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A note on three-quarters circulant digraphs

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ABSTRACT

We introduce and study a new family of circulant digraphs associated with the cyclic group $\mathbb{Z}_N$, obtained by restricting admissible combinations of two generators a and b to three coordinate sectors. The resulting distance-like function differs from the standard directed distance in circulant digraphs and gives rise to new geometric and combinatorial phenomena. Using planar lattice representations and periodic tessellations, we analyze the growth of reachable sets and derive Moore-type upper bounds for the corresponding order/diameter problem. We construct explicit infinite families of three-quarters circulant structures with the prescribed diameter and provide lattice-based methods for determining admissible generator pairs. Separate constructions are obtained for even and odd diameters. In addition, computational experiments for small and moderate orders suggest improved families for even diameters and motivate a conjectural asymptotic formula for the maximum attainable order. The paper highlights the interplay between constrained lattice representations, periodic tilings, and extremal problems for circulant networks.

Keywords: circulant digraphs, three-quarter digraphs, diameter, degree/diameter problem, integer groups, plane tessellations

2020 Mathematics Subject Classification: 05C10, 05C50.

1. Introduction

Circulant digraphs form a central class of Cayley digraphs on cyclic groups and have been extensively studied due to their rich algebraic structure and wide applicability in areas such as communication networks, parallel architectures, and combinatorial optimization. Their vertex-transitivity and regularity make them natural candidates for addressing extremal problems, most notably the *degree/diameter problem*, which seeks the largest pos-

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sible number of vertices in a graph or digraph with given maximum degree and diameter (see Bermond, Delorme, and Quisquater [1], and the comprehensive survey by Miller and Širáň [3]).

A classical approach to studying distance-related properties in such graphs relies on embedding them into geometric structures. In particular, the representation of circulant digraphs by means of plane tessellations has proved to be a powerful tool to analyze distances, diameters, and shortest paths. This methodology, which can be traced back to the work of Wong and Coppersmith [6], was further developed in a series of papers by Morillo, Fiol, and Fàbrega [5], Morillo, Comellas, and Fiol [4], and Yebra, Fiol, Morillo, and Alegre [7]. In these papers, optimal constructions for families of graphs associated with planar tilings were obtained. These techniques provide both an intuitive geometric interpretation and an effective combinatorial framework for obtaining extremal results.

Let $CD(N, a, b)$ denote the circulant digraph with vertex set $\mathbb{Z}_N$ and generating set $\{a, b\}$. Such digraphs are regular of degree two and vertex-transitive, and their (strongly) connectivity is determined by the condition $\gcd(N, a, b) = 1$. Traditionally, distances in $CD(N, a, b)$ are defined by directed paths using only the generators a and b. However, this standard notion does not exhaust all possible combinatorial structures that can arise from such generating sets.

In this paper, we introduce and investigate a new infinite family of circulant digraphs, which we call *three-quarters digraphs*. These digraphs are defined by allowing paths that use combinations of steps of the form $(+a, +b)$, $(+a, -b)$, and $(-a, +b)$. This modification induces a non-standard distance function that differs significantly from the classical directed distance. As a consequence, the resulting metric structure exhibits new geometric and combinatorial features that are not present in the usual circulant setting.

One of the main advantages of this model is that the vertices reachable from a fixed origin can be arranged in a planar pattern that resembles a discrete tessellation. In this representation, the number of vertices at distance exactly ℓ grows linearly as $3\ell + 1$, leading to cumulative counts of the form $1, 5, 12, 22, \ldots$. This observation enables us to derive upper bounds on the number of vertices attainable for a given diameter k. However, unlike in previously studied cases, these optimal patterns do not always extend to valid digraphs, since the corresponding tiles fail to tessellate the plane periodically. This phenomenon highlights a fundamental obstruction linking combinatorial optimality with geometric feasibility.

To overcome this difficulty, we adopt a lattice-based approach inspired by earlier work on plane tessellations [5, 4]. By identifying suitable translation vectors that generate periodic tilings, we derive systems of linear equations whose solutions yield admissible pairs of generators a and b. This approach allows us to construct infinite families of three-quarters digraphs and to distinguish between the cases of even and odd diameter. In particular, explicit formulas for the parameters are obtained by solving these systems over the integers, thereby providing constructive solutions to the degree/diameter problem within this framework.

The study of three-quarters digraphs thus combines algebraic, geometric, and combinatorial techniques. It contributes to the broader understanding of how modifications to path structures affect metric properties in Cayley digraphs and reveals new connections among planar tessellations, lattice theory, and extremal graph constructions. Moreover, these results complement previous work on circulant and related digraphs (see, for instance, Wong and Coppersmith [6],Morillo, Fiol, and Fàbrega [5] and Esqué, Aguiló and Fiol [2]), and may have further implications for the design of efficient interconnection networks and routing schemes.

The paper is organized as follows. In Section 2, we formally define three-quarters digraphs and describe their planar representation. We then analyze their distance structure and derive bounds on their diameter and order. Finally, we present constructive methods and computational results that illustrate the theoretical findings.

2. Three-quarters digraphs

A digraph is defined as an ordered pair $G = (V, A)$, where V is a non-empty set of objects called vertices, and A is a set of ordered pairs of vertices called arcs (or directed edges). A directed path in a digraph is a sequence of vertices where every adjacent pair is connected by a directed edge pointing from the first vertex to the second.

