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Vector spaces over finite commutative rings

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ABSTRACT

As a generalization of vector spaces over finite fields, we study vector spaces over finite commutative rings, and obtain Anzahl formulas and a dimensional formula for subspaces. By using these results, we discuss normalized matching (NM) property of a class of subspace posets.

Keywords: commutative ring, vector space, dimensional formula, singular linear space, NM property

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1. Introduction

Let R be a finite commutative ring and R^* denote its unit group. Throughout the paper, R always contains the identity $1 \neq 0$. Let $R^{m\times n}$ be the set of all $m \times n$ matrices over R, and $R^n = R^{1\times n}$. A matrix in R^n is also called an n-dimensional row vector over R. Let I_r (I for short) be the $r \times r$ identity matrix, $0_{m,n}$ (0 for short) the $m \times n$ zero matrix and $0_n = 0_{n,n}$. The set of all $n \times n$ invertible matrices forms a group under matrix multiplication, called the *general linear group* of degree n over R and denoted by $GL_n(R)$.

Let $A \in R^{m\times n}$. Denote by $I_k(A)$ the ideal in R generated by all $k \times k$ minors of A. Let $\mathrm{Ann}_R(I_k(A)) = \{x \in R : xa = 0 \text{ for all } a \in I_k(A)\}$ denote the annihilator of $I_k(A)$. The *McCoy rank* of A, denoted by $\mathrm{rk}(A)$, is $\max\{k : \mathrm{Ann}_R(I_k(A)) = \{0\}\}$. For $m \leq n$, let $R^{m\times n}(m)$ denote the set of all $m \times n$ matrices of McCoy rank m over R.

For $A \in R^{m\times n}$ and $B \in R^{n\times m}$, if $AB = I_m$, we say that A has a *right inverse* and B is a right inverse of A. The free module R^n of rank n is called the n-dimensional *row vector space* over R. Let $\alpha_i \in R^n$ for each $i \in [m] := \{1, \ldots, m\}$. The vector subset $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is called *unimodular* if the matrix $(\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top$ has a right inverse,

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where $X^\top$ is the transpose of X. Let $V \subseteq R^n$ be a *linear subset* (i.e. R-module). A *largest unimodular vector subset* of V is a unimodular vector subset of V which has maximum number of vectors. The *dimension* of V, denoted by $\dim(V)$, is the number of vectors in a largest unimodular vector subset of V. If a linear subset X of R^n has a unimodular basis with m vectors, then X is called an *m-dimensional vector subspace* (*m-subspace* for short) of R^n. We define the 0-subspace to be $\{0_{1,n}\}$. By Lemma 2.9 below, X is an m-subspace of R^n if and only if X is a free submodule of R^n of rank m.

Let $\mathbb{F}_q$ be a finite field with q elements. Let $\mathbb{F}_q^n$ be one of the n-dimensional classical spaces over $\mathbb{F}_q$, and let G_n be the corresponding classical group of degree n, see Wan's monograph [12]. Wan determined the orbits of subspaces of $\mathbb{F}_q^n$ under the action of G_n, discussed the following questions and found their the answers: (i) How should the orbits be described? (ii) What are the lengths of the orbits? (iii) What is the number of subspaces in any orbit contained in a given subspace? (iv) What is the number of subspaces in any orbit containing a given subspace?

As generalizations, Wang, the first present author and Li [17, 18] studied problems (i)-(iv) in the singular linear space over $\mathbb{F}_q$; Huang, Lv and Wang [9] studied problems (ii)-(iv) in the vector space over the residue class ring $\mathbb{Z}_{p^s}$, where p is a prime number; The first present author [6] studied problems (ii)-(iv) in the vector space over the residue class ring $\mathbb{Z}_m$; Sirisuk and Meemark [11] studied a problem (ii) by using a lifting technique in the vector space over the commutative ring R.

In this paper, we study problems (i)-(iv) in the vector space over the commutative ring R by using a matrix method, and discuss NM property of subspace posets. The structure is as follows. In Section 2, we obtain Anzahl formulas of matrices over R for later references. In Section 3, we find the answers of problems (i)-(iv) in the vector space (Theorems 3.1, 3.3 and 3.4), and obtain the dimensional formula for subspaces (Theorem 3.5). In Section 4, we find the answers of problems (i)-(iv) in the singular linear space (Theorems 4.1-4.3). Given a finite graded poset, knowing whether the poset has the NM property is crucial in understanding the combinatorics of the poset. As an application, the NM property of subspace posets in the vector space R^n (Theorem 5.2) is discussed in Section 5.

For convenience, we have compiled the main notations in Table 1.

2. Matrices over commutative rings

Suppose that R is a finite commutative ring. In this section, we obtain an equivalent condition for $A \in R^{m\times n}$ with $\mathrm{rk}(A) = m$, and compute the number of $m \times n$ matrices of McCoy rank m over R by using the matrix method, see Lemma 2.8 and Corollary 2.12. We begin with a useful proposition.

Proposition 2.1. (See [3]). *Let $A \in R^{m\times n}$. Then*

(i) $0 \le \mathrm{rk}(A) \le \min\{m, n\}$.

(ii) $\mathrm{rk}(A) = \mathrm{rk}(PAQ)$ *for all* $P \in GL_m(R)$ *and* $Q \in GL_n(R)$.

(iii) *If* $m \le n$, *then* $\mathrm{rk}(A) = m$ *if and only if the set of all the rows of A is linearly independent.*

Table 1. Notation

$R^{m\times n}$	the set of all $m \times n$ matrices over the ring R.
$R^{m\times n}(m)$	the set of all $m \times n$ matrices of McCoy rank m over R.
$GL_n(R)$	the general linear group of degree n over R.
$GL_{n+k,n}(R)$	the singular general linear group of degree $n+k$ over R.
π	the natural surjective homomorphism from the local ring R to the quotient ring R/M, where M is the maximal ideal of R.
ρ_i	the projection map from the commutative ring $R_1 \times R_2 \times \cdots \times R_\ell$ to the local ring R_i for all $i \in [\ell]$.
$\mathcal{M}_R(m,n)$	the set of all the m-subspaces of R^n.
$N_R(m,n)$	the size of $\mathcal{M}_R(m,n)$.
$\mathcal{M}_R(m,s;n+k,n)$	the set of all the (m,s)-subspaces of the singular linear space R^{n+k}.
$N_R(m,s;n+k,n)$	the size of $\mathcal{M}_R(m,s;n+k,n)$.
$\langle m,s\rangle$	the set $\mathcal{M}_R(m,s;n,n-k)$ in the singular linear space R^n.
$w_{G_0}(\langle m,s\rangle)$	the size of $\langle m,s\rangle$, i.e. $N_R(m,s;n,n-k)$.
$\mathcal{P}_n(R)$	the poset formed by all the subspaces of R^n ordered by inclusion.
$\mathcal{P}_n(R)_m$	the set of all the m-subspaces in $\mathcal{P}_n(R)$.

2.1. Over local rings

A *local ring* is a commutative ring which has a unique maximal ideal. For a local ring R, its unique maximal ideal is $M = R \setminus R^*$, and the residue field of R is the quotient ring R/M. In this subsection, we always assume that R is a local ring.

Let π be the natural surjective homomorphism from $(R,+,\cdot)$ to $(R/M,+,\cdot)$, i.e. $\pi(r) = r+M$ for all $r \in R$. For $A = (a_{uv}) \in R^{m\times n}$, let $\pi(A) = (\pi(a_{uv}))$. For matrices $A \in R^{m\times n}$ and $B \in R^{n\times k}$, it is easy to see that

$$\pi(AB) = \pi(A)\pi(B). \tag{1}$$

Lemma 2.2. (See [2, Lemma 2]). *Let $A \in R^{m\times n}$. Then* $\mathrm{rk}(A) = \mathrm{rk}(\pi(A))$.

Observe that $\{\alpha\}$ is unimodular, where $\alpha = (r_1,\ldots,r_n) \in R^n$, if and only if the ideal generated by $r_1,\ldots,r_n$ is equal to R. So, we obtain the following result.

Lemma 2.3. (See [11, Proposition 1.1]). *Let $\alpha = (r_1,\ldots,r_n) \in R^n$. Then the following statements are equivalent.*

(i) *$\{\alpha\}$ is unimodular.*
(ii) *r_i is a unit for some $i \in [n]$.*
(iii) *$\{\alpha\}$ is linearly independent.*

Lemma 2.4. *Let $m \le n$ and $A \in R^{m\times n}$. Then* $\mathrm{rk}(A) = m$ *if and only if there exists an*

$S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$.

Proof. If there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$, by Proposition 2.1, $\mathrm{rk}(A) = \mathrm{rk}(I_m, 0_{m,n-m})$, which implies that $\mathrm{rk}(A) = m$.

