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Combinatorial Interpretations of q-Identities containing negative exponent of q

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ABSTRACT

We give combinatorial interpretations of some Rogers−Ramanujan type identities, also known as sum-product identities in terms of $(n+t)-$color partitions and split $(n+t)-$color partitions. The identities discussed in this study contains negative exponent of q. These interesting results reveal rich structure and great potential for further research because they reveal intricate mathematical structures, and link various other fields.

Keywords: q-identities; combinatorial interpretations, $n-$color partitions

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1. Introduction

The Rogers−Ramanujan identities and Ramanujan's mock theta functions of various orders are well established in the literature [4, 18]. These works has motivated researchers over several decades to discover additional q-identities and new classes of mock theta functions. In parallel to this study, combinatorial interpretations of these identities have been extensively explored in terms of ordinary partitions [7], overpartitions [12], $n-$color partitions [14] and plane partitions [10].

Rogers−Ramanujan type identities have gained significant attention analytically and combinatorially due to their wide ranging applications in various fields such as representation theory [16], conformal field theory [6], statistical mechanics [5], vertex operator algebra [15], modular forms [17], computer algebra [20]. Several authors Connor [9], Subbarao [22], Agarwal and Andrews [1], Goyal and Agarwal [11], Sood and Agarwal [21], Gupta et al. [13], Alanazi et al. [3], and Gu and Wang [12] have provided a broad spectrum of partition-theoretic interpretations for these identities.

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In this paper, we present the partition-theoretic interpretations of certain Rogers–Ramanujan type identities, also known as sum-product identities. Before discussing these identities, we recall the standard definition of a $q-$series. A $q-$series, also known as basic series, is a series where the summands contain the $q-$Pochhammer symbol defined as:

$$(t;q)_0 = 1, \quad (t;q)_n = \prod_{k=0}^{n-1}(1-tq^k), \quad (t;q)_\infty = \prod_{k=0}^{\infty}(1-tq^k), \tag{1}$$

where t and q are complex numbers with $|q|< 1$.

While most identities studied in the literature involve only positive powers of q, this study focuses on identities involving q-series with negative exponents. The combinatorial aspect of such series remain unexplored in the existing literature. In the present paper, we focus on the following identities derived by Chu and Zhang [8] [Identities 19, 30, 62, 66]:

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n(-q^{-2};q^4)_n q^n}{(q;q)_{2n}(-q^{-2};q^2)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^8,q^2,q^6;q^8]_\infty, \tag{2}$$

$$\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q^{-1};q^2)_n q^{\frac{n(n+1)}{2}}}{(q;q)_n(-q^{-1};q)_n(q;q^2)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^8,q^3,q^5;q^8]_\infty, \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+2)}}{(-q^{-1};q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,q^2,q^3;q^5]_\infty, \tag{4}$$

$$\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n q^{n(n-2)}}{(q^4;q^4)_n(-q^{-1};q^2)_n} = \frac{2(q^2;q^4)_\infty}{(q^4;q^4)_\infty}[q^4,-q^3,-q^5;q^4]_\infty. \tag{5}$$

We provide their combinatorial interpretations in Section 2 and some preliminary definitions.

2. Combinatorial interpretations

We start this section by recalling the definition of $n-$color partitions [1], an $n-$color partition (also called a partition with 'n copies of n'), is a partition in which a part of size n, $n \geq 0$, can come in n different colors denoted by $n_1, n_2, \ldots, n_n$.

Example 2.1. The number of $n-$color partitions of 3 are 6 and the relevant partitions are

$$3_3, 3_2, 3_1, 2_2+1_1, 2_1+1_1, 1_1+1_1+1_1$$

Further, the study of $n-$color partitions has been extended to $(n+t)-$color partitions. An $(n+t)$–color partition (also called a partition with '$(n+t)$ copies of n'), is a partition of a positive integer, in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by $n_1, n_2, \ldots, n_{n+t}$. Note that, zeros are permissable if and only if $t > 0$, but only one copy of 0, that is, 0_t is allowed.

Example 2.2. The number of $(n+1)-$color partitions of 3 are 28 and the relevant partitions are

$3_4, 3_3, 3_2, 3_1, 3_4+0_1, 3_3+0_1, 3_2+0_1, 3_1+0_1, 2_3+1_2, 2_2+1_2, 2_1+1_2, 2_3+1_1, 2_2+1_1, 2_1+1_1,$

$2_3+1_2+0_1, 2_2+1_2+0_1, 2_1+1_2+0_1, 2_3+1_1+0_1, 2_2+1_1+0_1, 2_1+1_1+0_1, 1_2+1_2+1_2,$

$1_2+1_2+1_1, 1_2+1_1+1_1, 1_1+1_1+1_1, 1_2+1_2+1_2+0_1, 1_2+1_2+1_1+0_1, 1_2+1_1+1_1+0_1,$

$1_1+1_1+1_1+0_1.$

A split $(n+t)-$color partition [2] (where $t \geq 0$) is a partition where the color b of a part a_b is split into two components: a 'green part' g and 'red part' r such that $1 \leq g \leq b$, $0 \leq r \leq b-1$, and $g+r=b$. We represent a generalized $n-$color partition as: $(a_1)_{b_1}+(a_2)_{b_2}+\cdots+(a_s)_{b_s}$, where $(a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_s)_{b_s}$.

For t, q to be complex numbers with $|q|< 1$, we have

$$(t;q)_{-n} = \frac{(\frac{-q}{t})^n q^{\frac{n(n-1)}{2}}}{(\frac{q}{t};q)_n}. \tag{6}$$

In order to establish the interpretations of (2)-(5), by using constructive method, we introduce the following notations.

Remark 2.3. (i) In the coming text, wherever necessary we shall be denoting the weighted difference $((a_{i-1})_{b_{i-1}}-(a_i)_{b_i})$ by $(w.d.)_i$, where

$$(w.d.)_i = a_{i-1} - b_{i-1} - a_i - b_i.$$

(ii) The notation for the largest part and the smallest part of the partition is $(a_1)_{b_1}$ and $(a_s)_{b_s}$, respectively.