The *circulant digraph* $CD(N, a, b)$ is the Cayley digraph on $\mathbb{Z}_N$ with generating set a, b. Then, its N vertices are identified with the integers modulo N, and each vertex i is connected to the vertices $i + a \pmod N$ and $i + b \pmod N$ for the integers (called *steps)* a and b, with $1 \le a, b \le N - 1$. See Figure 1 for the circulant digraph $CD(10, 1, 4)$.

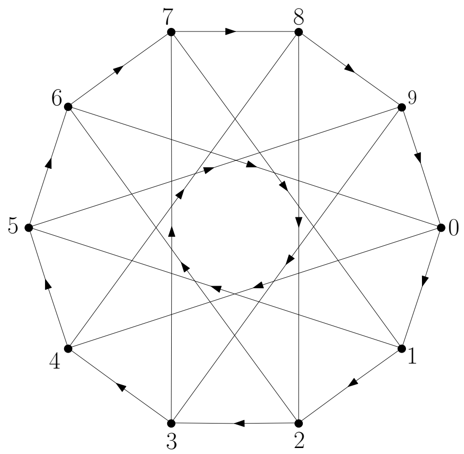


Fig. 1. The circulant digraph $CD(10, 1, 4) \cong TQ(10, 1, 4)$

Some simple properties of these digraphs are the following:

P1. The digraphs $CD(N, a, b)$ are regular of degree 2.

P2. A necessary and sufficient condition for $CD(N, a, b)$ to be strongly connected is $\gcd(N, a, b) = 1$.

P3. The digraphs $CD(N, a, b)$ are vertex-symmetric. Thus, the study of its diameter can be done from any arbitrary vertex, say, 0.

Instead of considering only directed paths (with steps a and b), as usual, in this paper, we allow all paths between two vertices that consist of steps (a, b), $(a, -b)$, or $(-a, b)$.

Thus, intuitively, the 'distance' between two vertices i, j is defined as the minimum number of these steps in such a path from i to j, and the diameter is defined consistently. More rigorously, for $x, y \in \mathbb{Z}_N$, we define $d(x, y) := d_{TQ}(x, y)$ as the minimum integer $\ell \geq 0$ such that $y - x$ can be written modulo N as

$$y - x \equiv ma + nb, \qquad y - x \equiv -ma + nb, \qquad \text{or} \qquad y - x \equiv ma - nb,$$

for some integers $m, n \geq 0$ with $m + n = \ell$. Equivalently, $d(x, y)$ is the minimum length of a word from x to y using one of the three admissible directions determined by the pairs $(+a, +b)$, $(-a, +b)$, and $(+a, -b)$. Notice that this distance is translation invariant:

$$d(x, y) = d(0, y - x).$$

However, it is not necessarily symmetric, since the admissible directions do not include the pair $(-a, -b)$. Therefore, d should be understood as a sector-constrained distance, rather than as a metric in the usual undirected sense. The diameter of $TQ(N, a, b)$ is then defined by

$$k := k(TQ(N, a, b)) = \max_{x,y \in \mathbb{Z}_N} d(x, y) = \max_{z \in \mathbb{Z}_N} d(0, z).$$

These digraphs can be represented as congruent tiles that tessellate the plane periodically. More precisely, if each vertex is represented by a numbered modulo N unit square, the vertices reached at distances $0, 1, 2, \ldots$ from any given vertex can be arranged in a planar pattern, as shown in Figure 2 starting from vertex 0. From now on, we call these digraphs *three-quarters digraphs* (because only three of the four directions in the plane from a vertex 0 are admissible). Then, Figure 3 shows the distances from 0 to 4 of the different vertices and, in Figure 4, there is the plane tessellation representing the three-quarter digraph $TQ(27, 2, 7)$.

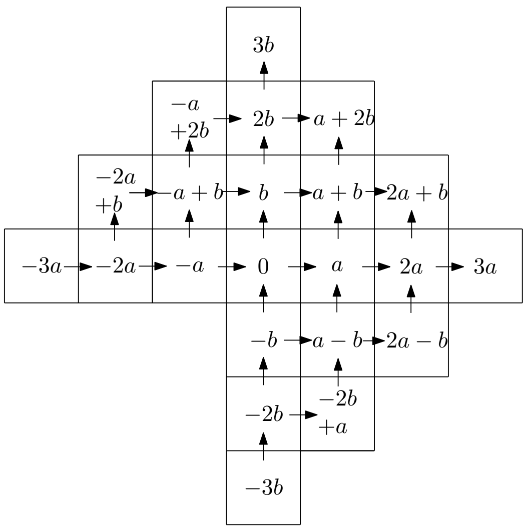


Fig. 2. Adjacencies and planar pattern of the vertices in the digraph $CD(N, a, b)$ (with vertices at distance at most three from vertex 0)

