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Coupon coloring number of inflated graphs

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ABSTRACT

Let G be a graph with no isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex contains all colors from $[k]$. The maximum integer k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$. In this paper, we investigate coupon coloring in inflated graphs arising from various classes of graphs. In addition, we introduce new graph operations based on inflation and study their effect on the existence and behavior of coupon colorings. Our results contribute to a deeper understanding of how inflation based graph operations influence coupon coloring.

Keywords: coupon coloring number, paths, cycles, unary operations

2020 Mathematics Subject Classification: 05C15, 05C76.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. In general we follow [1] and [16] for graph terminology and notations. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, the neighborhood of x, denoted by $\mathcal{N}(x)$, is the set of all vertices adjacent to x. The degree of a vertex x is the cardinality of $\mathcal{N}(x)$, and the minimum degree of G is denoted by $\delta(G)$. A vertex of degree 1 is called a pendant vertex, whereas a universal vertex is one that is adjacent to every other vertex of the graph.

Graph coloring is a central topic in graph theory with a wide range of theoretical and practical applications. A particularly interesting variant, introduced by Chen et al. [3], is the coupon coloring of graphs, which models situations where each vertex must have access to all resources (colors) through its neighbors. Let G be a graph with no isolated

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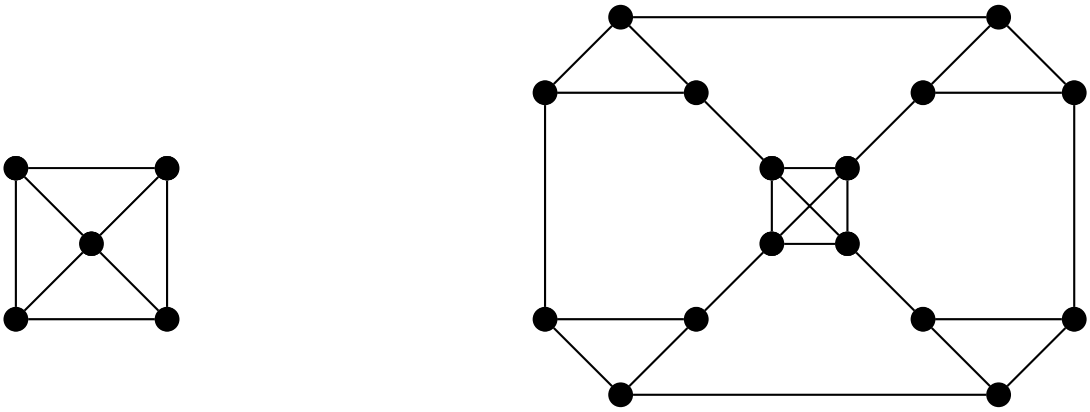
vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \ldots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G, and is denoted by $\chi_c(G)$ [3]. Clearly, $\chi_c(G) \leq \delta(G)$ for any graph G. In a k-coloring c, a vertex v is considered to be a bad vertex if its neighborhood does not contain vertices of all colors from $[k]$ and it is clear that there are no bad vertices in a coupon coloring [15]. The notion of coupon coloring is closely related to graph domination. Let $G = (V, E)$ be a graph without isolated vertices. $D^{'} \subseteq V$ is a total dominating set if every vertex of G is adjacent to at least one vertex in $D^{'}$. The coupon coloring number is also referred to as the total domatic number introduced in [12], which is the maximum number of disjoint total dominating sets. Total domatic number and coupon coloring have been extensively studied. The coupon coloring numbers of wheels, cycles, unicyclic and bicyclic graphs, complete graphs, and complete k-partite graphs were determined by Y Shi et al. in [15]. Additionally, coupon coloring has been examined in [14, 4] and [12]. The coupon coloring number of a few binary products, including the lexicographic and cartesian products is studied in [6, 13]. Also in [10] the authors have demonstrated that for every $k \geq 3$, it is NP-complete to decide whether $d_t(G) \geq k$, where G is split graph. Similarly, [10] shows that determining whether a bipartite planar graph has $d_t(G) \geq 3$ is NP-complete.

Inflation is a unary graph operation that leaves the degree sequence unchanged while increasing the number of vertices and edges, and it has been examined in relation to total domination [7], double domination [11], as well as in various other studies [9, 8]. Cubic inflation was previously investigated in [2]. This paper studies coupon coloring in inflated graphs and introduces new inflation-based graph operations to examine their effect on coupon coloring.

2. Inflation of graphs

For the notation of inflated graphs, we follow [5]. The inflation G_I of a graph G is obtained by replacing each vertex v_i of degree $d_G(v_i)$ with a clique K_i of size $d_G(v_i)$. Each edge $v_iv_j \in E(G)$ is replaced by an edge uv such that $u \in K_i$ and $v \in K_j$, and distinct edges of G are replaced by nonadjacent edges in G_I. An immediate consequence of this definition is that $n(G_I) = \sum\limits_{v_i \in V(G)} d_G(v_i) = 2m(G)$, $\Delta(G_I) = \Delta(G)$, $\delta(G_I) = \delta(G)$. Moreover, G_I can be seen as the line graph of the subdivision of G. In G_I, each vertex v_i of degree k in the original graph G is replaced by a clique of k vertices labeled $v_i^1, v_i^2, \ldots, v_i^k$. Figure 1 illustrates Inflation of wheel.

Observation 2.1. *Let G be any graph having a pendant vertex. After applying the inflation operation, the graph G_I also contains pendant vertex and hence $\delta(G_I) = 1$. We know that $\chi_c(G_I) \leq \delta(G_I)$. This implies that $\chi_c(G_I) \leq 1$. Also G_I admits 1-coupon coloring. Thus $\chi_c(G_I) = 1$*



(a) (b)

Fig. 1. (a) Wheel graph of order 5 (b) Inflation of wheel graph

Observation 2.2. *Let G be a cycle graph on n vertices, then $(C_n)_I = C_{2n}$. Therefore,*

$$\chi_c(G_I) = \begin{cases} 1 & \textit{if } n \equiv 1 \pmod 2, \\ 2 & \textit{otherwise.} \end{cases}$$

Theorem 2.3. *Let $G = W_n$ be a wheel graph on n vertices. Then*

$$\chi_c(G_I) = \begin{cases} 2 & \textit{if } n = 5 \\ 3 & \textit{otherwise.} \end{cases} \tag{1}$$

Proof. Let $G = W_n$ be a wheel graph on n vertices with vertex set $v_1, v_2, \cdots, v_n$ and let v_1 be the universal vertex. Here v_1, having degree $n-1$, is replaced by the set $\{v_1^1, v_1^2, \ldots, v_1^{n-1}\}$, where each vertex v_1^i is adjacent to v_{i+1}^1 in G_I, corresponding to its neighbor v_{i+1} in G. For each vertex v_i with $i = 2, 3, \ldots, n$, the vertices v_i^2 and v_i^3 are connected in such a way that the sequence $v_2^2 - v_2^3 - v_3^2 - v_3^3 - \cdots - v_n^2 - v_n^3 - v_2^2$ forms a cycle in G_I, reflecting the cyclic structure among the vertices $v_2, v_3, \ldots, v_n$ in G. Since $\delta(G_I) = 3$, we have $\chi_c(G_I) \leq 3$.