Suppose $\mathrm{rk}(A) = m$. Let $A = (a_{ij}) = (\alpha_1^\top, \alpha_2^\top, \dots, \alpha_m^\top)^\top$. By Proposition 2.1 (iii), $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is linearly independent, which implies that $\{\alpha_1 = (a_{11}, a_{12}, \dots, a_{1n})\}$ is linearly independent. Without loss of generality, by Lemma 2.3, we may assume that a_{11} is a unit. Then

$$S_1 = \left(\begin{array}{c|cccc} a_{11}^{-1} & -a_{11}^{-1}a_{12} & \cdots & -a_{11}^{-1}a_{1n} \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right) \in GL_n(R),$$

such that

$$AS_1 = \begin{pmatrix} 1 & 0_{1,n-1} \\ B_1 & B_2 \end{pmatrix},$$

where $B_1 = (a_{11}^{-1}a_{21}, \dots, a_{11}^{-1}a_{m1})^\top$ and $B_2 \in R^{(m-1)\times(n-1)}$. By Proposition 2.1 (ii), $\mathrm{rk}(AS_1) = m$, which implies that the set of all the rows of AS_1 is linearly independent. It follows that $\{\beta_2, \dots, \beta_m\}$ is linearly independent, where $B_2 = (\beta_2^\top, \dots, \beta_m^\top)^\top$. By Proposition 2.1 (iii), $\mathrm{rk}(B_2) = m - 1$. By induction, there exists an $S_2 \in GL_{n-1}(R)$ such that

$$B_2S_2 = \begin{pmatrix} I_{m-1} & 0_{m-1,n-m} \end{pmatrix}.$$

Then

$$AS_1 \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0_{1,m-1} & 0_{1,n-m} \\ B_1 & I_{m-1} & 0_{m-1,n-m} \end{pmatrix}.$$

Write

$$S_3 = \left(\begin{array}{cc|c} 1 & 0_{1,m-1} & \\ -B_1 & I_{m-1} & \\ \hline & & I_{n-m} \end{array}\right).$$

Then

$$S = S_1 \begin{pmatrix} 1 & 0_{1,n-1} \\ 0_{n-1,1} & S_2 \end{pmatrix} S_3 \in GL_n(R),$$

and $AS = (I_m, 0_{m,n-m})$. Therefore, the desired result follows. □

Lemma 2.5. *Let $m \le n$ and $A = (\alpha_1^\top, \alpha_2^\top, \dots, \alpha_m^\top)^\top \in R^{m\times n}$. Then the following statements are equivalent.*

(i) $\mathrm{rk}(A) = m$.

(ii) *$\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is linearly independent.*

(iii) *There exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$.*

(iv) *$\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is unimodular.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By Proposition 2.1 (iii) and Lemma 2.4.

(iii) $\Leftrightarrow$ (iv). If there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$, then $A = (I_m, 0_{m,n-m})S^{-1}$. Choose

$$B = S \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix}.$$

Then $AB = I_m$. So, $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is unimodular.

Conversely, if $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is unimodular, then there exists some $B \in R^{n\times m}$ such that $AB = I_m$. By (1), $\pi(I_m) = \pi(AB) = \pi(A)\pi(B)$, which implies that $\mathrm{rk}(\pi(A)) = m$ since R/M is a field. By Lemma 2.2, $\mathrm{rk}(A) = m$. So, there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$ by Lemma 2.4. □

For $T \in GL_m(R), S \in GL_n(R)$ and $\mathcal{M} \subseteq R^{m\times n}(m)$, we write $T\mathcal{M} = \{TA : A \in \mathcal{M}\}$ and $\mathcal{M}S = \{AS : A \in \mathcal{M}\}$.

Lemma 2.6. *Let $0 \le m \le n$. Then*

$$|R^{m\times n}(m)| = |R|^{\frac{m(m-1)}{2}} \prod_{i=0}^{m-1} (|R|^{n-i} - |M|^{n-i}).$$

In particular,

$$|GL_n(R)| = |R|^{\frac{n(n-1)}{2}} \prod_{i=0}^{n-1} (|R|^{n-i} - |M|^{n-i}).$$

Proof. Let $A = (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top \in R^{m\times n}(m)$. We enumerate how many A's there are. By Lemma 2.5, $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent, which implies that $\{\alpha_1\}$ is linearly independent. By Lemma 2.3, there are $|R|^n - |M|^n$ choices for α_1. Once α_1 is chosen, let

$$R^{m\times n}(m, \alpha_1) = \left\{ \begin{pmatrix} \alpha_1 \\ A_2 \end{pmatrix} \in R^{m\times n}(m) : A_2 \in R^{(m-1)\times n}(m-1) \right\}.$$

Similar to the proof of Lemma 2.4, there exists $S_1 \in GL_n(R)$ such that $R^{m\times n}(m, \alpha_1)S_1 = R^{m\times n}(m, e_1)$, where e_1 is the row vector in R^n in which 1st coordinate is 1 and all other coordinates are 0. Then $|R^{m\times n}(m, \alpha_1)| = |R^{m\times n}(m, e_1)|$, which implies that $|R^{m\times n}(m, \alpha_1)|$ is independent of the particular choice of α_1. Without loss of generality, we may assume that $\alpha_1 = e_1$ and $A_2 = (\alpha_2^\top, \ldots, \alpha_m^\top)^\top$. Since $\{e_1, \alpha_2\}$ is linearly independent, by Lemma 2.3 again, there are $|R|(|R|^{n-1} - |M|^{n-1})$ choices for α_2. Proceeding in this way, finally after $\alpha_1, \ldots, \alpha_{m-1}$ are chosen, there are $|R|^{m-1}(|R|^{n-m+1} - |M|^{n-m+1})$ choices for α_m. Therefore,

$$|R^{m\times n}(m)| = \prod_{i=0}^{m-1} |R|^i (|R|^{n-i} - |M|^{n-i}) = |R|^{\frac{m(m-1)}{2}} \prod_{i=0}^{m-1} (|R|^{n-i} - |M|^{n-i}),$$

as desired. □

2.2. Over commutative rings

In this subsection, we always assume that R is a commutative ring. It is well known that R is a product of local rings (cf. [1]). Write $(R, +, \cdot) = (R_1 \times R_2 \times \cdots \times R_\ell, +, \cdot)$, where $R_1, R_2, \ldots, R_\ell$ are local rings with maximal ideals $M_1, M_2, \ldots, M_\ell$, respectively. Let

$$\begin{array}{rccl} \rho_i : & R & \longrightarrow & R_i, \\ & (r_1, r_2, \ldots, r_\ell) & \longmapsto & r_i \end{array}$$

be the projection map for all $i \in [\ell]$. For $A = (a_{uv}) \in R^{m\times n}$ and $B \in R^{n\times k}$, let $\rho_i(A) = (\rho_i(a_{uv}))$ for all $i \in [\ell]$. Then

$$\rho_i(AB) = \rho_i(A)\rho_i(B) \text{ for all } i \in [\ell]. \tag{2}$$

From [10], we deduce that

$$(GL_n(R), \cdot) \cong (GL_n(R_1) \times GL_n(R_2) \times \cdots \times GL_n(R_\ell), \cdot), \tag{3}$$

and

$$(R^{m\times n}, +) \cong (R_1^{m\times n} \times R_2^{m\times n} \times \cdots \times R_\ell^{m\times n}, +). \tag{4}$$

Lemma 2.7. *(See [11, Proposition 1.3]) Let $\alpha_1, \alpha_2, \ldots, \alpha_s \in R^n$. Then $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is linearly independent over R if and only if $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_s)\}$ is linearly independent over R_i for all $i \in [\ell]$.*

For convenience, we write the identity (resp. the zero) of R_i as 1 (resp. 0) for all $i \in [\ell]$, and the identity and the zero of R also as 1 and 0,[1] respectively.

Lemma 2.8. *Let $m \le n$ and $A \in R^{m\times n}$. Then $\mathrm{rk}(A) = m$ if and only if there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$.*

Proof. If there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$, then $\mathrm{rk}(A) = \mathrm{rk}(I_m, 0_{m,n-m}) = m$ by Proposition 2.1.

Suppose $\mathrm{rk}(A) = m$. Let $A = (a_{ij}) = (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top$. By Proposition 2.1 (iii), $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent over R. By Lemma 2.7, $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_m)\}$ is linearly independent over R_i for all $i \in [\ell]$. For every $i \in [\ell]$, by Lemma 2.5, there exists an $S_i \in GL_n(R_i)$ such that $\rho_i(A)S_i = (I_m, 0_{m,n-m})$. From (3), we deduce that there exists the unique $S \in GL_n(R)$ such that $\rho_i(S) = S_i$ for all $i \in [\ell]$. By (2), we obtain

$$\rho_i(AS) = \rho_i(A)\rho_i(S) = (I_m, 0_{m,n-m}) = \rho_i(I_m, 0_{m,n-m}) \text{ for all } i \in [\ell].$$

By (4), $AS = (I_m, 0_{m,n-m})$. $\square$

[1] In fact, the identity and the zero of R are $\underbrace{(1, \ldots, 1)}_{\ell}$ and $\underbrace{(0, \ldots, 0)}_{\ell}$, respectively.

Lemma 2.9. *Let $m \leq n$ and $A = (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top \in R^{m\times n}$. Then the following statements are equivalent.*

(i) $\mathrm{rk}(A) = m$.

(ii) *$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent.*

(iii) *There exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$.*

(iv) *$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is unimodular.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). By Proposition 2.1 (iii) and Lemma 2.8.

(iii) $\Rightarrow$ (iv). The proof is similar to that of Lemma 2.4, and will be omitted.

(iv) $\Rightarrow$ (iii). If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is unimodular, then there exists some $B \in R^{n\times m}$ such that $AB = I_m$. By (2), $I_m = \rho_i(I_m) = \rho_i(AB) = \rho_i(A)\rho_i(B)$ for all $i \in [\ell]$, which implies that $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_m)\}$ is unimodular for all $i \in [\ell]$. By Lemma 2.5, $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_m)\}$ is linearly independent over R_i for all $\in [\ell]$, which implies that $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent over R by Lemma 2.7. From (ii) $\Leftrightarrow$ (iii), we deduce that there exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$. □

By Lemmas 2.5, 2.7 and 2.9, we obtain the following result.

