6-cycles. When v is contained in exactly $\lfloor \frac{d(v)-2}{2} \rfloor$ 3-faces, v is contained in at most two 6-faces since G has no special 6-cycles. So $ch^*(v) \geq d(v) - 4 + 1 - \lfloor \frac{d(v)-2}{2} \rfloor \times \frac{1}{3} - \frac{1}{6} \times 2 > 0$. If $v \in C_0$ and C_0 is a 9^+ -face, let x be the number of 3-faces incident to v . Then we have that $x \leq \lfloor \frac{d(v)}{2} \rfloor$. By (R1) and (R3), $ch^*(v) \geq d(v) - 4 + 1 - \frac{x}{3} - (x - 1) \times \frac{1}{6} > 0$ or $ch^*(v) \geq d(v) - 4 + 1 - d(v) \times \frac{1}{6} > 0$. If $v \notin C_0$, by (R1), v gives at most $\frac{1}{3} \times \lfloor \frac{d(v)}{2} \rfloor$ to the incident 3-faces and we have that $ch^*(v) \geq d(v) - 4 - \lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{3} > 0$. \square

Lemma 3.2. *Every internal face in G has nonnegative final charge.*

Proof. Let f be an internal face in G . Recall that G contains no 4-, 5-, 8-cycles and special 6-cycles. If $d(f) = 3$, f gets $\frac{1}{3}$ from each incident vertex by (R1) and thus $ch^*(f) \geq 3 - 4 + \frac{1}{3} \times 3 = 0$.

Suppose that $d(f) = 6$. Since G contains no special 6-cycles, f is not adjacent to a 3-face. Hence, f has no bad 3-vertex. If $V(f) \cap C_0 = \emptyset$, by (R2), $ch^*(f) \geq 6 - 4 - \frac{1}{3} \times 6 = 0$. If $V(f) \cap C_0 \neq \emptyset$, then $1 \leq |V(f) \cap C_0| \leq 5$ by Lemma 2.3. Further, we can easily see that $V(f) \cap C_0$ contains at most two external 3^+ -vertices and at most three external 2-vertices. When $V(f) \cap C_0$ contains exactly three 2-vertices, it must contain exactly two external 3^+ -vertices. The degree set of the two external 3^+ -vertices can have the following cases $\{3, 3\}$, $\{3, 4^+\}$, $\{4^+, 4^+\}$. For all the cases, since f is not adjacent to a 3-face, by (R2)

and (R3), $ch^*(f) \geq 6 - 4 - \frac{2}{3} \times 3 - \frac{1}{3} + \frac{1}{6} \times 2 = 0$. When $V(f) \cap C_0$ contains less than three 2-vertices, the discussion is easy.

Suppose that $d(f) = 7$. Since G contains no 8-cycle, f is not adjacent to a 3-face. Hence, f has no bad 3-vertex. If $V(f) \cap C_0 = \emptyset$, by (R2), $ch^*(f) \geq 7 - 4 - \frac{1}{3} \times 7 \geq 0$. If $V(f) \cap C_0 \neq \emptyset$, then $1 \leq |V(f) \cap C_0| \leq 6$ by Lemma 2.3 and $V(f) \cap C_0$ contains at most four 2-vertices. If there are exactly four 2-vertices in $V(f) \cap C_0$, then there must be two external 3^+ -vertices in $V(f) \cap C_0$. The degree sets of the two external 3^+ -vertices are $\{3, 3\}$, $\{3, 4^+\}$ and $\{4^+, 4^+\}$. For all the cases, since f is not adjacent to a 3-face, by (R2), $ch^*(f) \geq 7 - 4 - \frac{2}{3} \times 4 - \frac{1}{3} \geq 0$. If there are at most three 2-vertices in $V(f) \cap C_0$, the discussion is the same as above.

Suppose that $d(f) = 9$. If $V(f) \cap C_0 = \emptyset$, by Lemma 2.4, f contains at most seven bad 3-vertices. In addition, we may assume that the other two vertices are internal 4-vertices (see Figure 4). Then, by (R2), f gives $\frac{2}{3}$ to each bad 3-vertex and gives $\frac{1}{3}$ to the internal 4-vertex which are in two triangles. So $ch^*(f) \geq 9 - 4 - \frac{2}{3} \times 7 - \frac{1}{3} = 0$. When there are less than seven bad 3-vertices on f , the discussion is easy. If $V(f) \cap C_0 \neq \emptyset$, then $1 \leq |V(f) \cap C_0| \leq 8$ by Lemma 2.3 and the discussion is the same as above.