Case 1. $n = 5$. For the case $n = 5$, we have $\delta(G_I) = 3$, so by the definition of coupon coloring, $\chi_c(G_I) \leq 3$. Assume, for contradiction, that a valid 3-coupon coloring of G_I exists. Consider the vertex v_3^1, which is adjacent to v_3^2, v_3^3, and v_1^2. Without loss of generality, assign colors $c(v_3^2) = 1$, $c(v_3^3) = 2$, and $c(v_1^2) = 3$. Now consider the vertex v_4^2, whose neighbors are v_4^1, v_4^3, and v_3^3. Since $c(v_3^3) = 2$, and three colors must appear in the neighborhood, the possible colors for v_4^1 are $\{1, 3\}$.

Case 1.1. Suppose $c(v_4^1) = 3$. This forces $c(v_4^3) = 1$. Now, focusing on v_5^2, which is adjacent to v_5^1, v_5^3, and v_4^3, and since $c(v_4^3) = 1$, we must have $c(v_5^3) \in \{2, 3\}$.

Case 1.1.1. First, consider $c(v_5^3) = 3$. This forces $c(v_5^1) = 2$. Since v_5^1 already has a neighbor colored 3, we assign $c(v_5^2) = 1$. Otherwise if we assign $c(v_5^2) = 2$, then v_5^3 will have two vertices in its neighborhood with same color 2, which makes v_5^3 a bad vertex. Next,we consider the vertex v_5^3, we get $c(v_2^2) = 2$. Now we consider the neighboring vertices of the following vertices $v_5^1, v_4^3, v_3^3, v_3^2, v_2^1, v_4^1$ and finally v_2^3 in the order. This forces a cascade: $c(v_2^2) = 3$, $c(v_1^4) = 2$, $c(v_4^2) = 2$, $c(v_3^1) = 3$, $c(v_2^3) = 1$, $c(v_1^1) = 2$, $c(v_1^3) = 3$, $c(v_2^1) = 3$. As a result, all vertices v_1^i for $1 \leq i \leq 4$ become "bad" since their neighborhoods such as the vertices v_i^j where $2 \leq i \leq 5$ and $1 \leq j \leq 3$ lack the color 1.

Case 1.1.2. Let $c(v_5^3) = 2$, then $c(v_5^1) = 3$. Given that v_5^1 has a neighbor colored 2, we must set $c(v_5^2) = 1$, otherwise v_5^3 will become the bad vertex. This again leads to: $c(v_2^2) = 2$, $c(v_1^4) = 3$, $c(v_4^2) = 2$, $c(v_3^1) = 3$, $c(v_2^3) = 1$, $c(v_1^1) = 3$, $c(v_1^3) = 3$, $c(v_2^1) = 3$. Once again, all vertices v_1^i for $1 \leq i \leq 4$ become bad since they lack the color 1 in its neighboring vertices which is v_i^j where $2 \leq i \leq 5$ and $1 \leq j \leq 3$.

Case 1.2. Suppose $c(v_4^1) = 1$. Then, we must have $c(v_4^3) = 3$. Now consider the vertex v_5^2, which is adjacent to v_5^1, v_5^3, and v_4^3. Since $c(v_4^3) = 3$, the possible colors for v_5^3 are 1 and 2.

Case 1.2.1. Assume $c(v_5^3) = 2$. This forces $c(v_5^1) = 1$. Since v_5^1 already has a neighbor with color 2, we assign $c(v_5^2) = 3$, otherwise if we assign $c(v_5^2) = 1$ this will force v_5^3 to become a bad vertex. This leads to the following color assignments: $c(v_2^2) = 2$ and $c(v_1^4) = 1$. Also, $c(v_4^2) = 2$, which implies $c(v_1^3) = 1$. Considering the vertex v_3^3, we get $c(v_3^1) = 3$, and subsequently $c(v_2^3) = 1$, which leads to $c(v_1^1) = 3$, $c(v_2^1) = 3$. As a result, all vertices v_1^i for $1 \leq i \leq 4$ become "bad" since their neighborhoods fail to contain the color 2.

Case 1.2.2. Alternatively, assume $c(v_5^3) = 1$. Then $c(v_5^1) = 2$. Since v_5^3 has a neighbor with color 2, we get $c(v_5^2) = 3$. This forces: $c(v_2^2) = 1$ and $c(v_1^4) = 2$. Also, $c(v_4^2) = 2$, which implies $c(v_1^3) = 1$. Considering v_3^3, we get $c(v_3^1) = 3$, and then $c(v_2^3) = 1$. In this scenario, the vertex v_2^1 becomes a bad vertex, since its neighborhood contain two vertices with same color 1 and hence all the required three colors are not present in the neighborhood.

In both cases, a contradiction arises, thereby showing that a valid 3-coupon coloring is not possible when $n = 5$. To prove this we define $C : V(G_I) \to [2]$ as follows. Let $c(v_1^1) = c(v_1^2) = c(v_2^1) = c(v_3^1) = c(v_4^2) = c(v_4^3) = c(v_5^2) = c(v_5^3) = 1$ and color all the other vertices with the color 2. This is indeed a coupon coloring. Thus $\chi_c(G_I) = 2$ when $n = 5$.

Case 2. $n \neq 5$. Define a coloring function $c : V(G_I) \to [3]$ as follows.

Case 2.1. $n \equiv 0 \pmod 3$. Assign $c(v_1^{n-1}) = c(v_1^{n-2}) = c(v_{n-1}^1) = c(v_n^1) = 3$, $c(v_{n-1}^2) = c(v_n^3) = 2$, and $c(v_{n-1}^3) = c(v_n^2) = 1$. For $1 \leq j \leq n-3$, define:

$$c(v_1^j) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 3, \\ 2 & \text{if } j \equiv 2 \pmod 3, \\ 3 & \text{if } j \equiv 0 \pmod 3. \end{cases} \tag{2}$$

Then, for $2 \leq i \leq n-2$ and $1 \leq j \leq 3$ define $c(v_i^j) = (i + j - 3) \pmod 3 + 1$.

Case 2.2. $n \equiv 1 \pmod 3$. For $1 \leq j \leq n-1$, define

$$c(v_1^j) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 3, \\ 2 & \text{if } j \equiv 2 \pmod 3, \\ 3 & \text{if } j \equiv 0 \pmod 3. \end{cases} \tag{3}$$

For $2 \leq i \leq n$ and $1 \leq j \leq 3$ we define $c(v_i^j) = (i + j - 3) \pmod 3 + 1$.