Corollary 2.10. *Let $m \leq n$ and $A = (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top \in R^{m\times n}$. Then the following hold.*

(i) *$\mathrm{rk}(A) = m$ if and only if $\mathrm{rk}(\rho_i(A)) = m$ for all $i \in [\ell]$.*

(ii) *$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent over R if and only if $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_m)\}$ is linearly independent over R_i for all $i \in [\ell]$.*

(iii) *There exists an $S \in GL_n(R)$ such that $AS = (I_m, 0_{m,n-m})$ over R if and only if there exists an $S_i \in GL_n(R_i)$ such that $\rho_i(A)S_i = (I_m, 0_{m,n-m})$ over R_i for all $i \in [\ell]$.*

(iv) *$\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is unimodular over R if and only if $\{\rho_i(\alpha_1), \rho_i(\alpha_2), \ldots, \rho_i(\alpha_m)\}$ is unimodular over R_i for all $i \in [\ell]$.*

Lemma 2.11. *Let $m \leq n$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in R^n$. If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent, then $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ can be extended to a unimodular basis of R^n.*

Proof. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent, by Lemma 2.9, there is an $S \in GL_n(R)$ such that $A = (I_m, 0_{m,n-m})S^{-1}$, where $A = (\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top$. Then

$$\widetilde{A} = \begin{pmatrix} A \\ (0_{n-m,m}, I_{n-m})S^{-1} \end{pmatrix} = S^{-1},$$

is invertible. Write $\widetilde{A} = (\alpha_1^\top, \ldots, \alpha_m^\top, \alpha_{m+1}^\top, \ldots, \alpha_n^\top)^\top$. Then $\{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n\}$ is a unimodular basis of R^n. □

By (3), (4), Lemma 2.6 and Corollary 2.10, we obtain the following result.

Corollary 2.12. *Let $0 \leq m \leq n$. Then*

$$|R^{m\times n}(m)| = |R|^{\frac{m(m-1)}{2}} \prod_{j=1}^{\ell} \prod_{i=0}^{m-1} (|R_j|^{n-i} - |M_j|^{n-i}).$$

In particular,

$$|GL_n(R)| = |R|^{\frac{n(n-1)}{2}} \prod_{j=1}^{\ell} \prod_{i=0}^{n-1} (|R_j|^{n-i} - |M_j|^{n-i}).$$

3. Vector spaces

Suppose that $R = R_1 \times R_2 \times \cdots \times R_\ell$ is a commutative ring, where $R_1, R_2, \ldots, R_\ell$ are local rings with maximal ideals $M_1, M_2, \ldots, M_\ell$, respectively. In this section, we obtain Anzahl formulas and a dimensional formula for subspaces of R^n by using the matrix method, see Theorems 3.1, 3.3-3.5.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq R^n$, and $\langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ denote the R-module generated by $\alpha_1, \alpha_2, \ldots, \alpha_m$. If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is linearly independent, then $X = \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ is an m-subspace of R^n by Lemma 2.9, and the matrix $(\alpha_1^\top, \alpha_2^\top, \ldots, \alpha_m^\top)^\top$ is called a *matrix representation* of X. The matrix representation is not unique. We use an $m \times n$ matrix A with $\mathrm{rk}(A) = m$ to represent an m-subspace of R^n. For $A, B \in R^{m\times n}$ with $\mathrm{rk}(A) = \mathrm{rk}(B) = m$, both A and B represent the same m-subspace if and only if there exists a $T \in GL_m(R)$ such that $B = TA$. For convenience, if A is a matrix representation of an m-subspace X, then we write $X = A$.

For an m-subspace X of R^n, by Lemma 2.9, there exists an $S \in GL_n(R)$ such that X has matrix representations

$$X = (I_m, 0_{m,n-m})S = T(I_m, 0_{m,n-m})S \text{ for all } T \in GL_m(R). \tag{5}$$

There is an action of $GL_n(R)$ on R^n defined as follows:

$$\begin{array}{ccc} R^n \times GL_n(R) & \longrightarrow & R^n, \\ ((r_1, \ldots, r_n), S) & \longmapsto & (r_1, \ldots, r_n)S. \end{array}$$

The above action induces an action on the set of subspaces of R^n; i.e., a subspace P is carried by $S \in GL_n(R)$ to the subspace PS.

For $0 \le m \le n$, let $\mathcal{M}_R(m,n)$ denote the set of all the m-subspaces of R^n, and let $N_R(m,n)$ denote the size of $\mathcal{M}_R(m,n)$. Sirisuk and Meemark [11] determined the size of $\mathcal{M}_R(m,n)$ by using a lifting technique.

Theorem 3.1. *$\mathcal{M}_R(m,n)$ is non-empty if and only if $0 \le m \le n$. Moreover, if $\mathcal{M}_R(m,n)$ is non-empty, then it forms an orbit of subspaces under the action of $GL_n(R)$ and*

$$N_R(m,n) = \prod_{j=1}^{\ell} \prod_{i=0}^{m-1} \frac{|R_j|^{n-i} - |M_j|^{n-i}}{|R_j|^{m-i} - |M_j|^{m-i}}.$$

Proof. Clearly, $\mathcal{M}_R(m,n)$ is non-empty if and only if $0 \le m \le n$. For any m-subspace $P \in \mathcal{M}_R(m,n)$, by Lemma 2.9, there exists an $S \in GL_n(R)$ such that $PS = (I_m, 0_{m,n-m})$. By (5), $\mathcal{M}_R(m,n)$ forms an orbit of subspaces under the action of $GL_n(R)$.

Recall that $R^{m\times n}(m)$ is the set of all the $m \times n$ matrices of McCoy rank m over R. By

Corollary 2.12 and (5),

$$N_R(m,n) = \frac{|R^{m\times n}(m)|}{|GL_m(R)|} = \prod_{j=1}^{\ell}\prod_{i=0}^{m-1} \frac{|R_j|^{n-i} - |M_j|^{n-i}}{|R_j|^{m-i} - |M_j|^{m-i}},$$

as desired. □

Example 3.2. (See [12, Theorem 1.7], [9, Theorem 3.5] and [6, Theorem 3.5]).

(i) If R is the finite field $\mathbb{F}_q$, then

$$N_{\mathbb{F}_q}(m,n) = \prod_{i=0}^{m-1} \frac{q^{n-i}-1}{q^{m-i}-1}.$$

(ii) If R is the residue class ring $\mathbb{Z}_{p^s}$, where p is a prime number, then

$$N_{\mathbb{Z}_{p^s}}(m,n) = p^{(s-1)m(n-m)} \prod_{i=0}^{m-1} \frac{p^{n-i}-1}{p^{m-i}-1}.$$

(iii) If R is the residue class ring $\mathbb{Z}_{p^s q^t}$, where p and q are two distinct prime numbers, then

$$N_{\mathbb{Z}_{p^s q^t}}(m,n) = p^{(s-1)m(n-m)} q^{(t-1)m(n-m)} \prod_{i=0}^{m-1} \frac{(p^{n-i}-1)(q^{n-i}-1)}{(p^{m-i}-1)(q^{m-i}-1)}.$$

For a fixed m-subspace P of R^n, let $\mathcal{M}_R(m_1,m,n)$ denote the set of all the m_1-subspaces contained in P, and let $N_R(m_1,m,n)$ denote the size of $\mathcal{M}_R(m_1,m,n)$. By Theorem 3.1, $N_R(m_1,m,n)$ is independent of the particular choice of the subspace P.

Theorem 3.3. *$\mathcal{M}_R(m_1,m,n)$ is non-empty if and only if $0 \le m_1 \le m \le n$. Moreover, if $\mathcal{M}_R(m_1,m,n)$ is non-empty, then*

$$N_R(m_1,m,n) = N_R(m_1,m) = \prod_{j=1}^{\ell}\prod_{i=0}^{m_1-1} \frac{|R_j|^{m-i} - |M_j|^{m-i}}{|R_j|^{m_1-i} - |M_j|^{m_1-i}}.$$

Proof. By Theorem 3.1, we may assume that the fixed m-subspace $P = (I_m, 0_{m,n-m})$. Then $N_R(m_1,m,n) = N_R(m_1,m)$. □

For a fixed m_1-subspace P of R^n, let $\mathcal{M}'_R(m_1,m,n)$ denote the set of all the m-subspaces containing P, and let $N'_R(m_1,m,n)$ denote the size of $\mathcal{M}'_R(m_1,m,n)$. By Theorem 3.1, $N'_R(m_1,m,n)$ is independent of the particular choice of the subspace P.

Theorem 3.4. *$\mathcal{M}'_R(m_1,m,n)$ is non-empty if and only if $0 \le m_1 \le m \le n$. Moreover, if $\mathcal{M}'_R(m_1,m,n)$ is non-empty, then*

$$N'_R(m_1,m,n) = N_R(m-m_1,n-m_1) = \prod_{j=1}^{\ell}\prod_{i=0}^{m-m_1-1} \frac{|R_j|^{n-m_1-i} - |M_j|^{n-m_1-i}}{|R_j|^{m-m_1-i} - |M_j|^{m-m_1-i}}.$$

Proof. By counting the number of couples (P, Q), where $P \in \mathcal{M}_R(m_1, n)$ and $Q \in \mathcal{M}_R(m, n)$ with $P \subseteq Q$, by Theorems 3.1 and 3.3,

$$N'_R(m_1, m, n) = \frac{N_R(m, n)N_R(m_1, m, n)}{N_R(m_1, n)} = N_R(m - m_1, n - m_1).$$

Therefore, the desired result follows. □

Let X and Y be two subspaces of R^n. The set of vectors belonging to both X and Y is a linear subset of R^n, called the *intersection* of X and Y and denoted by $X \cap Y$. The set of vectors which can be written as sums of a vector of X and a vector of Y is also a linear subset of R^n, called the *join* of X and Y and denoted by $X + Y$. In general, the intersection and the join of two subspaces in R^n are not subspaces. Note that $X \cap Y = Y \cap X, X + Y = Y + X$, and $X \cap Y = X, X + Y = Y$ if $X \subseteq Y$.