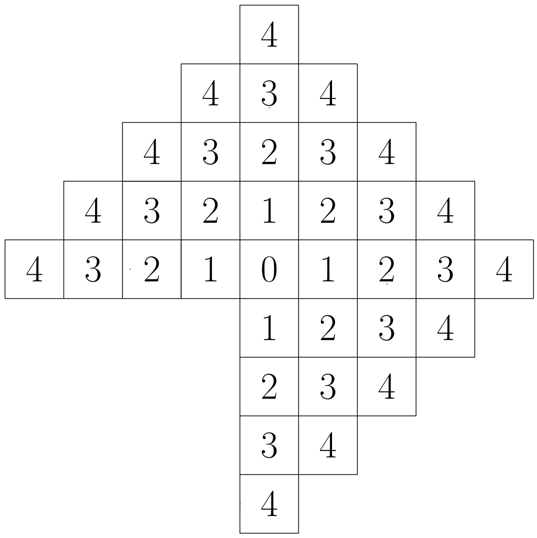


Fig. 3. The distances from 0 to different vertices

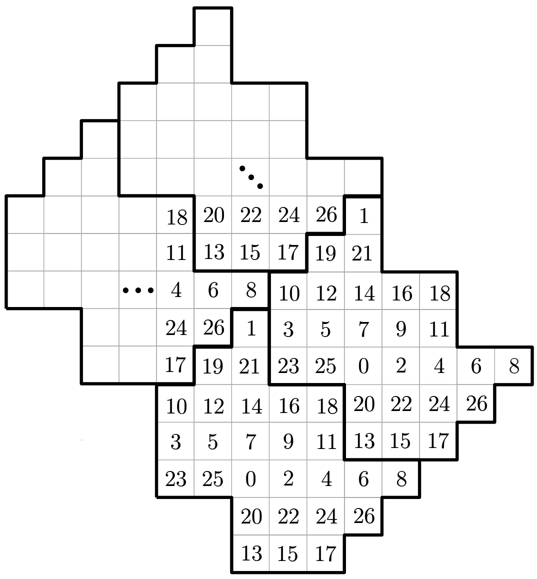


Fig. 4. The plane tessellation representing the three-quarters digraph $TQ(27, 2, 7)$

Since there are $3\ell + 1$ vertices at distance $\ell(> 0)$ from vertex 0, given a diameter k, the number of vertices reached by exactly i steps is $1, 4, 7, 10, \dots, 3k + 1$ (see Figure 3). Then, the maximum number of vertices of a three-quarters digraph with diameter k (ball size) would be

$$N(k) = \sum_{r=0}^{k}(1 + 3r) = \frac{1}{2}\left(3k^2 + 5k + 2\right), \tag{1}$$

if all the numbers $ma + nb$, $-ma + nb$, and $ma - nb$ (with $m, n \geq 0$ and $m + n \leq k$) were distinct modulo $N(k)$. However, such a maximum cannot be attained for $k > 1$. The reason is that, for $k > 1$, the optimal tiles do not tessellate the plane.

The method used in [2, 5, 7] to obtain circulant digraphs with the maximum number of vertices consists of the following two steps:

(1) Using the planar pattern, find the 'optimal tiles' (that is, containing the maximum number of vertices for a given diameter k), and check that they periodically tessellate the plane.

(2) From the two basic translation vectors of the periodic tiling (generating the lattice of the positions of the 0 vertices), solve a linear system of equations to find the steps a and b satisfying P2.

In other words, the lattice construction can be described as follows. In the planar representation, moving one unit in the horizontal direction corresponds to adding a, while moving one unit in the vertical direction corresponds to adding b. Hence, a translation vector (u, v) in the plane corresponds to the residue

$$ua + vb \pmod{N}.$$

If two copies of the tile differ by a lattice translation, then they must represent the same vertex labels modulo N. Therefore, for each translation vector (u, v) of the lattice one must have

$$ua + vb \equiv 0 \pmod{N}.$$

If the lattice is generated by two independent translation vectors

$$(u_1, v_1), \qquad (u_2, v_2),$$

then the generators a, b must satisfy

$$u_1a + v_1b \equiv 0 \pmod{N}, \qquad u_2a + v_2b \equiv 0 \pmod{N}.$$

Equivalently, there exist integers α, β such that

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = N \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The parameters α and β record how many full periods modulo N are produced by the two fundamental lattice translations. Once the tile is fixed, the determinant

$$N = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

which is the area of the fundamental parallelogram spanned by the two lattice vectors, gives the number of distinct residue classes (labels of vertices) in one fundamental tile, see Esqué, Aguiló, and Fiol [2]. This is why the order N of the digraph is directly determined by the lattice. Solving the above linear system then yields admissible values of steps a and b provided that α and β are chosen in such a way that $\gcd(N, a, b) = 1$. This last condition **P2** assures that all the residues modulo N are obtained. Moreover, the value of the diameter k is imposed by the form of the tile (for instance, $k = 4$ if and only if the tile is a sub-tile of the one in Figure 3 containing some squares labeled with 4). Given a diameter $k > 1$, a good choice for the tiles that periodically tessellate the plane is the W-shaped tiles shaded in Figure 5 for $k = 4$ and $k = 5$. (The case $k = 1$ is trivial, giving the cross with arms of one unit). In such a figure, the vertices at the farthest distance from 0 are indicated by black dots.