To represent these partitions systematically, we use a two-line array:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{s-1} & a_s, \\ b_1 & b_2 & \dots & b_{s-1} & b_s \end{pmatrix},$$

where a_i represents the part and b_i represents its color, such that $a_1 \geq a_2 \geq \ldots \geq a_s$. Each column in the array represents a colored part, mapped as follows:

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \mapsto (a_i)_{b_i}$$

Consider the L.H.S. of (2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1;q^2)_n(-q^{-2};q^4)_n q^n}{(q;q)_{2n}(-q^{-2};q^2)_n} &= \sum_{n=0}^{\infty} \frac{(1+1)(-q^2;q^2)_{n-1}(1+q^{-2})(-q^2;q^4)_{n-1}q^n}{(q;q)_{2n}(1+q^{-2})(-1;q^2)_{n-1}} \\ &= \sum_{n=0}^{\infty} \frac{2(-q^2;q^2)_{n-1}(-q^2;q^4)_{n-1}q^n}{(q;q)_{2n}(1+1)(-q^2;q^2)_{n-2}}\end{aligned}$$

$$=\sum_{n=0}^{\infty}\frac{(1+q^{2n-2})(-q^2;q^4)_{n-1}q^n}{(q;q)_{2n}}$$
$$=\frac{(1+q^{-2})(-q^2;q^4)_{-1}q^0}{(q;q)_0}+\sum_{n=1}^{\infty}\frac{(1+q^{2n-2})(-q^2;q^4)_{n-1}q^n}{(q;q)_{2n}}.$$

Now using (6), we have

$$(-q^2;q^4)_{-1}=\frac{(\frac{-q^4}{-q^2})^1q^{\frac{4\cdot1(1-1)}{2}}}{(\frac{q^4}{-q^2};q^4)_1}=\frac{q^2}{(-q^2;q^4)_1}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1;q^2)_n(-q^{-2};q^4)_nq^n}{(q;q)_{2n}(-q^{-2};q^2)_n}=&\frac{(1+q^2)}{q^2}\cdot\frac{q^2}{(1+q^2)}+\sum_{n=1}^{\infty}\frac{(1+q^{2n-2})(-q^2;q^4)_{n-1}q^n}{(q;q^2)_n(q^2;q^2)_n}\\ =&1+\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^n}{(q;q^2)_n(q^2;q^2)_n}+\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^{3n-2}}{(q;q^2)_n(q^2;q^2)_n}\\ \sum_{n=0}^{\infty}\frac{(-1;q^2)_n(-q^{-2};q^4)_nq^n}{(q;q)_{2n}(-q^{-2};q^2)_n}=&1+\sum_{\nu\geq1}S_1(\nu)q^\nu+\sum_{\nu\geq1}S_2(\nu)q^\nu.\end{aligned}$$

Interpretations of $S_1(\nu)$ and $S_2(\nu)$ given below:

Theorem 2.4. *For $\nu\geq1$, let $S_1(\nu)$ denote the number of split $n-$color partitions of ν such that*

(i) $r_s=0$, $r_i=0$ *or* 2 *for* $i\neq s$,

(ii) *parts and their subscripts have same parity,*

(iii) $(w.d.)_i\geq-2$ *and is even* $\forall\ 1<i\leq s$.

Then

$$\sum_{\nu\geq1}S_1(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^n}{(q;q^2)_n(q^2;q^2)_n}.$$

Theorem 2.5. *For $\nu\geq1$, let $S_2(\nu)$ denote the number of split $n-$color partitions of ν such that*

(i) $r_s=0$ *and* $r_i=0$ *or* 2 *for* $i\neq s$,

(ii) *parts and their subscripts have same parity,*

(iii) $(w.d.)_s\geq0$ *and even,* $(w.d.)_i\geq-2$ *and even* $\forall\ 1<i<s$.

Then

$$\sum_{\nu\geq1}S_2(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^{3n-2}}{(q;q^2)_n(q^2;q^2)_n}.$$

Proof of Theorem 2.4. Consider the expression,

$$\sum_{\nu\geq1}S_1(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^n}{(q;q^2)_n(q^2;q^2)_n},\tag{7}$$

in above q^n generates n parts each equal to 1. The factor $(-q^2;q^4)_{n-1}$ generates natural numbers say, $k_1\cdot 2, k_2\cdot 6,\ldots,k_{n-1}\cdot(4n-6)$, $k_i=0$ or 1, $1\le i\le n-1$. $(q^2;q^2)_n^{-1}$ generates even natural numbers say, $m_1\cdot 2, m_2\cdot 4,\ldots,m_n\cdot(2n)$, $m_j\ge 0$, where $1\le j\le n$. And, $(q;q^2)_n^{-1}$ generates odd natural numbers say, $l_1\cdot 1, l_2\cdot 3,\ldots,l_n\cdot(2n-1)$, $l_j\ge 0$, where $1\le j\le n$.
Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix} \begin{matrix}1+2(m_n+\cdots+m_1)+2(l_n+\cdots+l_2)\\ +l_1+4(k_{n-1}+\cdots+k_2)+2k_1\end{matrix} & \ldots & \begin{matrix}1+2(m_n+m_{n-1})\\ +2l_n+l_{n-1}+2k_{n-1}\end{matrix} & \begin{matrix}1+2m_n\\ +l_n\end{matrix} \\ \\ \\ 1+l_1+2k_1 & \ldots & 1+l_{n-1}+2k_{n-1} & 1+l_n \end{pmatrix}.$$

Here, $1+l_j$ and $2k_i$ represent the green and red color respectively, where $1\le j\le n$, $1\le i\le n-1$.

From the array, $s=n\ge 1$ and it is clear that r_i can be 0 or 2, $\forall\ i\ne s$ and $b_s=1+l_n$, that contributes only green color to the part so for the smallest part or if there is only one part in the partition, r_s is always 0, which proves (i).