Theorem 3.5. (Dimensional formula). *Let A and B be two subspaces of R^n. Then the following hold.*

(a)

$$\max\{\dim(A), \dim(B)\} \le \dim(A + B) \le \min\{n, \dim(A) + \dim(B) - \dim(A \cap B)\}.$$

(b) *The following statements are equivalent:*

(i) $\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$*;*

(ii) *$A + B$ is a subspace of R^n;*

(iii) *$A \cap B$ is a subspace of R^n.*

Proof. Suppose $\dim(A \cap B) = t, \dim(A) = m_1, \dim(B) = m_2$ and $\dim(A+B) = k$, where both $A \cap B$ and $A + B$ are linear subsets of R^n.

(a). By $A + B \subseteq R^n$, $k \le n$. Since $\dim(A \cap B) = t$, there is a largest unimodular vector subset $\{\alpha_1, \dots, \alpha_t\}$ of $A \cap B$. By Lemma 2.9, $\{\alpha_1, \dots, \alpha_t\}$ is linearly independent. By Lemma 2.11, $\{\alpha_1, \dots, \alpha_t\}$ can be extended to a unimodular basis $\{\alpha_1, \dots, \alpha_{m_1}\}$ of A and a unimodular basis $\{\alpha_1, \dots, \alpha_t, \beta_{t+1}, \dots, \beta_{m_2}\}$ of B. Then

$$A + B = \langle \alpha_1, \dots, \alpha_{m_1}, \beta_{t+1}, \dots, \beta_{m_2} \rangle. \tag{6}$$

Let $\{\gamma_1, \dots, \gamma_k\}$ be a largest unimodular vector subset of $A+B$. Write $A_1 = (\alpha_1^\top, \dots, \alpha_{m_1}^\top, \beta_{t+1}^\top, \dots, \beta_{m_2}^\top)^\top \in R^{(m_1+m_2-t)\times n}$ and $A_2 = (\gamma_1^\top, \dots, \gamma_k^\top)^\top \in R^{k\times n}$. Since $\{\gamma_1, \dots, \gamma_k\} \subseteq A+B$, there exists a $k\times(m_1+m_2-t)$ matrix M such that $A_2 = MA_1$. Since $\{\gamma_1, \dots, \gamma_k\}$ is unimodular, by Lemma 2.9, there is an $n \times k$ matrix D_2 such that $A_2D_2 = I_k$. Therefore, we have $M(A_1D_2) = I_k$, which implies that $\mathrm{rk}(M) = k$ by Lemma 2.9. By Proposition 2.1, $k \le m_1 + m_2 - t$. So, $k \le \min\{n, m_1 + m_2 - t\}$. Since $A \subseteq A + B$ and $B \subseteq A + B$, $\dim(A + B) \ge \max\{\dim(A), \dim(B)\}$.

(b). Suppose that A_1, A_2 and M are as the above.

(i) $\Leftrightarrow$ (ii). Suppose $k = m_1 + m_2 - t$. Then M is invertible, which implies that $M^{-1}A_2 = A_1$. If $X \in R^k$ such that $XA_1 = XM^{-1}A_2 = 0_{1,n}$, then $XM^{-1} = 0_{1,k}$ since $A_2D_2 = I_k$. Therefore, $X = 0_{1,k}$ and $\{\alpha_1, \dots, \alpha_{m_1}, \beta_{t+1}, \dots, \beta_{m_2}\}$ is linearly independent. From (6), we deduce that $A + B$ is a k-subspace.

Conversely, suppose that $A+B$ is a k-subspace of R^n. By Theorem 3.1, there is an $S\in GL_n(R)$ such that $(A+B)S=(I_k,0_{k,n-k})$. So, we may assume that

$$AS=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{m_1}\end{pmatrix}S=(I_{m_1},0_{m_1,n-m_1}),$$

and

$$BS=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_t\\ \beta_{t+1}\\ \vdots\\ \beta_{m_2}\end{pmatrix}S=\begin{pmatrix}I_t & 0_{t,m_1-t} & 0_{t,n-m_1}\\ 0_{m_2-t,t} & B_2 & B_3\end{pmatrix},$$

where $B_2\in R^{(m_2-t)\times(m_1-t)}$, $B_3\in R^{(m_2-t)\times(n-m_1)}$ and $\mathrm{rk}(B_2,B_3)=m_2-t$. Then

$$\begin{pmatrix}I_t & & \\ & I_{m_1-t} & \\ & -B_2 & I_{m_2-t}\end{pmatrix}A_1S=\begin{pmatrix}I_{m_1} & 0_{m_1,n-m_1}\\ 0_{m_2-t,m_1} & B_3\end{pmatrix}.$$

Write $B_3=(B_{31},B_{32})$, where $B_{31}\in R^{(m_2-t)\times(k-m_1)}$ and $B_{32}\in R^{(m_2-t)\times(n-k)}$. Since all the rows of $(0_{m_2-t,m_1},B_3)$ are contained in the subspace $(I_k,0_{k,n-k})$, there exists an $(m_2-t)\times k$ matrix M_1 such that $(0_{m_2-t,m_1},B_3)=M_1(I_k,0_{k,n-k})$, which implies that $B_{32}=0_{m_2-t,n-k}$. Similarly, there exist a $(k-m_1)\times m_1$ matrix M_2 and a $(k-m_1)\times(m_2-t)$ matrix M_3 such that

$$(0_{k-m_1,m_1},I_{k-m_1},0_{k-m_1,n-k})=(M_2,M_3)\begin{pmatrix}I_{m_1} & 0_{m_1,k-m_1} & 0_{m_1,n-k}\\ 0_{m_2-t,m_1} & B_{31} & 0_{m_2-t,n-k}\end{pmatrix},$$

which implies that $M_2=0_{k-m_1,m_1}$ and $M_3B_{31}=I_{k-m_1}$. By Lemma 2.9, there exists a $T\in GL_{m_2-t}(R)$ such that

$$TB_{31}=\begin{pmatrix}I_{k-m_1}\\ 0_{m_1+m_2-t-k,k-m_1}\end{pmatrix}.$$

Then

$$\begin{pmatrix}I_t & \\ & T\end{pmatrix}BS=\begin{pmatrix}I_t & 0_{t,m_1-t} & 0_{t,k-m_1} & 0_{t,n-k}\\ 0_{m_2-t,t} & B_{21} & I_{k-m_1} & 0_{k-m_1,n-k}\\ 0_{m_1+m_2-t-k,t} & B_{22} & 0_{m_1+m_2-t-k,k-m_1} & 0_{m_1+m_2-t-k,n-k}\end{pmatrix},$$

where $TB_2=\begin{pmatrix}B_{21}\\ B_{22}\end{pmatrix}$. From $\mathrm{rk}(B_2,B_3)=m_2-t$, we deduce that $\mathrm{rk}(B_{22})=m_1+m_2-t-k$. It follows that $t=\dim(A\cap B)=\dim(AS\cap BS)\geq m_1+m_2-k$. Since $k\leq m_1+m_2-t$, $k=m_1+m_2-t$.

(i) $\Leftrightarrow$ (iii). Suppose $k = m_1+m_2-t$. From (6), we deduce that $\{\alpha_1, \dots, \alpha_{m_1}, \beta_{t+1}, \dots, \beta_{m_2}\}$ is a unimodular basis of $A+B$. For any $\alpha \in A\cap B$, there exist $r_1, \dots, r_{m_1}, s_1, \dots, s_{m_2} \in R$ such that

$$\alpha = \sum_{j=1}^{m_1} r_j\alpha_j = \sum_{j=1}^{t} s_j\alpha_j + \sum_{j=t+1}^{m_2} s_j\beta_j.$$

It follows that $\sum_{j=1}^{t}(r_j - s_j)\alpha_j + \sum_{j=t+1}^{m_1} r_j\alpha_j - \sum_{j=t+1}^{m_2} s_j\beta_j = 0_{1,n}$. Since $\{\alpha_1, \dots, \alpha_{m_1}, \beta_{t+1}, \dots, \beta_{m_2}\}$ is linearly independent, $r_j = s_j$ for all $1 \le j \le t$, $r_j = 0$ for all $t+1 \le j \le m_1$ and $s_j = 0$ for all $t+1 \le j \le m_2$. So, $\alpha = \sum_{j=1}^{t} r_j\alpha_j \in \langle\alpha_1, \dots, \alpha_t\rangle$. Therefore, $A\cap B = \langle\alpha_1, \dots, \alpha_t\rangle$ is a t-subspace of R^n.