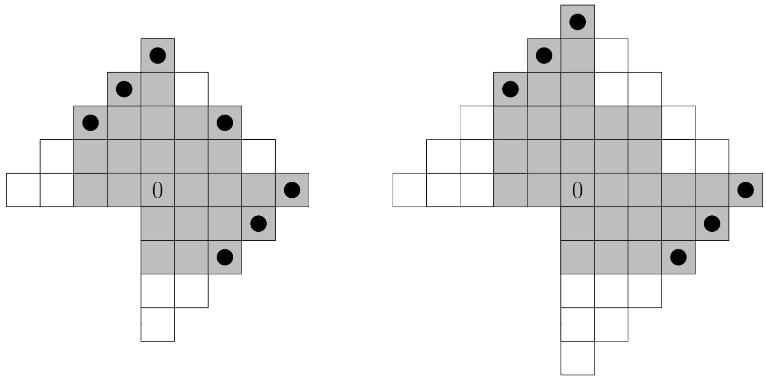


Fig. 5. In gray, there are the optimal W-tiles with even ($k = 4$) and odd diameter ($k = 5$). The farthest vertices from 0 are indicated by black dots

In our study of such tiles, we have to distinguish between even and odd diameters k.

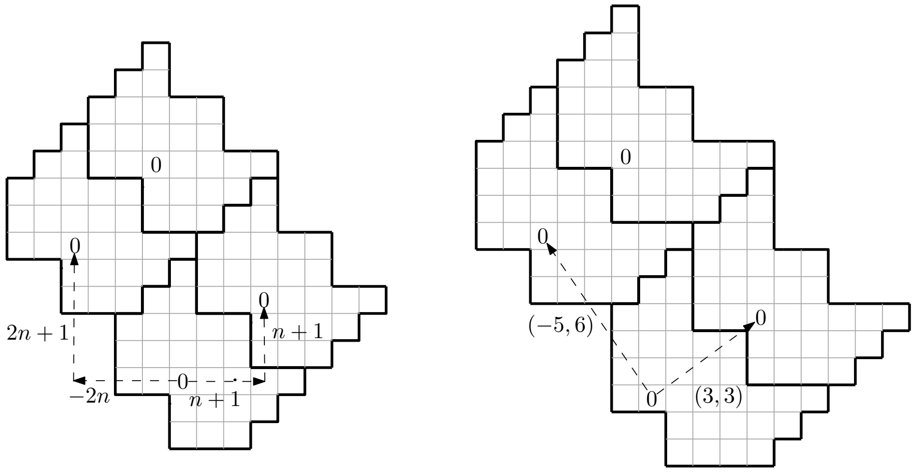


Fig. 6. Plane tessellation with W-tiles, and vectors generating the lattices. Left: $k = 2n = 4$; Right: $k = 2n + 1 = 5$

Proposition 2.1. *There exists a three-quarter digraph $TQ(N, a, b)$ for any even diameter $k = 2n$, $N = k^2 + \frac{5}{2}k + 1$ vertices, and steps $a = n$ and $b = 3n + 1$.*

Proof. For even k, that is, $k = 2n$, the two vectors generating the lattice are $(n+1, n+1)$ and $(-2n, 2n + 1)$. See Figure 6 (left). Then, the equations for the distribution of the zeros (of the lattice) are

$$(n + 1)a + (n + 1)b \equiv 0 \pmod{N}, \quad \text{and} \quad -2na + (2n + 1)b \equiv 0 \pmod{N}.$$

Then, the pair of steps a and b satisfies the system

$$\begin{pmatrix} n+1 & n+1 \\ -2n & 2n+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = N \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

for some integers α, β and $N = \det \begin{pmatrix} n+1 & n+1 \\ -2n & 2n+1 \end{pmatrix} = 4n^2 + 5n + 1 = k^2 + \frac{5}{2}k + 1.$

Solving the system, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2n+1 & -(n+1) \\ 2n & n+1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Taking, for instance, $\alpha = \beta = 1$, we get the steps $a = n$ and $b = 3n+1$ satisfying Property **P2**. In the case of diameter $k = 2$, or $n = 1$, Figure 1 shows the three-quarters digraph $TQ(10, 1, 4)$. □

Proposition 2.2. *There exists a three-quarter digraph $TQ(N, a, b)$ for any odd diameter $k = 2n+1 > 1$, $N = k^2 + \frac{3}{2}k + \frac{1}{2}$ vertices, and steps $a = n+1$ and $b = 3n+2$.*

Proof. For odd k, that is, $k = 2n+1$, the two vectors generating the lattice are $(n+1, n+1)$ and $(-(2n+1), 2n+2)$. See Figure 6 (right). Then, the pair of steps satisfies the system

$$\begin{pmatrix} n+1 & n+1 \\ -(2n+1) & 2n+2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = N \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where $N = \det \begin{pmatrix} n+1 & n+1 \\ -(2n+1) & 2n+2 \end{pmatrix} = 4n^2 + 7n + 3 = k^2 + \frac{3}{2}k + \frac{1}{2}$. Solving the system, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2n+2 & -(n+1) \\ 2n+1 & n+1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Taking, for instance, $\alpha = \beta = 1$, we get the steps $a = n+1$ and $b = 3n+2$. □

For the particular case $k = 1$, we have the three-quarter digraph $TQ(5, 1, 2)$.