Case 2.3. $n \equiv 2 \pmod 3$ and $n > 5$. Set $c(v_1^{n-1}) = c(v_{n-1}^2) = 1$, $c(v_1^{n-2}) = c(v_n^3) = 2$, and $c(v_{n-1}^3) = c(v_n^2) = 3$. For $1 \leq j \leq n - 3$:

$$c(v_1^j) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 3, \\ 2 & \text{if } j \equiv 2 \pmod 3, \\ 3 & \text{if } j \equiv 0 \pmod 3. \end{cases} \tag{4}$$

For $j \neq 1$ and $2 \leq i \leq n$:

$$c(v_i^j) = \begin{cases} 1 & \text{if } i + j \equiv 0 \pmod 3, \\ 2 & \text{if } i + j \equiv 1 \pmod 3, \\ 3 & \text{if } i + j \equiv 2 \pmod 3. \end{cases} \tag{5}$$

□

Theorem 2.4. *Let $G = K_{m,n}$ be a complete bipartite graph on $m + n$ vertices. Then $\chi_c(G_I) = min\{m, n\}$.*

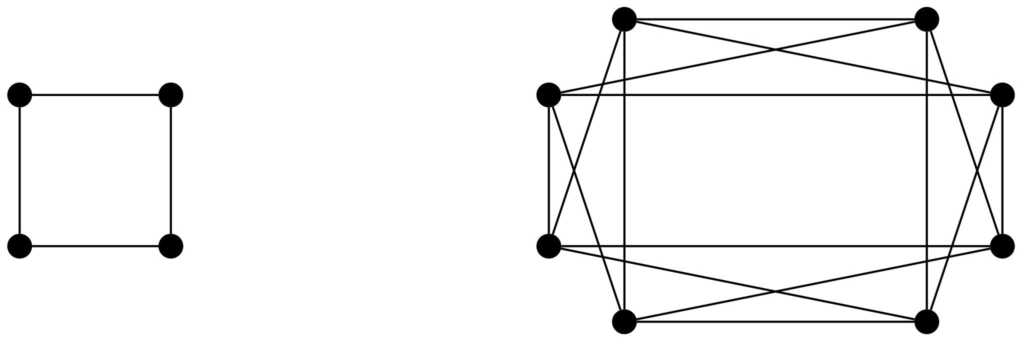
Proof. Let $G = K_{m,n}$ be a complete bipartite graph with $m + n$ vertices, and assume without loss of generality that $m \leq n$. In G_I, each vertex v_i of degree k is replaced by a clique of k vertices labeled $v_i^1, v_i^2, \ldots, v_i^k$. The graph G_I has a vertex set $V' = V_1' \cup V_2'$, where $V_1' = \{v_1^1, v_1^2, \ldots, v_1^n, v_2^1, v_2^2, \ldots, v_2^n, \ldots, v_m^1, v_m^2, \ldots, v_m^n\}$ and $V_2' = \{v_{m+1}^1, v_{m+1}^2, \ldots, v_{m+1}^m, v_{m+2}^1, v_{m+2}^2, \ldots, v_{m+2}^m, \ldots, v_{m+n}^1, v_{m+n}^2, \ldots, v_{m+n}^m\}$. Also each vertex $v_i^j \in V_1'$ is connected to the vertex $v_{m+j}^i \in V_2'$. Since every vertex in G_I has degree at least m, we have $\chi_c(G_I) \leq m$. We need to only show that $\chi_c(G_I) \geq m$. Define $c : V(G_I) \to [m]$ in the following way.

$$c(v_i^j) = \begin{cases} m & \textit{if } i + j - 1 \equiv 0 \pmod m, \\ i + j - 1 \pmod m & \textit{otherwise}. \end{cases} \tag{6}$$

This ensures that each vertex sees all m colors in its neighborhood. Consider V_1', the replacement vertices of $v_i \in V_1$ has the colors $1, 2, \cdots, m$ along with one more color from these m colors. So each vertex in V_1' will have $m-1$ colors among the replacement vertices of v_i itself since the replacement vertices form a clique due to inflation. The only color that is not present in its neighborhood would be its own color. The vertex $v_i^j \in V_1'$ is adjacent to $v_{m+j}^i \in V_2'$. When $i + j - 1 \neq m$, $c(v_i^j) = c(v_{m+j}^i) = i + j - 1 \pmod m$. Similarly the case when $i + j - 1 = m$. Hence the vertex $v_i^j \in V_1'$ will have all m colors in its neighborhood. Similarly consider the vertices $v_i^j \in V_2'$ where $m + 1 \leq i \leq m + n$ and $1 \leq j \leq m$, each of these vertices will have all the colors among its neighboring vertices in V_2' which is nothing but the replacemnt vertices of v_i except its own color and this missing color will be present in the neighboring vertex that belongs to V_1'. This ensures that the neighborhood of every vertex contains all m colors, satisfying the requirement for a valid m-coupon coloring. □

3. α-Operation

Definition 3.1. Let G be a graph of order n with no isolated vertices. Construct a new graph G_α by replacing each vertex $v_i \in V(G)$ of degree k with an independent set of k vertices, labelled $v_i^1, v_i^2, \cdots, v_i^k$. Whenever $v_i v_l \in E(G)$, a complete join is made between the replacement set of v_i and the replacement set of v_l. Order of G_α is $\sum_{i=1}^{n} \deg(v_i)$. Figure 2 illustrates cycle of order 4 and α-operation of cycle.



(a) (b)

Fig. 2. (a) Cycle of order 4 (b) α-operation of cycle

Theorem 3.2. *Let $G = P_n$ be a path on n vertices. Then*

$$\chi_c(G_\alpha) = \begin{cases} 1 & n = 2, \\ 2 & \textit{otherwise}. \end{cases} \tag{7}$$

Proof. Let $G = P_n$ be a path on n vertices with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Applying the alpha operation to G, we obtain a new graph G_α, whose vertex set is $V(G_\alpha) = \{v_1^1, v_2^1, v_2^2, \ldots, v_n^1\}$, where each original vertex v_i of degree k is replaced by k new vertices forming an independent set. In the case of a path, each internal vertex v_i (for $2 \leq i \leq n-1$) has degree 2 and is replaced by two vertices v_i^1 and v_i^2, while the endpoints v_1 and v_n have degree 1 and are replaced by a single vertex v_1^1 and v_n^1, respectively. The resulting graph G_α has minimum degree $\delta(G_\alpha) = 2$, which implies that the coupon coloring number satisfies $\chi_c(G_\alpha) \leq 2$.

Case 1. $n = 2$. When $n = 2$, the path graph P_2 consists of two vertices connected by a single edge. Applying the alpha operation to P_2 does not change its structure, since each vertex has degree 1 and is replaced by a single vertex, preserving the original edge. As a result $(P_2)_\alpha$ is the same as P_2. Therefore, the coupon coloring number remains unchanged, and we have $\chi_c(G_\alpha) = 1$ when $n = 2$.