Conversely, suppose that $A\cap B$ is a t-subspace of R^n. Then $\{\alpha_1, \dots, \alpha_t\}$ is a unimodular basis of $A\cap B$. If $\sum_{i=1}^{m_1} r_i\alpha_i + \sum_{j=t+1}^{m_2} s_j\beta_j = 0_{1,n}$, then $\sum_{i=1}^{m_1} r_i\alpha_i = -\sum_{j=t+1}^{m_2} s_j\beta_j \in A\cap B$, and there exist $s_1, \dots, s_t \in R$ such that $\sum_{i=1}^{t} s_i\alpha_i + \sum_{j=t+1}^{m_2} s_j\beta_j = 0_{1,n}$. Since $\{\alpha_1, \dots, \alpha_t, \beta_{t+1}, \dots, \beta_{m_2}\}$ is a unimodular basis of B, $s_1 = \cdots = s_{m_2} = 0$. It follows that $\sum_{i=1}^{m_1} r_i\alpha_i = 0_{1,n}$, which implies that $r_1 = \cdots = r_{m_1} = 0$ since $\{\alpha_1, \dots, \alpha_{m_1}\}$ is a unimodular basis of A. Therefore, $\{\alpha_1, \dots, \alpha_{m_1}, \beta_{t+1}, \dots, \beta_{m_2}\}$ is linearly independent. By (5), $A+B$ is an (m_1+m_2-t)-subspace and $k = m_1+m_2-t$. □

Remark 3.6. Let A and B be two subspaces of R^n. If R is a field, then $\dim(A+B) = \dim(A)+\dim(B)-\dim(A\cap B)$ always holds. Suppose that R is not a field. Since R is a finite ring, each non-zero element in $R\backslash R^*$ is a zero divisor. Let $1 \le m_1 \le m_2, m_1+m_2 \le n$ and $a \in R\backslash R^*$ be a fixed non-zero element. Pick $A = (0_{m_1,n-m_1-m_2}, aI_{m_1}, 0_{m_1,m_2-m_1}, I_{m_1})$ and $B = (0_{m_2,n-m_2}, I_{m_2})$. Then $\dim(A+B) = m_2$ and $\dim(A\cap B) = 0$. So, $\dim(B) = m_2 = \dim(A+B) < m_1+m_2 = \dim(A)+\dim(B)-\dim(A\cap B)$.

4. Singular linear spaces

In this section, we always assume that $R = R_1 \times R_2 \times \cdots \times R_\ell$ is a commutative ring, where $R_1, R_2, \dots, R_\ell$ are local rings with maximal ideals $M_1, M_2, \dots, M_\ell$, respectively.

For two non-negative integers n and k, R^{n+k} denotes the $(n+k)$-dimensional row vector space over R. The set of all $(n+k)\times(n+k)$ invertible matrices over R of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0_{k,n} & T_{22} \end{pmatrix},$$

where $T_{11} \in GL_n(R)$ and $T_{22} \in GL_k(R)$, forms a group under matrix multiplication, called the *singular general linear group* of degree $n+k$ over R and denoted by $GL_{n+k,n}(R)$. There is an action of $GL_{n+k,n}(R)$ on R^{n+k} defined as follows:

$$\begin{array}{ccc} R^{n+k} \times GL_{n+k,n}(R) & \longrightarrow & R^{n+k}, \\ ((x_1, x_2, \dots, x_{n+k}), T) & \longmapsto & (x_1, x_2, \dots, x_{n+k})T. \end{array}$$

The vector space R^{n+k} together with the above group action, is called the $(n+k)$-dimensional *singular linear space* over R.

For $1 \leq i \leq n+k$, let e_i denote the row vector in R^{n+k} in which ith coordinate is 1 and all other coordinates are 0. Define $E = \langle e_{n+1}, e_{n+2}, \ldots, e_{n+k} \rangle$. Then E is a k-subspace of R^{n+k}. An m-subspace P of R^{n+k} is called an (m,s)-*subspace* if $P \cap E$ is an s-subspace. Let $\mathcal{M}_R(m,s;n+k,n)$ denote the set of all the (m,s)-subspaces of R^{n+k}, and let $N_R(m,s;n+k,n)$ denote the size of $\mathcal{M}_R(m,s;n+k,n)$.

Theorem 4.1. *$\mathcal{M}_R(m,s;n+k,n)$ is non-empty if and only if $0 \leq s \leq k$ and $0 \leq m-s \leq n$. Moreover, if $\mathcal{M}_R(m,s;n+k,n)$ is non-empty, then it forms an orbit of subspaces under the action of $GL_{n+k,n}(R)$ and*

$$N_R(m,s;n+k,n) = |R|^{(m-s)(k-s)} N_R(m-s,n) N_R(s,k),$$

where $N_R(m-s,n)$ is given by Theorem 3.1.

Proof. If $0 \leq s \leq k$ and $0 \leq m-s \leq n$, then $\langle e_1, \ldots, e_{m-s}, e_{n+1}, \ldots, e_{n+s} \rangle$ is an (m,s)-subspace, which implies that $\mathcal{M}_R(m,s;n+k,n)$ is non-empty. Conversely, suppose $P \in \mathcal{M}_R(m,s;n+k,n)$. Since $P \cap E$ is an s-subspace of E, it has a matrix representation $(0_{s,n}, P_{22})$, where $P_{22} \in R^{s\times k}(s)$. So, $0 \leq s \leq k$. By Lemma 2.11, P has a matrix representation

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0_{s,n} & P_{22} \end{pmatrix}, \tag{7}$$

where $P_{11} \in R^{(m-s)\times n}$ and $P_{12} \in R^{(m-s)\times k}$. By Theorem 3.5 and (5), $P+E$ is an $(m-s+k)$-subspace and has a matrix representation

$$\begin{pmatrix} P_{11} & 0_{m-s,k} \\ 0_{s,n} & I_k \end{pmatrix},$$

where $P_{11} \in R^{(m-s)\times n}(m-s)$. So, $0 \leq m-s \leq n$.

For each $P \in \mathcal{M}_R(m,s;n+k,n)$ has a matrix representation as in (7) with $P_{11} \in R^{(m-s)\times n}(m-s)$ and $P_{22} \in R^{s\times k}(s)$. By Lemma 2.9 and (5), P_{11} is an $(m-s)$-subspace of R^n, and P_{22} is an s-subspace of E. By Theorem 3.1, there exist $T_{11} \in GL_n(R)$ and $T_{22} \in GL_k(R)$ such that $P_{11}T_{11} = (I_{m-s}, 0_{m-s,n-m+s})$ and $P_{22}T_{22} = (I_s, 0_{s,k-s})$. Let

$$T_1 = \begin{pmatrix} T_{11} & 0_{n,k} \\ 0_{k,n} & T_{22} \end{pmatrix}.$$

Then PT_1 has a matrix representation

$$PT_1 = \begin{pmatrix} I_{m-s} & 0_{m-s,n-m+s} & 0_{m-s,s} & P_{14} \\ 0_{s,m-s} & 0_{s,n-m+s} & I_s & 0_{s,k-s} \end{pmatrix}.$$

Let

$$T_2 = \left(\begin{array}{ccc|c} I_{m-s} & & & -P_{14} \\ \hline & I_{n-m+s} & & \\ & & I_s & \\ & & & I_{k-s} \end{array}\right).$$

Then $T_1T_2 \in GL_{n+k,n}(R)$ and

$$PT_1T_2 = \begin{pmatrix} I_{m-s} & 0_{m-s,n-m+s} & 0_{m-s,s} & 0_{m-s,k-s} \\ 0_{s,m-s} & 0_{s,n-m+s} & I_s & 0_{s,k-s} \end{pmatrix}.$$

Therefore, $\mathcal{M}_R(m, s; n+k, n)$ forms an orbit of subspaces under the action of $GL_{n+k,n}(R)$.

Denote by $n_R(m, s; n+k, n)$ the number of all the matrices P of the form (7) with $P_{11} \in R^{(m-s)\times n}(m-s)$ and $P_{22} \in R^{s\times k}(s)$. Suppose that P and Q represent the same subspace, then there is an $m \times m$ invertible matrix S such that $SP = Q$. It follows that S is necessarily of the form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ 0_{s,m-s} & S_{22} \end{pmatrix} \in GL_{m,m-s}(R).$$

Moreover, if $SP = P$, then $S = I_m$. Consequently,

$$N_R(m, s; n+k, n) = \frac{n_R(m, s; n+k, n)}{|GL_{m,m-s}(R)|}.$$

By Corollary 2.12 and Theorem 3.1, the desired result follows. □

For a fixed (m, s)-subspace P of R^{n+k}, let $\mathcal{M}_R(m_1, s_1; m, s; n+k, n)$ denote the set of all the (m_1, s_1)-subspaces contained in P, and let $N_R(m_1, s_1; m, s; n+k, n)$ denote the size of $\mathcal{M}_R(m_1, s_1; m, s; n+k, n)$. By Theorem 4.1, $N_R(m_1, s_1; m, s; n+k, n)$ is independent of the particular choice of the subspace P.

Theorem 4.2. *$\mathcal{M}_R(m_1, s_1; m, s; n+k, n)$ is non-empty if and only if*

$$0 \le s_1 \le s \le k \quad \text{and} \quad 0 \le m_1 - s_1 \le m - s \le n. \tag{8}$$

Moreover, if $\mathcal{M}_R(m_1, s_1; m, s; n+k, n)$ is non-empty, then

$$N_R(m_1, s_1; m, s; n+k, n) = |R|^{(m_1-s_1)(s-s_1)} N_R(m_1 - s_1, m - s) N_R(s_1, s),$$

where $N_R(s_1, s)$ is given by Theorem 3.1.