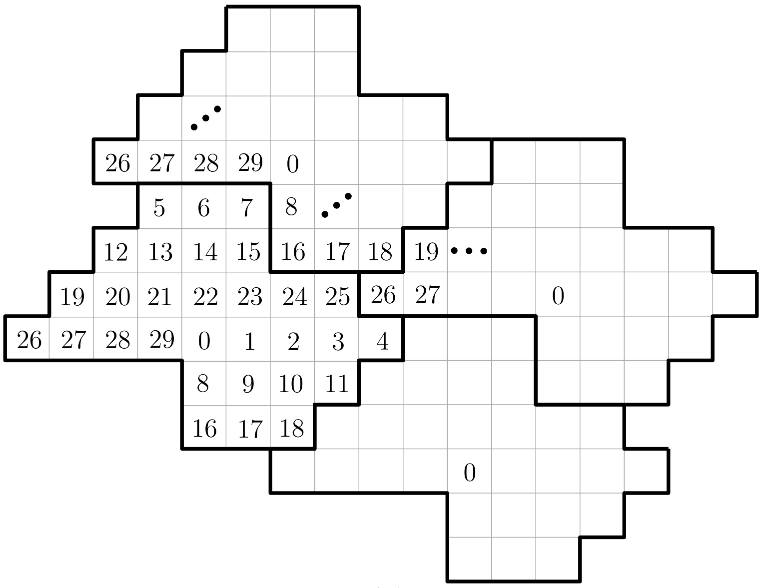
In Table 1, we show the minimum diameter k and steps a, b for each number of vertices $N \leq 208$ of three-quarters digraphs. The cases in which we get the maximum number of vertices for a given diameter are in boldface. Notice that such a maximum is greater than the values obtained with the W-shape tiles. For instance, for diameter $k = 4$, Proposition 2.1 gives $N = 27$ (see Figure 5), whereas, according to Table 1, we have a maximum of $N = 30$. The tiles that achieve the maximum values are more involved, as shown in Figure 7 for $N = 30$ and steps $a = 1$ and $b = 22 = -8$.

As it was given in (1), for diameter k, the number of vertices contained in the ideal ball of radius k is $\frac{3}{2}k^2 + \frac{5}{2}k + 1$. This gives the natural Moore-type upper bound for the three-quarter distance model, provided that all represented residues are distinct modulo N. Nevertheless, for $k > 1$, this ideal ball does not tessellate the plane. Hence, the upper bound is not achievable with a three-quarters circulant digraph. The constructions given in Propositions 2.1 and 2.2 should therefore be interpreted as explicit infinite families of large order, rather than as optimal constructions in general. More precisely, they give

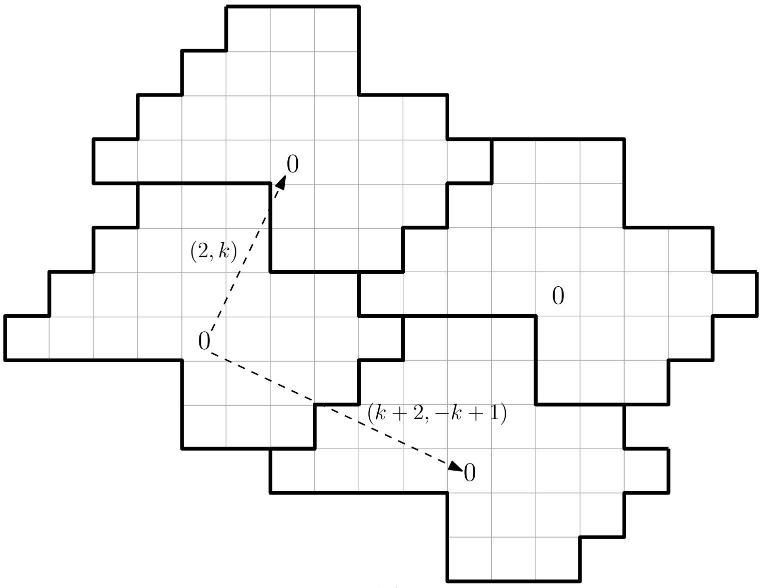
$$N(k) = k^2 + \frac{5}{2}k + 1 \quad \text{for even } k,$$

and

$$N(k) = k^2 + \frac{3}{2}k + \frac{1}{2} \quad \text{for odd } k.$$



(a)



(b)

Fig. 7. (a) Plane tessellation for the three-quarters digraph with diameter $k=4$, order $N=30$, and steps $a=1$ and $b=22$. (b) Conjectured vectors generating the lattices for a general tile

In contrast, according to Table 3, the maximum order for even diameter $k \le 30$ is $N(k)=\frac{9}{8}k^2+\frac{22}{8}k+1$, achieved with steps $a=1$ and $b(k)=7+23\left(\frac{k-2}{4}\right)+18\left(\frac{k-2}{4}\right)^2$ for $k=2r$ with odd r, and $b(k)=22+41\left(\frac{k-4}{4}\right)+18\left(\frac{k-4}{4}\right)^2$ for $k=2r$, with even r. Experimental results suggest that these values are valid for any even diameter. The difficulty is that the associated tiles exhibit similar but apparently non-generalizable shapes. Examples include the cases $k=6,12,18$ in Figure 8. Thus, one may add the following conjecture.