For $i\ne s$,
$a_i=1+2(m_n+m_{n-1}+\cdots+m_i)+2(l_n+l_{n-1}+\cdots+l_{i+1})+l_i+4(k_{n-1}+\cdots+k_{i+1})+2k_i$,
$b_i=1+l_i+2k_i$,

Also, $a_s=1+2m_n+l_n$, $b_s=1+l_n$, concludes that $a_i-b_i\equiv 0 \pmod 2$ $\forall\ 1\le i\le s$, which shows parts and their subscripts have same parity, and proves (ii). For consecutive parts, $(w.d.)_i=2m_{i-1}-2\ge -2$ and is even $\forall\ 1<i\le s$ which proves (iii). $\square$

Proof of Theorem 2.5. Consider the expression,

$$\sum_{\nu\ge 1} S_2(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q^2;q^4)_{n-1}q^{3n-2}}{(q;q^2)_n(q^2;q^2)_n}, \tag{8}$$

in above q^{3n-2} generates n parts as $\underbrace{3,3,\ldots,3}_{(n-1)\text{-times}},1$. The factor $(-q^2;q^4)_{n-1}$ generates natural numbers say, $k_1\cdot 2, k_2\cdot 6,\ldots,k_{n-1}\cdot(4n-6)$, k_i=0 or 1, $1\le i\le n-1$. $(q^2;q^2)_n^{-1}$ generates even natural numbers say, $m_1\cdot 2, m_2\cdot 4,\ldots,m_n\cdot(2n)$, $m_j\ge 0$, where $0\le j\le n$. And, $(q;q^2)_n^{-1}$ generates odd natural numbers say, $l_1\cdot 1, l_2\cdot 3,\ldots,l_n\cdot(2n-1)$, $l_j\ge 0$, where $1\le j\le n$.

Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix} \begin{matrix}3+2(m_n+\cdots+m_1)+2(l_n+\cdots+l_2)\\ +l_1+4(k_{n-1}+\cdots+k_2)+2k_1\end{matrix} & \ldots & \begin{matrix}3+2(m_n+m_{n-1})\\ +2l_n+l_{n-1}+2k_{n-1}\end{matrix} & \begin{matrix}1+2m_n\\ +l_n\end{matrix} \\ \\ \\ 1+l_1+2k_1 & \ldots & 1+l_{n-1}+2k_{n-1} & 1+l_n \end{pmatrix}.$$

Here, $1+l_j$ and $2k_i$ represent the green color and red color respectively, where $1 \le j \le n$, $1 \le i \le n-1$.

From the array, $s = n \ge 1$ and it is clear that r_i can be 0 or 2 $\forall\ i \ne s$ and $b_s = 1 + l_n$, that contributes only green color to the part so for the smallest part or if there is only one part in the partition, r_s is always 0, which proves (i).

For $i \ne s$,
$a_i = 3+2(m_n+m_{n-1}+\cdots+m_i)+2(l_n+l_{n-1}+\cdots+l_{i+1})+l_i+4(k_{n-1}+\cdots+k_{i+1})+2k_i$,
$b_i = 1 + l_i + 2k_i$,

Also, $a_s = 1+2m_n+l_n$, $b_s = 1+l_n$, concludes that $a_i - b_i \equiv 0 \pmod 2$ $\forall\ 1 \le i \le s$, which shows parts and their subscripts have same parity, which proves (ii). For consecutive parts, $(w.d.)_s = 2m_{n-1} \ge 0$ and is even. And we have $(w.d.)_i = 2m_{i-1} - 2 \ge -2$, $\forall\ 1 < i < s$, which proves (iii). $\square$

Thus, $a_1(\nu) = 1 + S_1(\nu) + S_2(\nu)$.

Example 2.6. The relevant partitions for $S_1(6) = 19$ are $6_2, 6_4, 6_6, 5_1 1_1, 5_3 1_1, 5_5 1_1, 5_{1+2} 1_1, 5_{3+2} 1_1, 4_2 2_2, 3_1 3_1, 4_2 1_1 1_1, 4_4 1_1 1_1, 4_{2+2} 1_1 1_1, 3_1 2_2 1_1, 3_1 1_1 1_1 1_1, 3_3 1_1 1_1 1_1, 3_{1+2} 1_1 1_1 1_1, 2_2 1_1 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1 1_1$.

The relevant partitions for $S_2(6) = 6$ are $6_2, 6_4, 6_6, 5_1 1_1, 5_3 1_1, 5_{1+2} 1_1$.

Interpretation of R.H.S. of (2)

$$P_1(\nu) = \sum_{k=0}^{\nu} A_1(\nu - k) B_1(k),$$

where $A_1(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 1, \pm 3,\ 4 \pmod 8$ and $B_1(\nu)$ denote the number of partitions of ν into distinct parts $\equiv 0, \pm 1, \pm 2, \pm 3,\ 4$ (mod 8). Partitions enumerated by $P_1(6)$ is presented in Table 1.

Table 1. Partitions enumerated by $P_1(6)$

k	$A_1(\nu - k)$	$B_1(k)$	$A_1(\nu - k)B_1(k)$
0	$5\ (5+1, 4+1^2, 3^2, 3+1^3, 1^6)$	1	5
1	$4\ (5, 4+1, 3+1^2, 1^5)$	1 (1)	4
2	$3\ (4, 3+1, 1^4)$	1 (2)	3
3	2 (2)	2 (2)	4
4	$1\ (1^2)$	$2\ (4, 3+1)$	2
5	1 (1)	$3\ (5, 4+1, 3+2)$	3
6	1	$4\ (6, 5+1, 4+2, 3+2+1)$	4
		Total	25

Proof of Identity (2). The combinatorial interpretation of L.H.S. of identity (2) can be easily seen from Theorems 2.4 and 2.5 while for proof of R.H.S. of (2), we use convolution and as provided in the aforementioned examples $S_1(6) + S_2(6) = 19 + 6 = 25$ while $P_1(6) = 25$. $\square$

To find the interpretation of (3), consider the L.H.S. of (3) and using (6), we get

$$\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q^{-1};q^2)_nq^{\frac{n(n+1)}{2}}}{(q;q)_n(-q^{-1};q)_n(q;q^2)_n}=\frac{(1+q^{-1})(-q;q^2)_{-1}}{(q;q)_0(q;q^2)_0}+\sum_{n=1}^{\infty}\frac{(1+q^{n-1})(-q;q^2)_{n-1}q^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q^2)_n}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q^{-1};q^2)_nq^{\frac{n(n+1)}{2}}}{(q;q)_n(-q^{-1};q)_n(q;q^2)_n}&=1+\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q^2)_n}+\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{\frac{n^2+3n-2}{2}}}{(q;q)_n(q;q^2)_n}\\&=1+\sum_{\nu\geq 1}S_3(\nu)q^\nu+\sum_{\nu\geq 1}S_4(\nu)q^\nu.\end{aligned}$$

Theorem 2.7. *For $\nu\geq 1$, let $S_3(\nu)$ denote the number of split $n-$color partitions of ν such that*

(i) $r_s=0$, *and* $r_i=0$ *or* 1 *for* $i\neq s$,

(ii) $(w.d.)_i\geq -1\ \forall\ 1<i\leq s$.