Proof. Suppose that P is a fixed (m, s)-subspace of R^{n+k}. Then we may assume that

$$P = \begin{pmatrix} I_{m-s} & 0_{m-s,n-m+s} & 0_{m-s,s} & 0_{m-s,k-s} \\ 0_{s,m-s} & 0_{s,n-m+s} & I_s & 0_{s,k-s} \end{pmatrix}.$$

By Theorem 4.1,

$$\mathcal{M}_R(m_1, s_1; m, s; n+k, n) \ne \emptyset \Leftrightarrow \mathcal{M}_R(m_1, s_1; m, m-s) \ne \emptyset,$$

and

$$N_R(m_1, s_1; m, s; n+k, n) = N_R(m_1, s_1; m, m-s).$$

Therefore, the desired result follows. □

For a fixed (m_1, s_1)-subspace P of R^{n+k}, let $\mathcal{M}'_R(m_1, s_1; m, s; n+k, n)$ denote the set of all the (m, s)-subspaces containing P, and let $N'_R(m_1, s_1; m, s; n+k, n)$ denote the size of $\mathcal{M}'_R(m_1, s_1; m, s; n+k, n)$. By Theorem 4.1, $N'_R(m_1, s_1; m, s; n+k, n)$ is independent of the particular choice of the subspace P.

Theorem 4.3. *$\mathcal{M}'_R(m_1, s_1; m, s; n+k, n)$ is non-empty if and only if (8) holds. Moreover, if $\mathcal{M}'_R(m_1, s_1; m, s; n+k, n)$ is non-empty, then*

$$N'_R(m_1, s_1; m, s; n+k, n) = |R|^{(k-s)(m-s-m_1+s_1)} N_R(m-s-m_1+s_1, n-m_1+s_1) N_R(s-s_1, k-s_1),$$

where $N_R(s - s_1, k - s_1)$ is given by Theorem 3.1.

Proof. By counting the number of couples (P, Q), where $P \in \mathcal{M}_R(m_1, s_1; n+k, n)$ and $Q \in \mathcal{M}_R(m, s; n+k, n)$ with $P \subseteq Q$, by Theorems 4.1 and 4.2,

$$N'_R(m_1, s_1; m, s; n+k, n) = \frac{N_R(m, s; n+k, n) N_R(m_1, s_1; m, s; n+k, n)}{N_R(m_1, s_1; n+k, n)}.$$

Therefore, the desired result follows. □

5. NM properties of subspace posets

The structure of this section is as follows: (i) define the restricted poset $P_R[t, P_0; n]$ in (9), (ii) pass $P_R[t, P_0; n]$ to the quotient poset $P_R[t, n]$ in (10), (iii) reduce NM property of $P_R[t, P_0; n]$ to the inequality (11), and (iv) prove (11) via Lemmas 5.6-5.9.

Let P be a finite graded poset, i.e., there is a function $r : P \to \mathbb{N}$ with $r(x) = 0$ if x is minimal in P, and $r(y) = r(x) + 1$ if y covers x, where $\mathbb{N}$ is the set of all the nonnegative integers. A *chain* of P is a subset of P whose elements are totally comparable. The maximum length of a chain of P is called the rank of P, denoted by $r(P)$. An *antichain* is a subset A of P which no two elements are comparable in P. A *k-family* is a subset of P that contains no chains of length k. For $i = 0, 1, \ldots, r(P)$, let $P_i = \{x \in P : r(x) = i\}$. Clearly, each P_i is an antichain in P, and each union of k rank sets of P is a k-family.

As stated in [4], some extremal problems can be considered in a weighted poset (P, w), which is a poset P together with a function $w : P \to \mathbb{N}$. The weight $w(A)$ of a subset A of P is defined by $w(A) = \sum_{a \in A} w(a)$. Every poset P can be considered a weighted poset (P, w) with $w \equiv 1$, that is, $w(x) = 1$ for all $x \in P$.

We say a weighted poset (P, w) has the *k-Sperner property* if a union of k rank sets of P is a maximal-weighted k-family of P. We say P has the *strong Sperner property* if it has the k-Sperner property for all k. We say (P, w) has the *LYM property* if, for any antichain A in P, the following inequality holds

$$\sum_{i=0}^{r(P)} \frac{w(A \cap P_i)}{w(P_i)} \leq 1.$$

We say (P, w) has the *normalized matching (NM) property* if, for any subset A of P_i, the following inequalities hold

$$\frac{w(\nabla(A))}{w(A)} \geq \frac{w(P_{i+1})}{w(P_i)}, \quad 0 \leq i \leq r(P) - 1,$$

where $\nabla(A) = \{b \in P_{i+1} : a \leq b \text{ for some } a \in A\}$. It is well known that the LYM property and the NM property are equivalent [5], which implies the strong Sperner property.

Let P be a finite graded poset and G a group acting on P which preserves the order relation and the weight on P, that is, for every $g \in G$, $x \leq y$ in P implies $g(x) \leq g(y)$ and $w(g(x)) = w(x)$. For $x \in P$, let $Gx = \{g(x) : g \in G\}$, which is a G-orbit. Then we have the quotient poset $(P/G, w_G)$, where P/G consists of all the G-orbits ordered as follows: $Gx \leq Gy$ in P/G if there exist $x' \in Gx$ and $y' \in Gy$ such that $x' \leq y'$ in P, and the weight function w_G is given by $w_G(Gx) = |Gx|w(x) = |Gx|$. Wang discovered the relationship between the NM property of a graded poset and that of its quotient posets.

Theorem 5.1. (See [13]). *(P, w) has the NM property if and only if $(P/G, w_G)$ does.*

Let $\mathbb{F}_q^n$ be the n-dimensional row vector space over the finite field $\mathbb{F}_q$. The linear lattice $L_n(q)$ consists of all subspaces of $\mathbb{F}_q^n$ ordered by inclusion. The NM property of subposets of $L_n(q)$ was studied in [7, 8, 14, 16, 15].

Recall that R^n is the n-dimensional row vector spaces over the finite commutative ring R. Then all the subspaces of R^n form a poset ordered by inclusion, denoted by $\mathcal{P}_n(R)$. For $0 \leq t \leq k \leq n-1$, let P_0 be a fixed k-subspace of R^n, and

$$\mathcal{P}_R[t, P_0; n] := \{Q \in \mathcal{P}_n(R) : Q \cap P_0 \in \mathcal{P}_n(R), \dim(Q \cap P_0) \geq t\}. \tag{9}$$

Then $\mathcal{P}_R[t, P_0; n]$ is a subposet of $\mathcal{P}_n(R)$.

In this section, we obtain the following result.

Theorem 5.2. *The poset $\mathcal{P}_R[t, P_0; n]$ (resp. $\mathcal{P}_n(R)$) has the NM property, which implies that $\mathcal{P}_R[t, P_0; n]$ (resp. $\mathcal{P}_n(R)$) has the strong Sperner property and the LYM property.*

For convenience, we assume that $P_0 = (0_{k,n-k}, I_k)$ and write $\langle m, s\rangle = \mathcal{M}_R(m, s; n, n-k)$ in this section. Let $G_0 = GL_{n,n-k}(R)$. Then $P_0G_0 = \{P_0\}$, and G_0 induces an order preserving permutation group acting on $\mathcal{P}_R[t, P_0; n]$. Using (9) and Theorem 4.1, we find

$$\mathcal{P}_R[t, P_0; n] = \bigcup_{\substack{t \leq s \leq k \\ 0 \leq m-s \leq n-k}} \langle m, s\rangle,$$

which implies that

$$\mathcal{P}_R[t, n] := \mathcal{P}_R[t, P_0; n]/G_0 = \{\langle m, s\rangle : t \leq s \leq k, 0 \leq m-s \leq n-k\}. \tag{10}$$

By Theorems 4.2 and 4.3, $\langle m, s\rangle \leq \langle m', s'\rangle$ in $\mathcal{P}_R[t, n]$ if and only if $s \leq s'$ and $m - s \leq m' - s'$.

Lemma 5.3. *$\mathcal{P}_R[t, P_0; n]$ (resp. $\mathcal{P}_R[t, n]$) is a finite graded poset.*

Proof. Define the function

$$\begin{array}{rcc} r : \mathcal{P}_R[t, P_0; n] & \longrightarrow & \mathbb{N}, \\ A & \longmapsto & \dim(A) - t. \end{array}$$

Clearly, $r(A) = 0$ if $A \in \langle t, t\rangle$. Let $A \in \langle m, s\rangle$ and $A' \in \langle m', s'\rangle$ such that A' covers A. Then $m' \geq m+1$, and $s \leq s'$ and $m - s \leq m' - s'$ by Theorems 4.2 and 4.3. In order to prove that $\mathcal{P}_R[t, P_0; n]$ is graded, we only need to show $m' = m+1$. Suppose $m' \geq m+2$. By Lemma 2.11, we may assume that $\{\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{m'}\}$ is a unimodular basis of A' such that $\{\alpha_1, \ldots, \alpha_m\}$ is a unimodular basis of A. Then $B = \langle \alpha_1, \ldots, \alpha_m, \alpha_{m+1}\rangle \in \langle m+1, s\rangle \cup \langle m+1, s+1\rangle$ satisfies $A < B < A'$, a contradiction since A' covers A. It follows that $m' = m+1$.

Since the group G_0 preserves the order relation of $\mathcal{P}_R[t, P_0; n]$, the quotient poset $\mathcal{P}_R[t, n]$ is also a finite graded poset. □

Next, we prove that $\mathcal{P}_n(R)$ has the NM property. We begin with a useful result.