Conjecture 2.3. *For every even diameter k, there exists a three-quarter circulant digraph*

$TQ(N,a,b)$ with

$$N(k)=\frac{9}{8}k^2+\frac{22}{8}k+1,$$

and steps indicated as above. More precisely: if $k=4s$ we have $N(k)=18s^2+11s+1$, $a=1$, and $b(k)=18s^2+5s-1$; and if $k=4s+2$ we have $N(k)=18s^2+29s+11$, $a=1$, and $b(k)=18s^2+23s+7$. Moreover, such numbers of vertices $N(k)$ are optimal among all three-quarter circulant digraphs of such diameters k.

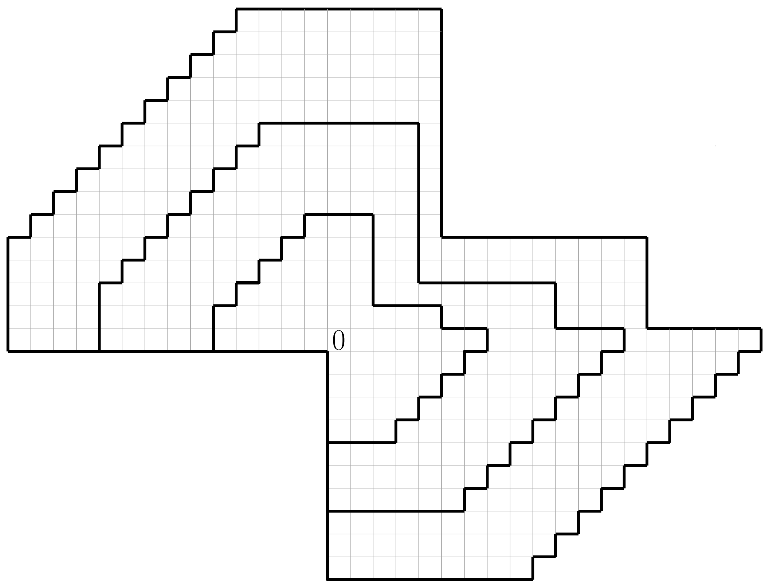


Fig. 8. Similar plane tiles for the three-quarters digraphs with diameters $k=6$, 12, and 18, and maximum number of vertices

From the case of $k=4$, we also have the following conjecture for general diameter, slightly improving that of Propositions 2.1 and 2.2.

Conjecture 2.4. *Given an integer $k\geq 1$, the lattice generated by the vectors $(u_1,v_1)=(k+2,-k+1)$ and $(u_2,v_2)=(2,k)$ corresponds to a tile of a three-quarters digraph with diameter k. Then, there exists a three-quarter digraph $TQ(N,a,b)$ with diameter k, $N=k^2+4k-2$ vertices, and steps $a=1$ and $b=-k-4$.*

Looking at the case for $k=4$ in the tables and also Figures 7 (a) and (b), the equations for the distribution of the zeros could be

$$(k+2)a+(-k+1)b\equiv 0 \pmod N, \quad \text{and} \quad 2a+kb\equiv 0 \pmod N.$$

Then, the pair of steps a and b satisfies the system

$$\begin{pmatrix} k+2 & -k+1 \\ 2 & k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=N\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

for some integers α,β and $N=\det\begin{pmatrix} k+2 & -k+1 \\ 2 & k \end{pmatrix}=k^2+4k-2$. Solving the system, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} k & k-1 \\ -2 & k+2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Taking, for instance, $\alpha = 1$ and $\beta = -1$, we get the steps $a = 1$ and $b = -k - 4$ satisfying property **P2**.

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Statements and Declarations

The authors have no competing interests. All the data generated in this paper is included here.

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Appendix

Table 1. The minimum diameter k for each number of vertices $N \leq 208$ with steps a and b in the three-quarters digraphs $TQ(N, a, b)$.