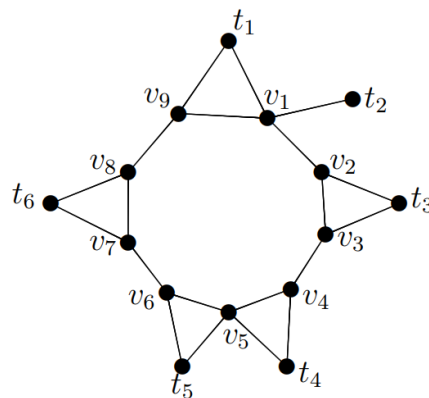


Fig. 4. A special 9-face

Suppose that $d(f) = 10$. If $V(f) \cap C_0 = \emptyset$, by Lemma 2.4, f contains at most eight bad 3-vertices. In addition, we may assume that the other two vertices are internal 4^- -vertices. By (R2), $ch^*(f) \geq 10 - 4 - \frac{2}{3} \times 8 - \frac{1}{3} \times 2 = 0$. When there are less than eight bad 3-vertices on f , the discussion is easy. If $V(f) \cap C_0 \neq \emptyset$, then $1 \leq |V(f) \cap C_0| \leq 9$ by Lemma 2.3 and the discussion is the same as above.

Suppose that $d(f) \geq 11$. If f contains 2-vertices, then f shares two 3^+ -vertices with C_0 which are not bad by definition. Then, by (R2), $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times (d(f) - 2) \geq 0$. So we may assume that f contains no 2-vertices. By Lemma 2.4, f contains at least two vertices which are not bad 3-vertices. It follows that $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times (d(f) - 2) - \frac{1}{3} \times 2 \geq 0$. □

Lemma 3.3. $ch^*(C_0) > 0$.

Proof. We may assume that there are only 2-vertices and 3-vertices on C_0 in the following.

Suppose not, we can easily get that $ch^*(C_0) > 0$. Then, by rough calculation using (R4), we have $ch^*(C_0) \geq |C_0| + 4 - \frac{4}{3}|C_0| = \frac{1}{3}(12 - |C_0|) \geq 0$ since $|C_0| \leq 12$. Further, we may assume that $|C_0| = 12$. Then the value of the external face C_0 is 0 before the related incident internal faces give surplus value to C_0 by (R5).

Without using the rule (R5) and under the assumptions above, the values of internal 6^+ -faces which have a common edge with C_0 are the following.

(a) The 6-faces sharing one common edge with C_0 : $6 - 4 - \frac{1}{3} \times 4 + \frac{1}{6} \times 2 = 1$ by (R2) and (R3); The 7-faces sharing one common edge with C_0 : $7 - 4 - \frac{1}{3} \times 5 = \frac{4}{3}$.

(b) The 6-faces sharing two common edges with C_0 : $6 - 4 - \frac{1}{3} \times 3 + \frac{1}{6} \times 2 - \frac{2}{3} = \frac{2}{3}$; The 7-faces sharing two common edge with C_0 : $7 - 4 - \frac{1}{3} \times 4 - \frac{2}{3} = 1$.

(c) The 6-faces sharing three common edges with C_0 : $6 - 4 - \frac{1}{3} \times 2 + \frac{1}{6} \times 2 - \frac{2}{3} \times 2 = \frac{1}{3}$; The 7-faces sharing three common edges with C_0 : $7 - 4 - \frac{1}{3} \times 3 - \frac{2}{3} \times 3 = \frac{2}{3}$.

(d) The 6-faces sharing four common edges with C_0 : $6 - 4 - \frac{2}{3} \times 3 + \frac{1}{6} \times 2 - \frac{1}{3} = 0$; The 7-faces sharing four common edges with C_0 : $7 - 4 - \frac{2}{3} \times 3 - \frac{1}{3} \times 2 = \frac{1}{3}$.

(e) By Lemma 2.3, it is impossible that a 6-face shares five common edges with C_0 . The 7-faces sharing five common edges with C_0 : $7 - 4 - \frac{2}{3} \times 4 - \frac{1}{3} = 0$. By Lemma 2.3, it is impossible that a 7-faces share six common edges with C_0 .

(f) When $d(f) \geq 9$ and f contains at least one 2-vertex, then f shares two 3^+ -vertices with C_0 which are not bad by definition. Then, by (R2), $ch^*(f) \geq d(f) - 4 - \frac{2}{3} \times (d(f) - 2) \geq 0$.

Finally, we analyze the incident internal faces of C_0 when $|C_0| = 12$. According to the above statements, $ch^*(C_0) = 0$ is possible only when 6-faces share four edges with C_0 or 7-faces share five edges with C_0 . When $|C_0| = 12$, since C_0 is not a bad 12-cycle, it is impossible that all the incident internal faces of C_0 are 6-faces which sharing four edges with C_0 or 7-faces which sharing five edges with C_0 by the assumption that there are only 2-vertices or 3-vertices on C_0 . So, we can always find a 6^+ -face that makes $ch^*(C_0) > 0$ after (R5). \square

The proof of Theorem 1.2 is completed by Lemmas 3.1, 3.2 and 3.3.

4. Conclusion

Mondal [6] showed that planar graphs without 4-, 5-, 8-cycles are 3-colorable. Kang et al. [4] asked that whether it is true that the planar graphs without 4-, 5- and 8-cycles are near-bipartite. In this paper, in order to ask the question above, we prove that planar graphs without 4-, 5-, 8-cycles and special 6-cycles are near-bipartite which is a step toward their problem.

Funding

This research was partially supported by National Nature Science Foundation of China (No 12361067), Nature Science Foundation of Guangxi Province (No. 2024GXNSFAA010506) and School-level talent programs (2022KJQD04 and 2023MDKJ001).

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