Case 2. $n \neq 2$. We define a 2-coloring $c : V(G_\alpha) \to [2]$ as follows: assign color 1 to all v_i^1 for $1 \leq i \leq n-1$, assign color 2 to v_n^1, and assign color 2 to all v_i^2 for $2 \leq i \leq n-1$. This coloring ensures that every vertex has neighbors receiving both colors, thereby satisfying the coupon coloring condition. □

Theorem 3.3. *Let $G = C_n$ be a cycle on n vertices. Then*

$$\chi_c(G_\alpha) = \begin{cases} 4 & n \equiv 0 \pmod 4, \\ 3 & \textit{otherwise}. \end{cases} \tag{8}$$

Proof. Let $G = C_n$ be a cycle on n vertices. Since $\delta(G_\alpha) = 4$, it follows that $\chi_c(G_\alpha) \leq 4$.

Case 1. When $n \equiv 0 \pmod 4$. We define a coloring function $c : V(G_\alpha) \to [4]$ as follows.

Case 1.1. If $\lceil \frac{i}{2} \rceil$ is odd, then assign

$$c(v_i^j) = \begin{cases} 1 & j \equiv 1 \pmod 2, \\ 2 & j \equiv 0 \pmod 2. \end{cases} \tag{9}$$

Case 1.2. If $\lceil \frac{i}{2} \rceil$ is even, then assign

$$c(v_i^j) = \begin{cases} 3 & j \equiv 1 \pmod 2, \\ 4 & j \equiv 0 \pmod 2. \end{cases} \tag{10}$$

Case 2. $n \not\equiv 0 \pmod 4$. Consider the vertex v_2^1, whose neighboring vertices are $\{v_1^1, v_1^2, v_3^1, v_3^2\}$. Assume that v_1^1 and v_1^2 receive colors 1 and 2, respectively or vice versa. This implies that v_3^1 and v_3^2 receive colors 3 and 4, as they must differ from the colors of the other neighbors of v_2. That is, if v_3^1 receives color 3, then v_3^2 must receive color 4, and vice versa. Continuing like this we get three cases.

Case 2.1. $n \equiv 1 \pmod 4$.

$$(c(v_i^1), c(v_i^2)) = \begin{cases} (1, 2) \text{ or } (2, 1), & \text{if } \lfloor \dfrac{i}{2} \rfloor \equiv 0 \pmod 2, \\ (3, 4) \text{ or } (4, 3), & \text{if } \lfloor \dfrac{i}{2} \rfloor \equiv 1 \pmod 2. \end{cases} \tag{11}$$

Thus we get a contradiction since v_n^1, v_n^2 will become the bad vertices. Hence, we have $\chi_c(G_\alpha) \leq 3$. Define a coloring function $c : V(G_\alpha) \to [3]$ as follows: assign $c(v_n^1) = 1$ and $c(v_n^2) = 3$. For each $1 \leq i \leq n - 1$, set $c(v_i^2) = 2$.

$$c(v_i^1) = \begin{cases} 1 & \textit{if } \lceil \frac{i}{2} \rceil \equiv 1 \pmod 2, \\ 3 & \textit{if } \lceil \frac{i}{2} \rceil \equiv 0 \pmod 2. \end{cases} \tag{12}$$

Case 2.2. $n \equiv 2 \pmod 4$.

$$(c(v_{2k+1}^1), c(v_{2k+1}^2)) = \begin{cases} (1, 2) \text{ or } (2, 1) & \text{if } k \equiv 0 \pmod 2, \\ (3, 4) \text{ or } (4, 3) & \text{if } k \equiv 1 \pmod 2. \end{cases} \tag{13}$$

From this we get v_n^1, v_n^2 are bad vertices. Thus $\chi_c(G_\alpha) \leq 3$. We define the following coloring $c : V(G_\alpha) \to [3]$ as follows. Set $c(v_{n-1}^1) = c(v_n^1) = 1$ and $c(v_{n-1}^2) = c(v_n^2) = 3$. Now for $1 \leq i \leq n-2$, we have $c(v_i^2) = 2$. Also when $j = 1$ we have

$$c(v_i^1) = \begin{cases} 1 & \textit{if } \lceil \frac{i}{2} \rceil \equiv 1 \pmod 2, \\ 3 & \textit{if } \lceil \frac{i}{2} \rceil \equiv 0 \pmod 2. \end{cases} \tag{14}$$

Case 2.3. $n \equiv 3 \pmod 4$.

$$(c(v_i^1), c(v_i^2)) = \begin{cases} (1,2) \text{ or } (2,1), & \text{if } \lceil \dfrac{i}{2} \rceil \equiv 1 \pmod 2, \\ (3,4) \text{ or } (4,3), & \text{if } \lceil \dfrac{i}{2} \rceil \equiv 0 \pmod 2. \end{cases} \tag{15}$$

This leads to a contradiction, as the final pair v_n^1, v_n^2 ends up with conflicting color requirements, thereby becoming *bad* vertices. Thus $\chi_c(G_\alpha) \leq 3$. We need to only prove that $\chi_c(G_\alpha) \geq 3$. Define $c : V(G_\alpha) \to [3]$ as follows.

$$c(v_i^j) = \begin{cases} 1 & i + j \equiv 2 \pmod 3, \\ 2 & i + j \equiv 0 \pmod 3, \\ 3 & i + j \equiv 1 \pmod 3. \end{cases} \tag{16}$$

The coloring c clearly satisfies the coupon coloring conditions. Thus, when $n \not\equiv 0 \pmod 4$, we conclude that $\chi_c(G_\alpha) = 3$. □

Theorem 3.4. *Let $G = K_n$ be a complete graph on n vertices. Then $\chi_c(G_\alpha) = \frac{n(n-1)}{2}$.*

Proof. Let $G = K_n$ be a complete graph on n vertices with vertex set $v_1, v_2, \cdots, v_n$. Here $\delta(G_\alpha) = (n-1)^2$ and the order of G_α is $n(n-1)$. Since by Lemma 4.4 and $\frac{n(n-1)}{2} < (n-1)^2$ for all $n > 0$ it follows that $\chi_c(G_\alpha) \leq \frac{n(n-1)}{2}$. We can prove $\chi_c(G_\alpha) \geq \frac{n(n-1)}{2}$ by defining a coloring $c : V(G_\alpha) \to [\frac{n(n-1)}{2}]$ as follows. We have two cases.

Case 1. $n \equiv 0 \pmod 2$. For all $1 \leq j \leq n-1$ we have

$$c(v_i^j) = \begin{cases} j + (i-1)(n-1) & 1 \leq i \leq \frac{n}{2}, \\ j + (i - \frac{n}{2} - 1)(n-1) & \frac{n}{2} + 1 \leq i \leq n. \end{cases} \tag{17}$$

Case 2. $n \equiv 1 \pmod 2$. For $i = \lceil \frac{n}{2} \rceil$ set $c(v_i^j) = j + (n-1)(i-1)$ for $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$ and $c(v_i^j) = j + \lfloor \frac{n-1}{2} \rfloor (2i - n - 2)$ for $\lfloor \frac{n-1}{2} \rfloor + 1 \leq j \leq n-1$. For all $1 \leq j \leq n-1$ we have

$$c(v_i^j) = \begin{cases} j + (i-1)(n-1) & 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1, \\ j + \lfloor \frac{n-1}{2} \rfloor (2i - n - 2) & \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases} \tag{18}$$

In Case 1, when $n \equiv 0 \pmod 2$, assign $c(v_1^1) = 1, c(v_1^2) = 2, \cdots, c(v_1^{n-1}) = n-1$. The replacement vertices of v_2 receive the colors $n, \ldots, 2(n-1)$, and, in general, the replacement vertices of v_i, $2 \leq i \leq \frac{n}{2}$, receive the colors $(i-1)(n-1)+1, \cdots, i(n-1)$.