Theorem 5.4. (See [13]). *Let P be a finite graded poset and G a group acting on P which preserves the order relation on P. Then for every i with $1 \leq i < r(P)$, there is a subset A of P_i which is invariant under the action of G and*

$$\frac{|\nabla(A)|}{|A|} = \min\left\{\frac{|\nabla(B)|}{|B|} : B \subseteq P_i\right\}.$$

Theorem 5.5. *The graded poset $\mathcal{P}_n(R)$ has the NM property.*

Proof. By Theorem 3.1, the group $GL_n(R)$ acts transitively on each set of subspaces with the same dimension in $\mathcal{P}_n(R)$, which implies that $\mathcal{P}_n(R)_m$ has no proper subset invariant under the action of $GL_n(R)$. By Theorem 5.4, for all $\mathcal{B} \subseteq \mathcal{P}_n(R)_m$, we have

$$\frac{|\nabla(\mathcal{B})|}{|\mathcal{B}|} \geq \frac{|\nabla(\mathcal{P}_n(R)_m)|}{|\mathcal{P}_n(R)_m|},$$

as desired. □

In the rest of this section, we always assume that $0 \leq t \leq k$. We shall show that $\mathcal{P}_R[t, P_0; n]$ has the NM property by considering its quotient poset $\mathcal{P}_R[t, n]$.

For each $\langle m, s\rangle \in \mathcal{P}_R[t, n]$, we have $w_{G_0}(\langle m, s\rangle) = N_R(m, s; n, n-k)$. For $t \leq m \leq n$, let $\mathscr{C}_m = \mathcal{P}_R[t, n]_{m-t}$.

By Theorem 5.1, in order to prove that $\mathcal{P}_R[t, P_0; n]$ has the NM property, it suffices to show that for any subset M of $\mathscr{C}_m$,

$$\frac{w_{G_0}(\nabla(M))}{w_{G_0}(M)} \geq \frac{w_{G_0}(\mathscr{C}_{m+1})}{w_{G_0}(\mathscr{C}_m)} \quad (t \leq m \leq n-1). \tag{11}$$

Lemma 5.6. *For $i = 1, 2$, let $\langle m, s_i\rangle, \langle m+1, s_i\rangle, \langle m+1, s_i+1\rangle \in \mathcal{P}_R[t, n]$. If $s_1 \leq s_2$, then*

$$\frac{w_{G_0}(\langle m+1, s_1\rangle)}{w_{G_0}(\langle m, s_1\rangle)} \leq \frac{w_{G_0}(\langle m+1, s_2\rangle)}{w_{G_0}(\langle m, s_2\rangle)} \quad \textit{and} \quad \frac{w_{G_0}(\langle m+1, s_1+1\rangle)}{w_{G_0}(\langle m, s_1\rangle)} \geq \frac{w_{G_0}(\langle m+1, s_2+1\rangle)}{w_{G_0}(\langle m, s_2\rangle)}.$$

Proof. Write $\theta = n - k - m$. By Theorem 4.1,

$$\frac{w_{G_0}(\langle m+1, s_i\rangle)}{w_{G_0}(\langle m, s_i\rangle)} = |R|^{k-s_i} \prod_{j=1}^{\ell} \frac{|R_j|^{\theta+s_i} - |M_j|^{\theta+s_i}}{|R_j|^{m+1-s_i} - |M_j|^{m+1-s_i}},$$

$$\frac{w_{G_0}(\langle m+1, s_i+1\rangle)}{w_{G_0}(\langle m, s_i\rangle)} = \frac{1}{|R|^{m-s_i}} \prod_{j=1}^{\ell} \frac{|R_j|^{k-s_i} - |M_j|^{k-s_i}}{|R_j|^{s_i+1} - |M_j|^{s_i+1}}.$$

Therefore,

$$\frac{\frac{w_{G_0}(\langle m+1,s_1\rangle)}{w_{G_0}(\langle m,s_1\rangle)}}{\frac{w_{G_0}(\langle m+1,s_2\rangle)}{w_{G_0}(\langle m,s_2\rangle)}} = |R|^{s_2-s_1} \prod_{j=1}^{\ell} \frac{(|R_j|^{\theta+s_1} - |M_j|^{\theta+s_1})(|R_j|^{m+1-s_2} - |M_j|^{m+1-s_2})}{(|R_j|^{\theta+s_2} - |M_j|^{\theta+s_2})(|R_j|^{m+1-s_1} - |M_j|^{m+1-s_1})} \leq 1,$$

and

$$\frac{\frac{w_{G_0}(\langle m+1,s_1+1\rangle)}{w_{G_0}(\langle m,s_1\rangle)}}{\frac{w_{G_0}(\langle m+1,s_2+1\rangle)}{w_{G_0}(\langle m,s_2\rangle)}} = |R|^{s_1-s_2} \prod_{j=1}^{\ell} \frac{(|R_j|^{k-s_1} - |M_j|^{k-s_1})(|R_j|^{s_2+1} - |M_j|^{s_2+1})}{(|R_j|^{k-s_2} - |M_j|^{k-s_2})(|R_j|^{s_1+1} - |M_j|^{s_1+1})} \geq 1,$$

as desired. □

For a given positive integer $t \leq m \leq n-1$, let $s_* = \max\{t, m+k-n\}$ and $s^* = \min\{m, k\}$. By (10),

$$\mathscr{C}_m = \{\langle m, s_*\rangle, \langle m, s_*+1\rangle, \ldots, \langle m, s^*\rangle\}. \tag{12}$$

Lemma 5.7. *For $0 \leq l \leq s^* - s_*$, let*

$$\mathscr{S}_l = \{\langle m, s_*+l\rangle, \ldots, \langle m, s^*\rangle\}, \quad \mathscr{T}_l = \{\langle m, s_*\rangle, \ldots, \langle m, s_*+l\rangle\}.$$

Then

$$\frac{w_{G_0}(\nabla(\mathscr{S}_{s^*-s_*}))}{w_{G_0}(\mathscr{S}_{s^*-s_*})} \geq \frac{w_{G_0}(\nabla(\mathscr{S}_{s^*-s_*-1}))}{w_{G_0}(\mathscr{S}_{s^*-s_*-1})} \geq \cdots \geq \frac{w_{G_0}(\nabla(\mathscr{S}_0))}{w_{G_0}(\mathscr{S}_0)} = \frac{w_{G_0}(\nabla(\mathscr{C}_m))}{w_{G_0}(\mathscr{C}_m)},$$

$$\frac{w_{G_0}(\nabla(\mathscr{T}_0))}{w_{G_0}(\mathscr{T}_0)} \geq \frac{w_{G_0}(\nabla(\mathscr{T}_1))}{w_{G_0}(\mathscr{T}_1)} \geq \cdots \geq \frac{w_{G_0}(\nabla(\mathscr{T}_{s^*-s_*}))}{w_{G_0}(\mathscr{T}_{s^*-s_*})} = \frac{w_{G_0}(\nabla(\mathscr{C}_m))}{w_{G_0}(\mathscr{C}_m)}.$$

Proof. By (12), $\nabla(\mathscr{C}_m) = \mathscr{C}_{m+1}$ is one of the following sets:

$$\begin{aligned}
\mathscr{C}_{m+1}^{(1)} &:= \{\langle m+1, s_*+1\rangle, \ldots, \langle m+1, s^*\rangle\} && \text{if } t \leq m+k-n \text{ and } m \geq k,\\
\mathscr{C}_{m+1}^{(2)} &:= \{\langle m+1, s_*+1\rangle, \ldots, \langle m+1, s^*+1\rangle\} && \text{if } t \leq m+k-n \text{ and } m < k,\\
\mathscr{C}_{m+1}^{(3)} &:= \{\langle m+1, s_*\rangle, \ldots, \langle m+1, s^*\rangle\} && \text{if } t > m+k-n \text{ and } m \geq k,\\
\mathscr{C}_{m+1}^{(4)} &:= \{\langle m+1, s_*\rangle, \ldots, \langle m+1, s^*+1\rangle\} && \text{if } t > m+k-n \text{ and } m < k.
\end{aligned}$$

For each $1 \leq l \leq s^* - s_* - 2$, we have $\nabla(\langle m, s_*+l\rangle) = \{\langle m+1, s_*+l\rangle, \langle m+1, s_*+l+1\rangle\}$ by (10). It follows that

$$\nabla(\mathscr{S}_l) = \{\langle m+1, s_*+l\rangle \,\dot\cup\, \nabla(\mathscr{S}_{l+1}) \quad (1 \leq l \leq s^* - s_* - 1),$$

and

$$\nabla(\mathscr{T}_l) \,\dot\cup\, \{\langle m+1, s_*+l+2\rangle\} = \nabla(\mathscr{T}_{l+1}) \quad (0 \leq l \leq s^* - s_* - 2).$$

By Lemma 5.6,

$$\begin{aligned}\frac{w_{G_0}(\nabla(\mathscr{S}_l))}{w_{G_0}(\mathscr{S}_l)} &= \frac{w_{G_0}(\langle m+1, s_*+l\rangle) + w_{G_0}(\nabla(\mathscr{S}_{l+1}))}{w_{G_0}(\langle m, s_*+l\rangle) + w_{G_0}(\mathscr{S}_{l+1})} \\ &\leq \frac{w_{G_0}(\nabla(\mathscr{S}_{l+1}))}{w_{G_0}(\mathscr{S}_{l+1})} \ (1 \leq l \leq s^* - s_* - 1),\end{aligned}$$

and

$$\begin{aligned}\frac{w_{G_0}(\nabla(\mathscr{T}_l))}{w_{G_0}(\mathscr{T}_l)} &= \frac{w_{G_0}(\nabla(\mathscr{T}_{l+1})) - w_{G_0}(\langle m+1, s_*+l+2\rangle)}{w_{G_0}(\mathscr{T}_{l+1}) - w_{G_0}(\langle m, s_*+l+1\rangle)} \\ &\geq \frac{w_{G_0}(\nabla(\mathscr{T}_{l+1}))}{w_{G_0}(\mathscr{T}_{l+1})} \ (0 \leq l \leq s^* - s_* - 2).\end{aligned}$$