N	a	b	k	N	a	b	k	N	a	b	k	N	a	b	k	N	a	b	k
4	1	2	1	45	1	20	6	86	1	20	8	127	1	24	10	168	1	15	12
5	**1**	**2**	**1**	46	1	14	6	87	1	12	8	128	1	30	10	169	1	27	12
6	1	2	2	47	1	9	6	88	1	14	8	129	1	14	10	170	1	14	12
7	1	2	2	48	1	9	6	89	1	41	8	130	1	60	10	171	1	54	12
8	1	3	2	49	1	15	6	90	1	66	8	131	1	18	10	172	1	67	12
9	1	4	2	50	1	8	6	91	1	28	8	132	1	16	10	173	1	17	12
10	1	4	2	51	1	30	6	92	1	76	8	133	1	62	10	174	1	17	12
11	**1**	**7**	**2**	52	1	12	6	93	1	13	9	134	1	12	11	175	1	19	12
12	1	4	3	53	1	10	6	94	1	82	8	135	1	115	10	176	1	19	12
13	1	3	3	54	1	24	6	**95**	**1**	**61**	**8**	136	1	32	10	177	1	69	12
14	1	4	3	55	1	35	6	96	1	18	9	137	1	15	11	178	1	42	12
15	1	6	3	56	1	9	7	97	1	30	9	138	1	124	10	179	1	16	12
16	1	10	3	57	1	45	6	98	1	12	9	139	1	15	11	180	1	84	12
17	1	5	3	**58**	**1**	**48**	**6**	99	1	12	9	140	1	122	10	181	1	82	12
18	2	15	3	59	1	11	7	100	1	16	9	**141**	**1**	**44**	**10**	182	1	22	12
19	**1**	**8**	**3**	60	1	14	7	101	1	11	9	142	1	14	11	183	1	13	13
20	1	6	4	61	1	14	7	102	1	14	9	143	1	14	11	184	1	18	12
21	1	4	4	62	1	10	7	103	1	24	9	144	1	34	11	185	1	20	12
22	1	5	4	63	1	10	7	104	1	32	9	145	1	13	11	186	1	137	12
23	1	7	4	64	1	12	7	105	1	77	9	146	1	20	11	187	1	59	12
24	1	7	4	65	1	9	7	106	1	20	9	147	1	16	11	188	1	30	13
25	1	11	4	66	1	20	7	107	1	13	9	148	1	16	11	189	1	36	12
26	1	6	4	67	1	26	7	108	1	42	9	149	1	18	11	190	1	140	12
27	1	8	4	68	1	40	7	109	1	15	9	150	1	42	11	191	1	17	13
28	1	12	4	69	1	11	7	110	1	96	9	151	1	24	11	192	1	42	13
29	1	23	4	70	1	58	7	111	1	10	10	152	1	84	11	193	1	114	12
30	**1**	**22**	**4**	71	1	52	7	112	1	72	9	153	1	15	11	194	1	16	13
31	1	5	5	72	1	10	8	113	1	18	10	154	1	48	11	195	1	177	12
32	1	6	5	73	1	8	8	114	1	14	10	155	1	128	11	**196**	**1**	**176**	**12**
33	1	10	5	74	1	9	8	115	1	27	9	156	1	136	11	197	1	15	13
34	1	8	5	**75**	**1**	**23**	**7**	116	1	14	10	157	1	17	11	198	1	24	13
35	1	8	5	76	1	12	8	**117**	**1**	**102**	**9**	158	1	140	11	199	1	24	13
36	1	14	5	77	1	18	8	118	1	33	10	159	1	141	11	200	1	32	13
37	1	7	5	78	1	18	8	119	1	13	10	160	1	50	11	201	1	18	13
38	1	24	5	79	1	11	8	120	1	13	10	161	1	22	12	202	1	18	13
39	1	9	5	80	1	11	8	121	1	23	10	162	2	117	11	203	1	22	13
40	1	12	5	81	1	13	8	122	1	12	10	163	1	31	11	204	1	20	13
41	1	26	5	82	1	10	8	123	1	15	10	164	1	16	12	205	1	20	13
42	1	8	6	83	1	47	8	124	1	15	10	165	1	18	12	206	1	49	13
43	**1**	**19**	**5**	84	1	38	8	125	1	35	10	166	1	18	12	207	1	17	13
44	1	7	6	85	1	16	8	126	1	20	10	**167**	**1**	**108**	**11**	208	1	88	13

Table 2. Values of N, a, b, and k with different steps a and b for $N = \frac{9k^2}{8} + \frac{22k}{8} + 1$ and even k in the three-quarters digraphs $TQ(N, a, b)$.

N	a	b	k	N	a	b	k	N	a	b	k	N	a	b	k	N	a	b	k
11	1	7	2	58	35	56	6	95	27	32	8	141	4	77	10	141	41	49	10
11	1	8	2	58	36	37	6	95	28	83	8	141	5	61	10	141	41	112	10
11	2	3	2	58	41	54	6	95	28	93	8	141	5	79	10	141	43	59	10
11	2	5	2	58	47	52	6	95	29	59	8	141	7	26	10	141	44	103	10
11	3	10	2	58	50	53	6	95	29	69	8	141	7	29	10	141	46	50	10
11	4	6	2	95	1	61	8	95	31	41	8	141	8	13	10	141	46	110	10
11	4	10	2	95	1	81	8	95	31	86	8	141	8	70	10	141	49	62	10
11	5	7	2	95	2	27	8	95	32	52	8	141	10	17	10	141	50	85	10
11	6	9	2	95	2	67	8	95	34	79	8	141	10	122	10	141	53	76	10
11	8	9	2	95	3	53	8	95	34	94	8	141	11	61	10	141	53	139	10
30	1	22	4	95	3	88	8	95	36	66	8	141	11	106	10	141	55	107	10
30	2	11	4	95	4	39	8	95	37	52	8	141	13	74	10	141	56	67	10
30	4	7	4	95	4	54	8	95	37	72	8	141	14	52	10	141	56	91	10
30	8	29	4	95	6	11	8	95	41	91	8	141	14	58	10	141	58	59	10
30	13	16	4	95	6	81	8	95	42	77	8	141	16	26	10	141	62	136	10
30	14	17	4	95	7	47	8	95	42	92	8	141	16	140	10	141	64	104	10
30	19	28	4	95	7	92	8	95	43	58	8	141	17	43	10	141	64	137	10
30	23	26	4	95	8	13	8	95	43	63	8	141	19	119	10	141	65	88	10
58	1	48	6	95	8	78	8	95	44	49	8	141	19	131	10	141	67	128	10
58	2	23	6	95	9	64	8	95	46	51	8	141	20	34	10	141	70	119	10
58	3	28	6	95	9	74	8	95	48	78	8	141	20	103	10	141	71	133	10
58	4	17	6	95	11	36	8	95	48	88	8	141	22	71	10	141	73	101	10
58	5	8	6	95	12	22	8	95	49	74	8	141	22	122	10	141	73	110	10
58	6	11	6	95	12	67	8	95	51	71	8	141	23	25	10	141	74	85	10
58	7	46	6	95	13	33	8	95	54	64	8	141	23	55	10	141	76	101	10
58	9	26	6	95	14	89	8	95	56	71	8	141	25	113	10	141	79	92	10
58	10	57	6	95	14	94	8	95	56	91	8	141	28	104	10	141	80	130	10
58	12	51	6	95	16	26	8	95	59	84	8	141	28	116	10	141	80	136	10
58	13	44	6	95	16	61	8	95	62	77	8	141	29	100	10	141	82	83	10
58	14	45	6	95	17	47	8	95	62	82	8	141	31	68	10	141	82	98	10
58	15	24	6	95	17	87	8	95	63	68	8	141	31	95	10	141	83	127	10
58	16	39	6	95	18	33	8	95	68	93	8	141	32	52	10	141	86	118	10
58	18	33	6	95	18	53	8	95	69	79	8	141	32	139	10	141	89	109	10
58	19	42	6	95	21	46	8	95	73	83	8	141	34	86	10	141	89	127	10
58	20	27	6	95	21	86	8	95	82	87	8	141	35	130	10	141	91	95	10
58	21	22	6	95	22	72	8	95	84	89	8	141	37	77	10	141	92	100	10
58	25	40	6	95	23	58	8	141	1	44	10	141	37	113	10	141	97	140	10
58	30	55	6	95	23	73	8	141	1	125	10	141	38	97	10	141	98	124	10
58	31	38	6	95	24	39	8	141	2	88	10	141	38	121	10	141	106	137	10
58	32	49	6	95	24	44	8	141	2	109	10	141	40	65	10	141	107	121	10
58	34	43	6	95	26	66	8	141	4	35	10	141	40	68	10	141	112	134	10