The replacement vertices of $v_{\frac{n}{2}+1}$ are assigned the colors $1, \cdots, n-1$, and the remaining replacement vertices are colored by continuing this pattern cyclically. Thus, each color appears exactly twice. Now let v_i be an arbitrary vertex. If $1 \leq i \leq \frac{n}{2}$, then the replacement vertices of v_j, where $\frac{n}{2}+1 \leq j \leq n$, together contain all $\frac{n(n-1)}{2}$ colors. Similarly, if $\frac{n}{2}+1 \leq i \leq n$, then the replacement vertices of v_j, where $1 \leq j \leq \frac{n}{2}$, together contain all $\frac{n(n-1)}{2}$ colors. Hence, the neighborhood of every arbitrary replacement vertex contains all $\frac{n(n-1)}{2}$ colors. Moreover, each color appears exactly twice among the replacement vertices, completing the required coupon coloring. In Case 2 when $n \equiv 1 \pmod 2$ we have assigned $c(v_1^1) = 1,\ c(v_1^2) = 2,\ \ldots,\ c(v_1^{n-1}) = n-1$. For $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, assign the replacement vertices of v_i the consecutive colors $(i-1)(n-1)+1,\ \ldots,\ i(n-1)$. The replacement vertices of $v_{\lceil \frac{n}{2} \rceil}$ are assigned the colors $\left(\lceil \frac{n}{2} \rceil - 2\right)(n-1)+1,\ \ldots,\ \frac{n(n-1)}{2},\ 1,\ \ldots,\ \frac{n-1}{2}$. For $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$, the color assignment continues cyclically, with the replacement vertices of v_n receiving the colors $\frac{(n-1)(n-2)}{2}+1,\ \ldots,\ \frac{n(n-1)}{2}$. Consequently, the neighborhood of every replacement vertex contains all $\frac{n(n-1)}{2}$ colors. Moreover, each color appears exactly twice among the replacement vertices, completing the required coupon coloring. Thus $\chi_c(G_\alpha) = \frac{n(n-1)}{2}$. □

Figure 3 illustrates coupon coloring of K_{4_α}.

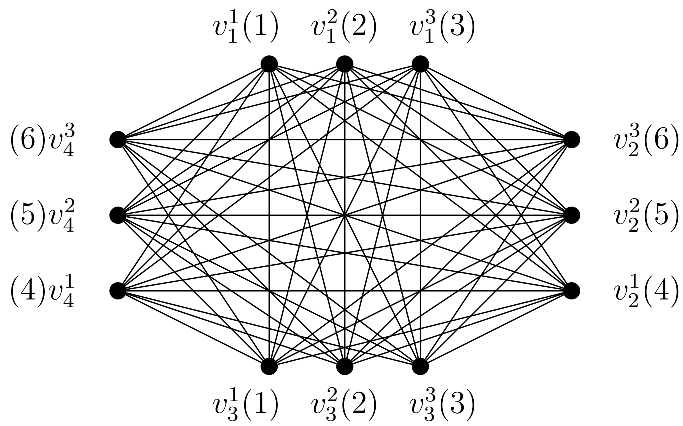


Fig. 3. Coupon coloring of K_{4_α}

Observation 3.5. *Let $G = K_{1,n}$ be a star graph on $n+1$ vertices. Then $\chi_c(G_\alpha) = n$.*

Proof. Let $G = K_{1,n}$ be a star graph on $n+1$ vertices with vertex set $\{v_1, v_2, \cdots, v_{n+1}\}$, where v_1 is the universal vertex. $(K_{1,n})_\alpha$ is the graph $K_{n,n}$. Thus $\chi_c(K_{1,n})_\alpha = n$. □

Theorem 3.6. *Let $G = S_{r,t}$ be a double star graph on $r+t+2$ vertices. Then $\chi_c(G_\alpha) = min\{r, t\} + 1$.*

Proof. Let $G = S_{r,t}$ denote a double star graph with $r+t+2$ vertices and vertex set $V(G) = \{x, w, u_1, u_2, \cdots, u_r, v_1, v_2, \ldots, v_t\}$. The vertices x and w are adjacent. The set $\{u_1, u_2, \ldots, u_r\}$ consists of leaves adjacent to x, forming a star centered at x, while

$\{v_1, v_2, \ldots, v_t\}$ consists of leaves adjacent to w, forming a star centered at w. Under the α-operation, every vertex u_i of degree 1 is replaced by a single vertex denoted u_i^1, and likewise for each vertex v_i. The vertices x and w, having degrees $r+1$ and $t+1$, respectively, are replaced by independent vertices labeled $x^1, x^2, \ldots, x^{r+1}$ and $w^1, w^2, \ldots, w^{t+1}$, respectively. Without loss of generality assume that $r \leq t$. Since $\delta(G_\alpha) = min\{r, t\} + 1$, we have $\chi_c(G_\alpha) \leq r+1$. We define $c : V(G_\alpha) \to [r+1]$ as follows. Set $c(w^i) = c(x^i) = i$ for all $1 \leq i \leq r+1$. Remaining vertices can be colored using any color from $1, 2, \cdots, r$. Consider the replacement vertices of leaves in G. In the case of u_i^1 where $1 \leq i \leq r$, these vertices are adjacent to x^h, where $1 \leq h \leq r+1$ which contains all the $r+1$ colors. Similarly in the case of v_j^1, where $1 \leq j \leq t$, these vertices are adjacent to w^g, where $1 \leq g \leq t+1$. The vertices x^h, where $1 \leq h \leq r+1$ are adjacent to w^g, where $1 \leq g \leq t+1$. Hence these vertices will also have $r+1$ colors in its neighborhood. □

Observation 3.7. [15] *Let $G = (X, Y)$ be a complete bipartite graph with $|X| = n_1$, $|Y| = n_2$ and $n_1 \leq n_2$. By the definition of the coupon coloring number, we can easily obtain that $\chi_c(G) = n_1$.*

Theorem 3.8. *Let $G = K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. Then $\chi_c(G_\alpha) = mn$.*

Proof. Let $G = K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. Applying the alpha operation to the complete bipartite graph $K_{m,n}$ results in the graph $K_{mn,mn}$. From Observation 3.7 we get $\chi_c(G_\alpha) = mn$. □

4. β-Operation

Definition 4.1. Let G be a graph of order n with no isolated vertices. Construct a new graph G_β by replacing each vertex $v_i \in V(G)$ of degree k with a Path P_{v_i} on k vertices, labeled $v_i^1, v_i^2, \ldots, v_i^k$. Whenever $v_iv_j \in E(G)$, complete join is made between the vertices of P_{v_i} and P_{v_j}. Order of G_β is $\sum_{i=1}^{n} \deg(v_i)$. Figure 4 illustrates cycle of order 4 and β-operation of cycle.