Then

$$\sum_{\nu\geq 1}S_3(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q^2)_n}.$$

Theorem 2.8. *For $\nu\geq 1$, let $S_4(\nu)$ denote the number of split $n-$color partitions of ν such that*

(i) $r_s=0$, *and* $r_i=0$ *or* $1\ \forall\ i\neq s$,

(ii) $(w.d.)_i\geq -1\ \forall\ 1<i<s$, $(w.d.)_s\geq 0$.

Then

$$\sum_{\nu\geq 1}S_4(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{\frac{n^2+3n-2}{2}}}{(q;q)_n(q;q^2)_n}.$$

Proof of Theorem 2.7. Consider the expression,

$$\sum_{\nu\geq 1}S_3(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1}q^{\frac{n(n+1)}{2}}}{(q;q)_n(q;q^2)_n},\tag{9}$$

in above $q^{\frac{n(n+1)}{2}}$ generates n parts i.e. $1,2,\dots,n$. The factor $(-q;q^2)_{n-1}$ generates natural numbers say, $k_1\cdot 1,k_2\cdot 3,\dots,k_{n-1}\cdot(2n-3)$, k_i=0 or 1, where $1\leq i\leq n-1$. $(q;q)_n^{-1}$ generates numbers say, $m_1\cdot 1,m_2\cdot 2,\dots,m_n\cdot n,m_j\geq 0$, where $1\leq j\leq n$. And, $(q;q^2)_n^{-1}$ generates natural numbers say, $l_1\cdot 1,l_2\cdot 3,\dots,l_n\cdot(2n-1)$, $l_j\geq 0$, where $1\leq j\leq n$. Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix}\begin{matrix}n+(m_n+\cdots+m_1)+2(l_n+\cdots+l_2)\\+l_1+2(k_{n-1}+k_{n-2}+\cdots+k_2)+k_1\end{matrix} & \dots & \begin{matrix}2+(m_n+m_{n-1})+\\2l_n+l_{n-1}+k_{n-1}\end{matrix} & \begin{matrix}1+m_n\\+l_n\end{matrix}\\ \\ \\ 1+l_1+k_1 & \dots & 1+l_{n-1}+k_{n-1} & 1+l_n\end{pmatrix}.$$

Here, $1+l_j$ and k_i represents the green and red color of the part respectively, where $1 \leq j \leq n$, $1 \leq i \leq n-1$.

From the array, $s=n\geq 1$ and it is clear that r_i can be 0 or 1, for $i \neq s$ and $b_s = 1+l_n$, it shows that for the smallest part or if there is only one part in the partition, r_s is always 0, which proves (i).

For $i \neq s$,

$$\begin{aligned} a_i &= (n-i+1)+(m_n+m_{n-1}+\cdots+m_i)+2(l_n+l_{n-1}+\cdots+l_{i+1})+l_i \\ &\quad +2(k_{n-1}+\cdots+k_{i+1})+k_i, \\ b_i &= 1+l_i+k_i \end{aligned}$$

and $a_s = 1+m_n+l_n,\ b_s = 1+l_n$. Therefore, $(w.d.)_i = m_{i-1}-1 \geq -1\ \forall\ 1<i\leq s$ which proves (ii). □

Proof of Theorem 2.8. Consider the expression,

$$\sum_{\nu\geq 1} S_4(\nu)q^\nu = \sum_{n=1}^{\infty} \frac{(-q;q^2)_{n-1}q^{\frac{n^2+3n-2}{2}}}{(q;q)_n(q;q^2)_n}, \tag{10}$$

in above $q^{\frac{n^2+3n-2}{2}}$ generates n parts i.e. $1,3,4,\ldots,(n+1)$. The factor $(-q;q^2)_{n-1}$ generates natural numbers say, $k_1\cdot 1, k_2\cdot 3,\ldots,k_{n-1}\cdot(2n-3)$, k_i=0 or 1, where $0\leq i\leq n-1$. $(q;q)_n^{-1}$ generates numbers say, $m_1\cdot 1, m_2\cdot 2,\ldots,m_n\cdot n, m_j\geq 0$, where $0\leq j\leq n$. And, $(q;q^2)_n^{-1}$ generates natural numbers say, $l_1\cdot 1, l_2\cdot 3,\ldots,l_n\cdot(2n-1)$, $l_j\geq 0$, where $0\leq j\leq n$.

Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix} \begin{matrix}(n+1)+(m_n+\cdots+m_1)+2(l_n+\cdots+l_2)\\ +l_1+2(k_{n-1}+k_{n-2}+\cdots+k_2)+k_1\end{matrix} & \ldots & \begin{matrix}3+(m_n+m_{n-1})+\\ 2l_n+l_{n-1}+k_{n-1}\end{matrix} & \begin{matrix}1+m_n\\ +l_n\end{matrix} \\ \\ \\ 1+l_1+k_1 & \ldots & 1+l_{n-1}+k_{n-1} & 1+l_n \end{pmatrix}$$

Here, $1+l_j$ and k_i represents the green color and red color of the part respectively, where $1\leq j\leq n$, $1\leq i\leq n-1$.
From the array, $s=n\geq 1$ and it is clear that r_i can be 0 or 1 for $i\neq s$ and $b_s = 1+l_n$, it shows that for the smallest part or if there is only one part in the partition, r_s is always 0, which proves (i).