Table 3. Values of N, a, b, and even k with $N(k) = \frac{9k^2}{8} + \frac{22k}{8} + 1$, $a = 1$, and $b(k) = 7 + 23\left(\frac{k-2}{4}\right) + 18\left(\frac{k-2}{4}\right)^2$ for $k = 2r$ with odd r, and $b(k) = 22 + 41\left(\frac{k-4}{4}\right) + 18\left(\frac{k-4}{4}\right)^2$ for $k = 2r$ with even r, in the three-quarters digraphs $TQ(N, a, b)$.

N	a	b	k
11	1	7	2
30	1	22	4
58	1	48	6
95	1	81	8
141	1	125	10
196	1	176	12
260	1	238	14
333	1	307	16
415	1	387	18
506	1	474	20
606	1	572	22
715	1	677	24
833	1	793	26
960	1	916	28
1096	1	1050	30

Computational procedure

The values in Tables 1-3 were obtained by exhaustive computation over all admissible triples (N, a, b) satisfying

$$\gcd(N, a, b) = 1.$$

For each triple, the diameter $k(TQ(N, a, b))$ was computed iteratively from the vertex 0, considering only the admissible directions

$$(+a, +b), \qquad (-a, +b), \qquad (+a, -b).$$

Since the direction $(-a, -b)$ is not allowed, representations containing simultaneous negative uses of both steps were excluded.

The computational procedure is summarized below.

Algorithm 1 Computation of $k(TQ(N,a,b))$

Input: Integers N, a, b
Output: The diameter $k(TQ(N,a,b))$
L_ℓ: vertices reached for the first time at level ℓ
S: set of all reached vertices
Initialize

$$L_0 = \{0\}, \qquad S = \{0\}, \qquad \ell = 0$$

repeat
  Set $L_{\ell+1} = \emptyset$
  for each $x \in L_\ell$ **do**
    Generate

$$x + a, \quad x - a, \quad x + b, \quad x - b \pmod{N}$$

    Discard paths using simultaneously $-a$ and $-b$
    Add to $L_{\ell+1}$ all vertices not contained in S
  end for
  Update

$$S \leftarrow S \cup L_{\ell+1}$$

  Increase

$$\ell \leftarrow \ell + 1$$

until $L_\ell = \emptyset$
return

$$k(TQ(N,a,b)) = \ell - 1$$

Algorithm 2 Search for minimum diameter

Input: Integer N
Output: Triples (N,a,b) attaining minimum diameter
Initialize

$$k_{\min} = \infty$$

for all pairs (a,b) satisfying $1 \le a < b \le N-1$ and $\gcd(N,a,b) = 1$ **do**
  Compute $k(TQ(N,a,b))$ using Algorithm A1
  if $k(TQ(N,a,b)) < k_{\min}$ **then**
    Update $k_{\min}$
    Record (N,a,b)
  end if
end for

Some comments:

- Table 1 lists all triples (N, a, b) with minimum diameter for $N \leq 208$.
- Table 2 contains the values corresponding to the family

$$N = \frac{9k^2}{8} + \frac{22k}{8} + 1,$$

  for even values of k.
- Table 3 contains the computations obtained from the explicit formulas for $b(k)$.

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