Theorem 4.2. *Let $G = P_n$ be a path on n vertices. Then*

$$\chi_c(G_\beta) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{otherwise}. \end{cases} \tag{19}$$

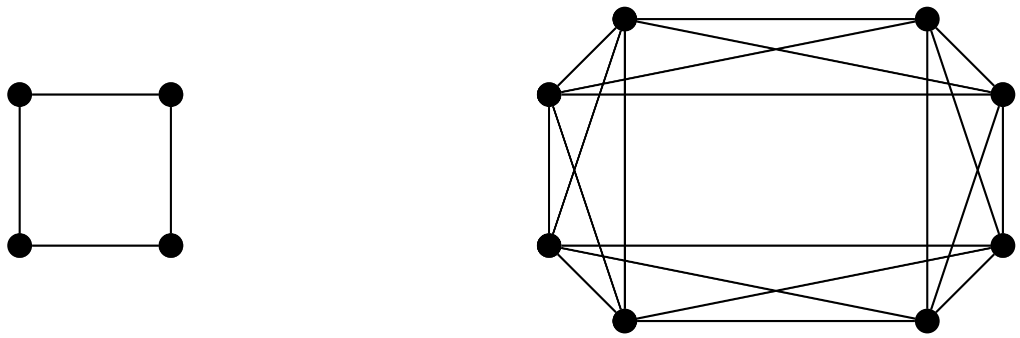
Proof. Let $G = P_n$ be a path on n vertices.

Case 1. $n = 2$. When $n = 2$, $P_{2_\beta} = P_2$.

Case 2. $n \neq 2$. We have $\chi_c(G_\beta) \leq 2$. Define the following coloring $c : V(G_\beta) \to [2]$ as follows. Set $c(v_1^1) = 1$ and $c(v_n^1) = 2$. For $2 \le i \le n-1$ and $j = 1, 2$, we have

$$c(v_i^j) = \begin{cases} 1 & \textit{if } j = 1, \\ 2 & \textit{otherwise}, \end{cases} \tag{20}$$

c can be easily verified to be a coupon coloring. □



(a) (b)

Fig. 4. (a) Cycle of order 4 (b) β-operation of cycle

Theorem 4.3. *Let $G = C_n$ be a cycle on n vertices. Then*

$$\chi_c(G_\beta) = \begin{cases} 4 & n \equiv 0 \pmod 4, \\ 3 & \textit{otherwise}. \end{cases} \tag{21}$$

Proof. Let $G = C_n$ be a cycle on n vertices with vertex set $v_1, v_2, \cdots, v_n$. Let $v_1^1, v_1^2, v_2^1, v_2^2, \cdots, v_n^1, v_n^2$ be the vertices of G_β after applying β operation. We have $\chi_c(G_\beta) \le 5$. Suppose there exists a 5-coupon coloring $c : V(G_\beta) \to [5]$. Consider the vertex v_1^1, the neighboring vertices are $v_1^2, v_2^1, v_2^2, v_n^1, v_n^2$. Without loss of generality, assume that $c(v_1^2) = 1, c(v_2^1) = 2, c(v_2^2) = 3, c(v_n^1) = 4, c(v_n^2) = 5$. Now consider v_1^2, we get $c(v_1^1) = 1$ which makes v_2^1 the bad vertex. Thus $\chi_c(G_\beta) \le 4$.

Case 1. $n \equiv 0 \pmod 4$. Define $c : V(G_\beta) \to [4]$ as follows.

$$c(v_i^j) = \begin{cases} 1 & \lfloor \frac{i}{2} \rfloor \equiv 0 \pmod 2 \text{ and } i + j \equiv 1 \pmod 2, \\ 4 & \lfloor \frac{i}{2} \rfloor \equiv 0 \pmod 2 \text{ and } i + j \equiv 0 \pmod 2, \\ 2 & \lfloor \frac{i}{2} \rfloor \equiv 1 \pmod 2 \text{ and } i + j \equiv 0 \pmod 2, \\ 3 & \lfloor \frac{i}{2} \rfloor \equiv 1 \pmod 2 \text{ and } i + j \equiv 1 \pmod 2. \end{cases} \tag{22}$$

It can be readily verified that c is a coupon coloring.

Case 2. $n \not\equiv 0 \pmod 4$. Consider the vertex v_1^1, the neighboring vertices are $v_1^2, v_2^1, v_2^2, v_n^1, v_n^2$. Without loss of generality, assume that $c(v_1^2) = 1, c(v_2^1) = 2, c(v_2^2) = 3, c(v_n^1) = 4$. Consider the vertex v_1^2 we have two cases. Either $c(v_1^1) = 1$ or $c(v_n^2) = 1$.

Case 2.1. $c(v_1^1) = 1$. Examining the vertex v_2^1 forces $c(v_3^1)$ to be either 2 or 4. Assume that $c(v_3^1) = 2$. This implies that $c(v_3^2) = 4$, which leads to a contradiction, since the neighborhood of v_2^2 does not contain the color 3. Similarly when $c(v_3^1) = 4$ implies $c(v_3^2) = 2$ and this also leads to contradiction since v_2^2 become the bad vertex.

Case 2.2. $c(v_n^2) = 1$. Consider the vertices v_2^2 and v_2^1. Their neighborhoods must contain the colors $2, 3$, and 4. We get three cases.

Case 2.2.1. $c(v_1^1) = 2$. Then we get either $c(v_3^1) = 4$ or $c(v_3^2) = 4$. if $c(v_3^1) = 4$, this implies $c(v_3^2) = 3$ and vice versa. In both these cases when we consider the vertices v_3^1 and v_3^2 we get $c(v_4^1), c(v_4^2) \in \{1, 4\}$. If $c(v_4^1) = 1$, then $c(v_4^2) = 4$ or viceversa. Hence for $1 \leq i \leq n-1$ we get

$$(c(v_i^1), c(v_i^2)) = \begin{cases} (1,4) \ \mathit{or} \ (4,1) \ \mathit{if} \ i \equiv 0 \pmod 4, \\ (1,2) \ \mathit{or} \ (2,1) \ \mathit{if} \ i \equiv 1 \pmod 4, \\ (2,3) \ \mathit{or} \ (3,2) \ \mathit{if} \ i \equiv 2 \pmod 4, \\ (3,4) \ \mathit{or} \ (4,3) \ \mathit{if} \ i \equiv 3 \pmod 4. \end{cases} \tag{23}$$

Now $n \equiv 1 \pmod 4$ then v_n^1 and v_n^2 doesnot contain the color 3 in ite neighborhood, hence we get a contradiction. When $n \equiv 2 \pmod 4$ v_n^1 and v_n^2 doesnot contain the color 3 and 4 in ite neighborhood and when $n \equiv 3 \pmod 4$, v_n^1 and v_n^2 doesnot contain the color 4 in ite neighborhood, hence this also leads to contradiction.