From array, for $i\neq s$

$$\begin{aligned} a_i &= (n-i+2)+(m_n+m_{n-1}+\cdots+m_i)+2(l_n+l_{n-1}+\cdots+l_{i+1})+l_i \\ &\quad +2(k_{n-1}+\cdots+k_{i+1})+k_i, \\ b_i &= 1+l_i+k_i. \end{aligned}$$

And $a_s = 1+m_n+l_n, b_s = 1+l_n$ Therefore, $(w.d.)_i = m_{i-1}-1\geq -1\ \forall\ 1<i<s$ and $(w.d.)_s\geq 0$ which proves (ii). □

Thus, $a_2(\nu) = 1+S_3(\nu)+S_4(\nu)$.

Example 2.9. The relevant partitions for $S_3(6)=18$ are $6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 5_11_1, 5_21_1, 5_31_1, 5_41_1, 5_{1+1}1_1, 5_{2+1}1_1, 5_{3+1}1_1, 4_12_1, 4_12_2, 4_22_1, 4_{1+1}2_1, 3_12_11_1$.

The relevant partitions for $S_4(6)=12$ are $6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 5_11_1, 5_21_1, 5_31_1, 5_{1+1}1_1, 5_{2+1}1_1, 4_12_1$.

Interpretation of R.H.S. of (3)

$$P_2(\nu)=\sum_{k=0}^{\nu} A_2(\nu-k)B_2(k),$$

where $A_2(\nu)$ denote the number of partitions of ν into parts $\equiv \pm1, \pm2,\ 4 \pmod 8$ and $B_2(\nu)$ denote the number of partitions of ν into distinct parts $\equiv 0, \pm1, \pm2, \pm3,\ 4 \pmod 8$.

Proof of Identity (3). The interpretation of sum in the L.H.S. of (3) is presented in Theorems 2.7 and 2.8 while the interpretation of the product provided in the R.H.S. is given by $P_2(\nu)$ mentioned above. □

Interpretation of L.H.S. of (4)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+2)}}{(-q^{-1};q^2)_n(q^4;q^4)_n} &= 1+\sum_{n=1}^{\infty}\frac{q^{(n+1)^2}}{(-q;q)_1(q^4;q^4)_n}+\sum_{n=1}^{\infty}\frac{q^{n^2+4n}}{(-q;q)_1(q^4;q^4)_n}\\ &= 1+\sum_{\nu\geq1}S_5(\nu)q^\nu+\sum_{\nu\geq1}S_6(\nu)q^\nu.\end{aligned}$$

Theorem 2.10. *For $n \geq 1$, consider the $n-$color partitions satisfying*

(i) $s>1$, *and* $(a_s)_{b_s}=1_1$

(ii) $b_i=1\ \forall\ i\neq 1$,

(iii) *If s is odd,*

$$a_i-b_i\equiv\begin{cases}0 \pmod 4 & i \text{ odd},\\ 2 \pmod 4 & i \text{ even}.\end{cases}$$

If s is even,

$$a_i-b_i\equiv\begin{cases}0 \pmod 4 & i \text{ even},\\ 2 \pmod 4 & i \text{ odd}.\end{cases}$$

(iv) $(w.d.)_i\geq 0$ *and is multiple of* $4\ \forall\ 1<i\leq s$,

(v) *Each partition is counted with weight* $(-1)^{b_1-1}$.

Then

$$\sum_{\nu\geq1}S_5(\nu)q^\nu=\sum_{n=1}^{\infty}\frac{q^{(n+1)^2}}{(-q;q)_1(q^4;q^4)_n}.$$

Theorem 2.11. *For $n \geq 1$, consider the $n-$color partitions satisfying*

(i) *If* $s>1$, $a_s=5+4k$, *and* $b_i=1\ \forall\ i\neq 1\ k\geq 0$,

(ii) *If* $s=1$, $a_1=5+4k+l$, *and* $b_1=1+l$, $k,l\geq 0$

(iii) *If s is odd,*

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ odd}, \\ 2 \pmod 4 & i \text{ even}. \end{cases}$$

If s is even,

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ even}, \\ 2 \pmod 4 & i \text{ odd}. \end{cases}$$

(iv) $(w.d.)_i \geq 0$ *and is multiple of* $4 \ \forall\ 1 < i \leq s$.

(v) *Each partition is counted with weight* $(-1)^{b_1-1}$. *Then*

$$\sum_{\nu\geq 1} S_6(\nu)q^\nu = \sum_{n=1}^{\infty} \frac{q^{n^2+4n}}{(-q;q)_1(q^4;q^4)_n}.$$

Proof of Theorem 2.10. From the expression,

$$\sum_{n=1}^{\infty} \frac{q^{(n+1)^2}}{(-q;q)_1(q^4;q^4)_n},$$

we initially consider its unsigned version

$$\sum_{n=1}^{\infty} \frac{q^{(n+1)^2}}{(q;q)_1(q^4;q^4)_n},$$

to know the relevant partitions, avoiding the negative signs in the expansion of the above series. Here, $q^{(n+1)^2}$ generates $(n+1)$ parts as $1, 3, 5, \ldots, (2n+1)$. $(q^4;q^4)_n^{-1}$ generates n parts as $m_1 \cdot 4, m_2 \cdot 8, \ldots, m_n \cdot (4n)$, where $m_i \geq 0 \ \forall\ 1 \leq i \leq n$. $(q;q)_1^{-1}$ generates one part $p \times 1$, where $p \geq 0$.

Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix} (2n+3) + 4(m_n + \cdots + m_1) + p & \cdots & 5 + 4(m_n + m_{n-1}) & 3 + 4m_n & 1 \\ 1 + p & \cdots & 1 & 1 & 1 \end{pmatrix},$$

From the given array, $s = n+1 > 1$, which implies (i). And each $b_i = 1 \ \forall\ i \neq 1$. Now, for $2 \leq i \leq n+1$, $a_i = 2(n-i) + 3 + 4(m_n + m_{n-1} + \cdots + m_i)$, $b_i = 1$,

Also, $a_1 = (2n+1) + 4(m_n + \cdots + m_1) + p$, $b_1 = 1 + p$,

The difference between the parts and their subscripts follows a pattern where $a_s - b_s \equiv 0 \pmod 4$, $a_{s-1} - b_{s-1} \equiv 2 \pmod 4$, then $\equiv 0 \pmod 4, \equiv 2 \pmod 4$, till the largest part. Now, if the number of parts in the partition are odd, or we can say if s is odd, then it follows

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ odd}, \\ 2 \pmod 4 & i \text{ even}, \end{cases}$$

and if s is even,

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ even}, \\ 2 \pmod 4 & i \text{ odd}, \end{cases}$$

that explicates condition (iii). For consecutive parts, $(w.d.)_i = 4m_{i-1} \geq 0 \;\; \forall\, 1 < i \leq s$ and is a multiple of 4, which proves (iv).
To explain condition (v), we consider the expansion of $\sum\limits_{n\geq 1} S_5(\nu)q^\nu$ which is given as

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^{(n+1)^2}}{(-q;q)_1(q^4;q^4)_n} &= \frac{1}{(1+q)}\left(q^4+q^9+q^{13}+q^{16}+q^{17}+q^{20}+q^{21}+2q^{24}+\cdots\right)\\ &= (1-q^1+(-1)^2q^{1\cdot 2}+(-1)^3q^{1\cdot 3}+(-1)^4q^{1\cdot 4}+\cdots)\cdot\\ &\quad\;\left(q^4+q^9+q^{13}+q^{16}+q^{17}+q^{20}+q^{21}+2q^{24}+\cdots\right)\\ &= q^4-q^5+q^6-q^7+2q^8-q^9+q^{10}-q^{11}+q^{12}-q^{13}+q^{14}+\cdots.\end{aligned} \tag{11}$$

Here, negative signs are coming due to the factor $(-q;q)_1^{-1}$, which generates the part 1 with multiplicity p, mentioned by $q^{1\cdot p}$, where $p \geq 0$ in the series. Now we will consider the expansion of unsigned version to know the number of relevant partitions which gives the following:

$$\sum_{n=1}^{\infty}\frac{q^{(n+1)^2}}{(q;q)_1(q^4;q^4)_n} = q^4+q^5+q^6+q^7+2q^8+3q^9+3q^{10}+3q^{11}+4q^{12}+5q^{13}+5q^{14}+\cdots. \tag{12}$$

To tackle this difference in the coefficients of two aforementioned series, we introduce some weight and this weight is due to the factor $(-q;q)_1^{-1}$. For reference, see [19].

It can be easily seen from the array, the factor p is responsible for the change in sign, so each partition is counted with weight $(-1)^p$. Also,

$$b_1 = 1 + p \implies p = b_1 - 1.$$

So, the weight can be given as $(-1)^{b_1-1}$. □

For perspicuity, we illustrate this notion by taking an example.

Table 2. Partitions enumerated by $S_5(14)$ with weights

Partition	Weight
$13_{11}+1_1$	$(-1)^{11-1}=+1$
13_3+1_1	$(-1)^{3-1}=+1$
13_7+1_1	$(-1)^{7-1}=+1$
$10_6+3_1+1_1$	$(-1)^{6-1}=-1$
$10_2+3_1+1_1$	$(-1)^{2-1}=-1$
Total weight	1

Example 2.12. For $\nu = 14$ there are 5 relevant partitions which can be easily seen from the unsigned version (12) where coefficient of q^{14} is 5 and the weighted sum of their number is 1 which is given in signed version (11) as obtained in Table 2.

Proof of Theorem 2.11. From the expression,

$$\sum_{n=1}^{\infty} \frac{q^{n^2+4n}}{(-q;q)_1(q^4;q^4)_n}.$$

we initially consider its unsigned version to know the relevant partitions, avoiding the negative signs in the expansion of the above series occurs due to $(-q;q)^{-1}$.

$$\sum_{n=1}^{\infty} \frac{q^{n^2+4n}}{(q;q)_1(q^4;q^4)_n}.$$

Here, q^{n^2+4n} generates n parts as $5, 7, 9, 11, \ldots, (2n+3)$. $(q^4;q^4)_n^{-1}$ generates n parts as $m_1 \cdot 4, m_2 \cdot 8, \ldots, m_n \cdot (4n)$, where $m_i \geq 0 \ \forall\ 1 \leq i \leq n$. $(q;q)_1^{-1}$ generates one part $p \times 1$, where $p \geq 0$.

Thus, the $n-$color partition can be written in two-line array as

$$\begin{pmatrix} (2n+3)+4(m_n+\cdots+m_1)+p & \ldots & 7+4(m_n+m_{n-1}) & 5+4m_n \\ 1+p & \ldots & 1 & 1 \end{pmatrix}.$$

From the array, $s = n \geq 1$. The condition (i) and (ii) can be easily proved by looking at the array.

For $2 \leq i \leq n$, $a_i = 2(n-i+2)+1+4(m_n+m_{n-1}+\cdots+m_i)$, $b_i = 1$, and, $a_1 = (2n+3)+4(m_n+\cdots+m_1)+p$, $b_1 = 1+p$.

The difference between the parts and their subscripts follows a pattern where $a_s - b_s \equiv 0 \pmod 4$, $a_{s-1} - b_{s-1} \equiv 2 \pmod 4$, then $\equiv 0 \pmod 4, \equiv 2 \pmod 4$, and the pattern continues till the largest part. Now, if the number of parts in the partition are odd, or we can say if s is odd, then

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ odd}, \\ 2 \pmod 4 & i \text{ even}, \end{cases}$$

and if s is even,

$$a_i - b_i \equiv \begin{cases} 0 \pmod 4 & i \text{ even}, \\ 2 \pmod 4 & i \text{ odd}. \end{cases}$$