In order to prove this lemma, we only need to show that

$$\frac{w_{G_0}(\nabla(\mathscr{S}_1))}{w_{G_0}(\mathscr{S}_1)} \geq \frac{w_{G_0}(\nabla(\mathscr{S}_0))}{w_{G_0}(\mathscr{S}_0)}, \quad \frac{w_{G_0}(\nabla(\mathscr{T}_{s^*-s_*-1}))}{w_{G_0}(\mathscr{T}_{s^*-s_*-1})} \geq \frac{w_{G_0}(\nabla(\mathscr{T}_{s^*-s_*}))}{w_{G_0}(\mathscr{T}_{s^*-s_*})}. \tag{13}$$

If $\mathscr{C}_{m+1} = \mathscr{C}_{m+1}^{(1)}$, then $\nabla(\mathscr{S}_0) = \nabla(\mathscr{S}_1)$ and $\nabla(\mathscr{T}_{s^*-s_*-1}) = \nabla(\mathscr{T}_{s^*-s_*})$. If $\mathscr{C}_{m+1} = \mathscr{C}_{m+1}^{(2)}$, then $\nabla(\mathscr{S}_0) = \nabla(\mathscr{S}_1)$ and $\nabla(\mathscr{T}_{s^*-s_*-1})\dot\cup\{\langle m+1, s^*+1\rangle\} = \nabla(\mathscr{T}_{s^*-s_*})$. If $\mathscr{C}_{m+1} = \mathscr{C}_{m+1}^{(3)}$, then $\nabla(\mathscr{S}_0) = \{\langle m+1, s_*\rangle\}\dot\cup\nabla(\mathscr{S}_1)$ and $\nabla(\mathscr{T}_{s^*-s_*-1}) = \nabla(\mathscr{T}_{s^*-s_*})$. If $\mathscr{C}_{m+1} = \mathscr{C}_{m+1}^{(4)}$, then $\nabla(\mathscr{S}_0) = \{\langle m+1, s_*\rangle\}\dot\cup\nabla(\mathscr{S}_1)$ and $\nabla(\mathscr{T}_{s^*-s_*-1})\dot\cup\{\langle m+1, s^*+1\rangle\} = \nabla(\mathscr{T}_{s^*-s_*})$. By Lemma 5.6, (13) hold in the above four cases. □

Lemma 5.8. *Let $M = \{\langle m, s_*+l\rangle, \ldots, \langle m, s_*+u\rangle\}$ be a subset of $\mathscr{C}_m$, where $0 \leq l \leq u \leq s^* - s_*$. Then*

$$\frac{w_{G_0}(\nabla(M))}{w_{G_0}(M)} \geq \frac{w_{G_0}(\mathscr{C}_{m+1})}{w_{G_0}(\mathscr{C}_m)}.$$

Proof. Since $M = \mathscr{S}_l \cap \mathscr{T}_u$ and $\nabla(M) = \nabla(\mathscr{S}_l) \cap \nabla(\mathscr{T}_u)$, we have $\mathscr{S}_l \cup \mathscr{T}_u = \mathscr{C}_m$ and $\nabla(\mathscr{S}_l) \cup \nabla(\mathscr{T}_u) = \mathscr{C}_{m+1}$, which implies that $w_{G_0}(M) = w_{G_0}(\mathscr{S}_l) + w_{G_0}(\mathscr{T}_u) - w_{G_0}(\mathscr{C}_m)$ and $w_{G_0}(\nabla(M)) = w_{G_0}(\nabla(\mathscr{S}_l)) + w_{G_0}(\nabla(\mathscr{T}_u)) - w_{G_0}(\nabla(\mathscr{C}_m))$. By Lemma 5.7, we have

$$\frac{w_{G_0}(\nabla(M))}{w_{G_0}(M)} = \frac{w_{G_0}(\nabla(\mathscr{S}_l)) + w_{G_0}(\nabla(\mathscr{T}_u)) - w_{G_0}(\nabla(\mathscr{C}_m))}{w_{G_0}(\mathscr{S}_l) + w_{G_0}(\mathscr{T}_u) - w_{G_0}(\mathscr{C}_m)} \geq \frac{w_{G_0}(\mathscr{C}_{m+1})}{w_{G_0}(\mathscr{C}_m)},$$

as desired. □

Lemma 5.9. *The poset $\mathcal{P}_R[t, P_0; n]$ has the NM property.*

Proof. By Lemma 5.3, $\mathcal{P}_R[t, n]$ is a finite graded poset. For any subset M of $\mathscr{C}_m$, set $M = M_0 \cup M_1 \cup \cdots \cup M_a$ $(a \geq 1)$, where $M_i = \{\langle m, s_*+l_i\rangle, \ldots, \langle m, s_*+u_i\rangle\}$ $(i = 0, 1, \ldots, a)$, $0 \leq l_0 \leq u_0 < l_1 \leq u_1 < \cdots < l_a \leq u_a \leq s^* - s_*$. Then $M_i \cap M_{i'} = \emptyset$ for $i \neq i'$. Since $\nabla(M_i \cap M_{i'}) = \nabla(M_i) \cap \nabla(M_{i'})$, $\nabla(M_i) \cap \nabla(M_{i'}) = \emptyset$ for $i \neq i'$. It follows that $w_{G_0}(M) = w_{G_0}(M_0) + \cdots + w_{G_0}(M_a)$ and $w_{G_0}(\nabla(M)) = w_{G_0}(\nabla(M_0)) + \cdots + w_{G_0}(\nabla(M_a))$. Then (11) holds by Lemma 5.8, and so the desired result follows. □

Combining Lemma 5.5 and Lemma 5.9, we complete the proof of Theorem 5.2.

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References

[1] M. F. Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Co., Mass.-London-Don Mills, Ontario, 1969.

[2] J. V. Brawley and L. Carlitz. Enumeration of matrices with prescribed row and column sums. *Linear Algebra and its Applications*, 6:165–174, 1973. https://doi.org/10.1016/0024-3795(73)90016-5.

[3] W. C. Brown. *Matrices Over Commutative Rings*. Marcel Dekker, New York/Basel/Hong Kong, 1993.

[4] K. Engel. *Sperner Theory*. Cambridge Univ. Press, Cambridge, 1997.

[5] J. R. Griggs. On chains and sperner k-families in ranked posets. *Journal of Combinatorial Theory, Series A*, 28:156–168, 1980. https://doi.org/10.1016/0097-3165(80)90082-5.

[6] J. Guo. Erdős-ko-rado theorem for vector spaces over residue class rings. *Journal of Algebra and Its Applications*, 23:2450166, 2024. https://doi.org/10.1142/S0219498824501664.

[7] J. Guo, F. Li, and K. Wang. Normalized matching property of posets generated by orbits of subspaces under finite symplectic groups. *Communications in Algebra*, 42:1711–1717, 2014. http://dx.doi.org/10.1080/00927872.2012.748783.

[8] J. Guo, K. Wang, and F. Li. Normalized matching property of subspace posets in finite classical polar spaces. *Finite Fields and Their Applications*, 19:67–72, 2013. http://dx.doi.org/10.1016/j.ffa.2012.08.004.

[9] L. Huang, B. Lv, and K. Wang. Erdős-ko-rado theorem, grassmann graphs and p^s-kneser graphs for vector spaces over a residue class ring. *Journal of Combinatorial Theory, Series A*, 164:125–158, 2019. https://doi.org/10.1016/j.jcta.2019.01.003.

[10] B. R. McDonald. *Linear Algebra over Commutative Rings*. Marcel Dekker, Inc., New York/Basel, 1984.

[11] S. Sirisuk and Y. Meemark. Generalized symplectic graphs and generalized orthogonal graphs over finite commutative rings. *Linear and Multilinear A.*, 67:2427–2450, 2019. https://doi.org/10.1080/03081087.2018.1494124.

[12] Z. Wan. *Geometry of Classical Groups over Finite Fields*. second edition. Science Press, Beijing/New York, 2002.

[13] J. Wang. Normalized matching property of the subgroup lattice of an abelian p-group. *Discrete Mathematics*, 257:559–574, 2002. https://doi.org/10.1016/S0012-365X(02)00514-9.

[14] J. Wang and H. J. Zhang. Normalized matching property of a class of subspace lattices. *Taiwanese Journal of Mathematics*, 11:43–50, 2007. https://www.jstor.org/stable/43833773.

[15] J. Wang and H. J. Zhang. q-weighted log-concavity and q-product theorem on the normality of posets. *Advances in Applied Mathematics*, 41:395–406, 2008. https://doi.org/10.1016/j.aam.2007.11.005.

[16] J. Wang and H. J. Zhang. Normalized matching property of restricted subspace lattices. *SIAM Journal on Discrete Mathematics*, 22:248–255, 2008. https://doi.org/10.1137/050635110.

[17] K. Wang, J. Guo, and F. Li. Association schemes based on attenuated spaces. *European Journal of Combinatorics*, 31:297–305, 2010. https://doi.org/10.1016/j.ejc.2009.01.002.

[18] K. Wang, J. Guo, and F. Li. Singular linear space and its applications. *Finite Fields and Their Applications*, 17:395–406, 2011. https://doi.org/10.1016/j.ffa.2011.02.001.

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