Case 2.2.2. $c(v_1^1) = 3$. Then we get either $c(v_3^1) = 4$ or $c(v_3^2) = 4$. If $c(v_3^1) = 4$, this implies $c(v_3^2) = 2$ and vice versa. In both these cases when we consider the vertices v_3^1 and v_3^2 we get $c(v_4^1), c(v_4^2) \in \{1, 4\}$. If $c(v_4^1) = 1$, then $c(v_4^2) = 4$ or viceversa. Hence for $1 \leq i \leq n-1$ we get

$$(c(v_i^1), c(v_i^2)) = \begin{cases} (1,4) \ \mathit{or} \ (4,1) \ \mathit{if} \ i \equiv 0 \pmod 4, \\ (1,3) \ \mathit{or} \ (3,1) \ \mathit{if} \ i \equiv 1 \pmod 4, \\ (2,3) \ \mathit{or} \ (3,2) \ \mathit{if} \ i \equiv 2 \pmod 4, \\ (2,4) \ \mathit{or} \ (4,2) \ \mathit{if} \ i \equiv 3 \pmod 4. \end{cases} \tag{24}$$

Now $n \equiv 1 \pmod 4$ then v_n^1 and v_n^2 doesnot contain the color 2 in ite neighborhood, hence we get a contradiction. When $n \equiv 2 \pmod 4$ v_n^1 and v_n^2 doesnot contain the color 2 and 4 in ite neighborhood and when $n \equiv 3 \pmod 4$, v_n^1 and v_n^2 doesnot contain the color 4 in ite neighborhood, hence this also leads to contradiction.

Case 2.2.3. $c(v_1^1) = 4$. Then we get either $c(v_3^1) = 2$ or $c(v_3^2) = 2$ if $c(v_3^1) = 2$, this implies $c(v_3^2) = 4$ and vice versa. In both these cases when we consider the vertices v_3^1 and v_3^2 we get $c(v_4^1), c(v_4^2) \in \{1, 4\}$. If $c(v_4^1) = 1$, then $c(v_4^2) = 4$ or viceversa. Hence for $1 \leq i \leq n-1$ we get

$$(c(v_i^1), c(v_i^2)) = \begin{cases} (1,4) \ \mathit{or} \ (4,1) \ \mathit{if} \ \lfloor \frac{i}{2} \rfloor \equiv 0 \pmod 2, \\ (2,3) \ \mathit{or} \ (3,2) \ \mathit{if} \ \lfloor \frac{i}{2} \rfloor \equiv 1 \pmod 2. \end{cases} \tag{25}$$

Now $n \equiv 1, 2 \pmod 4$ then v_n^1 and v_n^2 doesnot contain the colors 2 and 3 in its neighborhood, hence we get a contradiction and when $n \equiv 3 \pmod 4$, v_{n-1}^1 and v_{n-1}^2 doesnot contain the colors 2 and 3 in its neighborhood, hence this also leads to contradiction. Thus $\chi_c(G_\beta) \le 3$. Define $c : V(G_\beta) \to [3]$ as follows.

Case 1. $n \equiv 1 \pmod 4$. Assign $c(v_n^1) = 1$ and $c(v_n^2) = 3$. For each $1 \le i \le n-1$, set $c(v_i^2) = 2$.

$$c(v_i^1) = \begin{cases} 1 & \text{if } \lceil \frac{i}{2} \rceil \equiv 1 \pmod 2, \\ 3 & \text{if } \lceil \frac{i}{2} \rceil \equiv 0 \pmod 2. \end{cases} \tag{26}$$

Case 2. $n \equiv 2, 3 \pmod 4$.

$$c(v_i^j) = \begin{cases} 1 & i+j \equiv 2 \pmod 3, \\ 2 & i+j \equiv 0 \pmod 3, \\ 3 & i+j \equiv 1 \pmod 3. \end{cases} \tag{27}$$

In both cases if we consider an arbitary vertex v_i^j, where $2 \le i \le n-1$ and $j = 1, 2$, the vertices v_{i-1}^j and v_{i+1}^j where $j = 1, 2$ will contain all the required three colors. Hence all three colors will be present in the neighborhood of v_i^j. Similarly for v_1^j and v_n^j, where $j = 1, 2$. Thus $\chi_c(G_\beta) = 3$ when $n \not\equiv 0 \pmod 4$. □

Lemma 4.4. *In coupon coloring each color should appear atleast twice.*

Proof. Let G be a graph and for G to satisfy coupon coloring, each vertex should have all the colors in its neighboring vertices. If a color appears on exactly one vertex u, then u itself cannot see that color in $\mathcal{N}(u)$, so the coloring cannot be a coupon coloring. Similarly for all vertices. Thus each color should appear atleast twice to satisfy the coupon coloring condition. □

Theorem 4.5. *Let $G = K_n$ be a complete graph on n vertices. Then $\chi_c(G_\beta) = \frac{n(n-1)}{2}$.*

Proof. Let $G = K_n$ be a complete graph on n vertices with vertex set $v_1, v_2, \cdots, v_n$. Here $\delta(G_\beta) = (n-1)^2 + 1$ and the order of G_β is $n(n-1)$. Since by Lemma 4.4 and $\frac{n(n-1)}{2} < (n-1)^2 + 1$ for all $n > 0$ it follows that $\chi_c(G_\beta) \le \frac{n(n-1)}{2}$. We can prove $\chi_c(G_\beta) \ge \frac{n(n-1)}{2}$ by defining a coloring $c : V(G_\beta) \to [\frac{n(n-1)}{2}]$ as follows. We have two cases.

Case 1. $n \equiv 0 \pmod 2$. For all $1 \le j \le n-1$ we have

$$c(v_i^j) = \begin{cases} j + (i-1)(n-1) & 1 \le i \le \frac{n}{2}, \\ j + (i - \frac{n}{2} - 1)(n-1) & \frac{n}{2} + 1 \le i \le n. \end{cases} \tag{28}$$

Case 2. $n \equiv 1 \pmod 2$. For $i = \lceil \frac{n}{2} \rceil$ set $c(v_i^j) = j + (n-1)(i-1)$ for $1 \le j \le \lfloor \frac{n-1}{2} \rfloor$ and $c(v_i^j) = j + \lfloor \frac{n-1}{2} \rfloor (2i - n - 2)$ for $\lfloor \frac{n-1}{2} \rfloor + 1 \le j \le n-1$. For all $1 \le j \le n-1$ we have

$$c(v_i^j) = \begin{cases} j + (i-1)(n-1) & 1 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ j + \lfloor \frac{n-1}{2} \rfloor (2i - n - 2) & \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases} \tag{29}$$