For consecutive parts, $(w.d.)_i = 4m_{i-1} \geq 0 \ \forall\ 1 < i \leq s$ and is a multiple of 4, which proves (iii). To explain part (v) in the theorem, we consider the expansion of $\sum\limits_{\nu \geq 1} S_5(\nu) q^\nu$ in the expanded form is given as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^{n^2+4n}}{(-q;q)_1(q^4;q^4)_n} &= \frac{1}{(1+q)}\left(q^5+q^9+q^{12}+q^{13}+q^{16}+q^{17}+2q^{20}+2q^{21}+\cdots\right) \\ &= (1-q^1+(-1)^2q^{1\cdot 2}+(-1)^3q^{1\cdot 3}+(-1)^4q^{1\cdot 4}+\cdots)\cdot \\ &\qquad \left(q^5+q^9+q^{12}+q^{13}+q^{16}+q^{17}+2q^{20}+2q^{21}+\cdots\right) \\ &= q^5-q^6+q^7-q^8+2q^9-2q^{10}+2q^{11}-q^{12}+2q^{13}-2q^{14}+\cdots. \end{aligned}$$

Here, negative signs are coming due to the factor $(-q;q)_1^{-1}$, which generates the part 1 with multiplicity p, mentioned by $q^{1\cdot p}$, where $p \geq 0$ in the series. Now we will consider the expansion of unsigned version to know the number of relevant partitions which gives the following:

$$\sum_{n=1}^{\infty} \frac{q^{n^2+4n}}{(q;q)_1(q^4;q^4)_n} = q^5+q^6+q^7+q^8+2q^9+2q^{10}+2q^{11}+3q^{12}+4q^{13}+4q^{14}+4q^{15}+5q^{16}+\cdots.$$

To tackle this difference in the coefficients of two aforementioned series, we introduce some weight and this weight is due to the factor $(-q;q)_1^{-1}$. It can be easily seen from the array, the factor p is responsible for the change in sign, so each partition is counted with weight $(-1)^p$. Also,

$$b_1 = 1 + p \implies p = b_1 - 1.$$

So, the weight can be given as $(-1)^{b_1-1}$. □

For perspicuity, we illustrate this notion by an example.

Example 2.13. For $\nu = 14$ there are 4 relevant partitions which can be easily seen from the unsigned version where coefficient of q^{14} is 4 and the weighted sum is -2 which is given in signed version as obtained in Table 3.

Table 3. Partitions enumerated by $S_6(14)$ with weights

Partition	Weight
14_6	$(-1)^{6-1} = -1$
14_{10}	$(-1)^{10-1} = -1$
14_{14}	$(-1)^{14-1} = -1$
$9_3 + 5_1$	$(-1)^{3-1} = +1$
Total weight	-2

Thus, $a_3(\nu) = S_5(\nu) + S_6(\nu)$.
Interpretation of R.H.S. of (4)

$$P_3(\nu) = \sum_{k=0}^{\nu} \sum_{j=0}^{k} A_3(\nu-k)B_3(k-j)C_3(j),$$

where $A_3(\nu)$ denote the number of partitions of ν into parts $\equiv \pm 4 \pmod{20}$, $B_3(\nu)$ denote the number of partitions of ν into distinct parts $\equiv \pm 9, 19, 21 \pmod{20}$ and $C_3(\nu)$ denote the number of partitions with distinct parts $\equiv 10 \pmod{20}$ with weight -1.

Example 2.14. For $\nu = 14$, $A_3(4) = 1$ (Partition: 4), $B_3(0) = 1$, $C_3(10) = (-1)^1$ (Partition: 10), Therefore $P_3(14) = 1 \cdot 1 \cdot (-1)^1 = -1$.

Proof of Identity (4)**.** The interpretation of the sum in the L.H.S. of (4) is presented in Theorem 2.10 and 2.11 while the interpretation for the product provided in the R.H.S. is given by $P_3(\nu)$ mentioned above. In the examples shown above $S_5(14)+S_6(14) = 1+(-2) = -1$ while $P_3(14) = -1$. □

Interpretation of L.H.S. of (5)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q^2;q^4)_n q^{n(n-2)}}{(q^4;q^4)_n(-q^{-1};q^2)_n} &= 1+\sum_{n=1}^{\infty}\frac{(q^2;q^4)_n q^{(n-1)^2}}{(-q;q)_1(q^4;q^4)_n(-q;q^2)_{n-1}}\\ &= 1+\sum_{n=1}^{\infty}\frac{(q;q^2)_n(1+q^{2n-1})q^{(n-1)^2}}{(-q;q)_1(q^4;q^4)_n}\\ &= 1+\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{(n-1)^2}}{(-q;q)_1(q^4;q^4)_n}+\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{n^2}}{(-q;q)_1(q^4;q^4)_n}\\ &= 1+\sum_{\nu\geq 0}S_7(\nu)q^{\nu}+\sum_{\nu\geq 1}S_8(\nu)q^{\nu}. \end{aligned} \tag{13}$$

Theorem 2.15. *For $\nu \geq 0$, consider the split $(n+t)-$color partition satisfying:*

(i) $g_i = 1\ \forall\ i \neq 1,\ \ g_1 \geq 1,$

(ii) $r_i = 0$ *or* $1\ \forall\ i,$

(iii) *for $i \neq s$, parts and their subscripts have same parity, and particularly $a_s - b_s \equiv 3 \pmod 4$,*

(iv) $(w.d.)_i \geq 0$ *and multiple of* $4\ \forall\ 1 < i < s$, *and* $(w.d.)_s \geq -1,\ (w.d.)_s \equiv 3 \pmod 4$,

(v) *each partition is counted with weight* $(-1)^{\sum_{i=1}^{s} r_i + g_1 - 1}$. *Then*

$$\sum_{\nu\geq 0}S_7(\nu)q^{\nu} = \sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{(n-1)^2}}{(-q;q)_1(q^4;q^4)_n}.$$

Theorem 2.16. *For $\nu \geq 1$, consider the split $n-$color partition satisfying:*

(i) $g_i = 1\ \forall\ i \neq 1,\ \ g_1 \geq 1,$

(ii) $r_i = 0$ *or* $1\ \forall\ 1 \leq i \leq s,$

(iii) *parts and their subscripts have same parity, and particularly $a_s - b_s \equiv 0 \pmod 4$,*

(iv) $(w.d.)_i \geq 0$ *and multiple of* $4\ \forall\ 1 < i \leq s,$

(v) *each partition is counted with weight* $(-1)^{\sum_{i=1}^{s} r_i + g_1 - 1}$. *Then*

$$\sum_{\nu\geq 1}S_8(\nu)q^{\nu} = \sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{n^2}}{(-q;q)_1(q^4;q^4)_n}.$$

Sketch Proof of Theorem 2.15 and 2.16. The Simplification of $S_7(\nu)$ and $S_8(\nu)$ follows same steps as defined earlier. In the expression of $\sum_{\nu\geq 0} S_7(\nu)q^{\nu}$, consider its unsigned version to know the relevant partitions,

$$\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{(n-1)^2}}{(q;q)_1(q^4;q^4)_n}.$$

Here, $q^{(n-1)^2}$ generates n parts as $0,1,3,5,\ldots,(2n-3)$. $(-q;q^2)_n$ generates n parts as $l_1\cdot 1, l_2\cdot 3, l_3\cdot 5,\ldots,l_n\cdot(2n-1)$, where $l_j=0$ or $1\ \forall\ 1\le j\le n$. And $(q^4;q^4)_n^{-1}$ generates n parts as $m_1\cdot 4, m_2\cdot 8,\ldots,m_n\cdot(4n)$, where $m_i\ge 0\ \forall\ 1\le i\le n$. $(q;q)_1^{-1}$ generates one part $p\times 1$, where $p\ge 0$.

Thus, the $(n+t)-$split color partition can be written in two-line array as

$$\begin{pmatrix} \begin{matrix}(2n-3)+4(m_n+\cdots+m_1)\\ +2(l_n+\cdots+l_2)+l_1+p\end{matrix} & \ldots & \begin{matrix}3+4(m_n+m_{n-1}+m_{n-2})\\ +2(l_n+l_{n-1})+l_{n-2}\end{matrix} & \begin{matrix}1+4(m_n+m_{n-1})\\ +l_{n-1}\end{matrix} & \begin{matrix}0+4m_n\\ +l_n\end{matrix} \\ \\ \\ 1+p+l_1 & \ldots & 1+l_{n-2} & 1+l_{n-1} & 1+l_n \end{pmatrix}.$$

Here, l_j contributes to the red color in split partition and the factor p and l_j contributes to the weight in each partition. On similar lines, we present the array for unsigned version of $S_8(\nu)$ which is shown as:

$$\begin{pmatrix} \begin{matrix}(2n-1)+4(m_n+\cdots+m_1)\\ +2(l_n+\cdots+l_2)+l_1+p\end{matrix} & \ldots & \begin{matrix}3+4(m_n+m_{n-1}+m_{n-2})\\ +2l_n+l_{n-1}\end{matrix} & \begin{matrix}1+4(m_n+m_{n-1})\\ +l_n\end{matrix} \\ \\ \\ 1+p+l_1 & \ldots & 1+l_{n-1} & 1+l_n \end{pmatrix}.$$

The conditions of $S_7(\nu)$ and $S_8(\nu)$ can be extracted from the given array. □

Interpretation of R.H.S. of (5)

$$P_4(\nu)=\sum_{k=0}^{\nu}A_4(\nu-k)B_4(k),$$

where $A_4(\nu)$ denote twice the number of partitions of ν into distinct parts $\equiv\pm1\pmod 4$, $B_4(\nu)$ denote twice the number of partitions of ν into distinct parts $\equiv 2\pmod 4$ counted with weight -1.

Example 2.17. Partitions have been provided in Tables 4, 5 and 6, to verify the count of partitions and weight corresponding to $S_7(5)$ and $S_8(\nu)$.

Proof of Identity (5). The interpretation of the sum in the L.H.S. of (5) is presented in Theorems 2.15 and 2.16 while the interpretation for the product provided in the R.H.S. is given by $P_4(\nu)$ mentioned above. In the examples shown above $S_7(5)+S_8(5)=-1+1=0$ while $P_4(5)=0$. □

Table 4. Partitions enumerated by $S_7(5)$ with weights

Partition	Weight
5_6	$(-1)^{6-1} = -1$
5_{5+1}	$(-1)^{1+5-1} = -1$
5_{1+1}	$(-1)^{1+1-1} = -1$
5_2	$(-1)^{2-1} = -1$
$5_1 0_1$	$(-1)^{1-1} = +1$
$5_5 0_1$	$(-1)^{5-1} = +1$
$5_{4+1} 0_1$	$(-1)^{1+4-1} = +1$
$4_{1+1} 1_{1+1}$	$(-1)^{1+1+1-1} = +1$
$4_2 1_{1+1}$	$(-1)^{1+2-1} = +1$
$4_2 1_1 0_1$	$(-1)^{2-1} = -1$
$4_{1+1} 1_1 0_1$	$(-1)^{1+1-1} = -1$
Total weight	-1

Table 5. Partitions enumerated by $S_8(5)$ with weights

Partition	Weight
5_1	$(-1)^{1-1} = +1$
5_5	$(-1)^{5-1} = +1$
5_{4+1}	$(-1)^{1+4-1} = +1$
$4_2 1_1$	$(-1)^{2-1} = -1$
$4_{1+1} 1_1$	$(-1)^{1+1-1} = -1$
Total weight	$+1$

Table 6. Partitions enumerated by $P_4(5)$

k	$A_4(\nu - k)$	$B_4(k)$	$A_4(\nu - k)B_4(k)$
0	2×1 (5)	2×1	4
1	2×1 $(3 + 1)$	2×0 (ϕ)	0
2	2×1 (3)	$2 \times (-1)^1$ (2)	-4
3	2×0 (ϕ)	2×0 (ϕ)	0
4	2×1 (1)	2×0 (ϕ)	0
5	2×0 (ϕ)	2×0 (ϕ)	0
		Total	0

3. Conclusion

The combinatorial interpretations of the q-series offer a rich comprehension of their structure by connecting analytic expressions with discrete objects like partitions and lattice configurations. Ongoing research provide combinatorial proofs using $n-$color partitions for q-series having negative exponents. The study of Rogers$-$Ramanujan type identities provides an interplay between combinatorics and series identities is probably going to continue to be a dynamic and productive area of mathematical study.

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