In Case 1, when $n \equiv 0 \pmod 2$, assign $c(v_1^1) = 1,\ c(v_1^2) = 2,\ \ldots,\ c(v_1^{n-1}) = n-1$. The replacement vertices of v_2 receive the colors $n, \ldots, 2(n-1)$, and, in general, the replacement vertices of v_i, $2 \le i \le \frac{n}{2}$, receive the colors $(i-1)(n-1)+1,\ \ldots,\ i(n-1)$. The replacement vertices of $v_{\frac{n}{2}+1}$ are assigned the colors $1, \ldots, n-1$, and the remaining replacement vertices are colored by continuing this pattern cyclically. Thus, each color appears exactly twice. Now let v_i be an arbitrary vertex. If $1 \le i \le \frac{n}{2}$, then the replacement vertices of v_j, where $\frac{n}{2}+1 \le j \le n$, together contain all $\frac{n(n-1)}{2}$ colors. Similarly, if $\frac{n}{2}+1 \le i \le n$, then the replacement vertices of v_j, where $1 \le j \le \frac{n}{2}$, together contain all $\frac{n(n-1)}{2}$ colors. Hence, the neighborhood of every arbitrary replacement vertex contains all $\frac{n(n-1)}{2}$ colors. Moreover, each color appears exactly twice among the replacement vertices, completing the required coupon coloring. In Case 2 when $n \equiv 1 \pmod 2$ we have assigned $c(v_1^1) = 1,\ c(v_1^2) = 2,\ \ldots,\ c(v_1^{n-1}) = n-1$. For $2 \le i \le \lceil \frac{n}{2} \rceil - 1$, assign the replacement vertices of v_i the consecutive colors $(i-1)(n-1)+1,\ \ldots,\ i(n-1)$. The replacement vertices of $v_{\lceil \frac{n}{2} \rceil}$ are assigned the colors $\left(\lceil \frac{n}{2} \rceil - 2\right)(n-1)+1,\ \ldots,\ \frac{n(n-1)}{2},\ 1,\ \ldots,\ \frac{n-1}{2}$. For $\lceil \frac{n}{2} \rceil + 1 \le i \le n$, the color assignment continues cyclically, with the replacement vertices of v_n receiving the colors $\frac{(n-1)(n-2)}{2}+1,\ \ldots,\ \frac{n(n-1)}{2}$. Consequently, the neighborhood of every replacement vertex contains all $\frac{n(n-1)}{2}$ colors. Moreover, each color appears exactly twice among the replacement vertices, completing the required coupon coloring. Thus $\chi_c(G_\beta) = \frac{n(n-1)}{2}$. □

Theorem 4.6. *Let $G = K_{1,n}$ be a star graph on $n+1$ vertices. Then $\chi_c(G_\beta) = n$.*

Proof. Let $G = K_{1,n}$ be a star graph on $n+1$ vertices, where v_1 is the universal vertex and $v_2, v_3, \ldots, v_{n+1}$ are the pendant vertices. We have $\chi_c(G_\beta) \le n$, since $\delta(G_\beta) = n$. Define a coloring $c : V(G_\beta) \to [n]$ as follows: $c(v_1^j) = j$ where $1 \le j \le n$, and $c(v_i^1) = i-1$ where $2 \le i \le n+1$. Then $\chi_c(G_\beta) = n$. □

Theorem 4.7. *Let $G = S_{r,t}$ be a double star graph on $r+t+2$ vertices. Then $\chi_c(G_\beta) = min\{r,t\}+1$.*

Proof. Let $G = S_{r,t}$ be a double star graph with $r+t+2$ vertices and vertex set $V(G) = \{x, w, u_1, u_2, \cdots, u_r, v_1, v_2, \ldots, v_t\}$. The vertices x and w are adjacent. The set $\{u_1, u_2, \ldots, u_r\}$ consists of leaves adjacent to x, forming a star centered at x, while $\{v_1, v_2, \ldots, v_t\}$ consists of leaves adjacent to w, forming a star centered at w. Under the β-operation, every vertex u_i of degree 1 is replaced by a single vertex denoted u_i^1, and likewise for each vertex v_i. The vertices x and w, having degrees $r+1$ and $t+1$, respectively, are replaced by paths on $r+1$ and $t+1$ vertices labeled $x^1, x^2, \ldots, x^{r+1}$ and $w^1, w^2, \ldots, w^{t+1}$, respectively. Without loss of generality, assume that $r \le t$. Since $\delta(G_\beta) = min\{r,t\}+1$, we have $\chi_c(G_\beta) \le r+1$. We define $c : V(G_\beta) \to [r+1]$ as follows. Similar to Theorem 3.6 we set $c(w^i) = c(x^i) = i$ for all $1 \le i \le r+1$. Remaining vertices can be colored using any color from $1, 2, \cdots, r$. In the case of u_i^1 where $1 \le i \le r$, these vertices are adjacent to x^h, where $1 \le h \le r+1$ which contains all the $r+1$

colors. Similarly in the case of v_j^1, where $1 \leq j \leq t$, these vertices are adjacent to w^g, where $1 \leq g \leq t+1$. The vertices x^h, where $1 \leq h \leq r+1$ are adjacent to w^g, where $1 \leq g \leq t+1$. Hence these vertices will also have $r+1$ colors in its neighborhood. □

Theorem 4.8. *Let $G = K_{m,n}$ be a complete bipartite graph on $m+n$ vertices. Then $\chi_c(G_\beta) = mn$.*

Proof. Let $G = K_{m,n}$ be a complete bipartite graph on $m+n$ vertices with vertex set $u_1, u_2, \cdots, u_m$ on one partition and $v_1, v_2, \cdots, v_n$ on another partition. We have $\chi_c(G_\beta) \leq mn+1$. Assume there exists an $mn+1$-coupon coloring, $c : V(G_\beta) \to [mn+1]$. Now consider the vertex u_1^1, the neighboring vertices are u_1^2 and v_i^k for all $1 \leq i \leq m$ and $1 \leq k \leq n$. Without loss of generality, assume that $c(v_1^1) = 1, c(v_1^2) = 2, c(v_1^3) = 3, \cdots, c(v_1^m) = m, c(v_2^1) = m+1 \cdots, c(v_n^m) = mn$. We get $c(v_i^j) = mi+j-m$. Hence $c(u_1^2) = mn+1$. Now consider the vertex v_1^1, the neighboring vertices are v_1^2, which has color 2 and u_i^k for all $1 \leq i \leq m$ in which u_1^2 has color $mn+1$. Without loss of generality, assume that $c(u_1^1) = 1, c(u_1^3) = 3, \cdots, c(u_1^n) = n, c(u_2^1) = n+1, \cdots, c(u_m^n) = mn$. We get $c(u_i^j) = ni+j-n$ for all $i \neq 1$ and $j \neq 2$. Two of the neighboring vertices of v_1^2 have same color 1 and remaining there are $mn-1$ neighboring vertices. We need to give mn colors to them and this is a contradiction since the vertex v_1^2 will become a bad vertex. Consequently, $\chi_c(G_\beta) \leq mn$. Define a coloring $c : V(G_\beta) \to [mn]$ by assigning $c(u_i^j) = ni+j-n$ and $c(v_i^j) = mi+j-m$. Since this assignment yields a valid coupon coloring with mn colors, we conclude that $\chi_c(G_\beta) = mn$. □

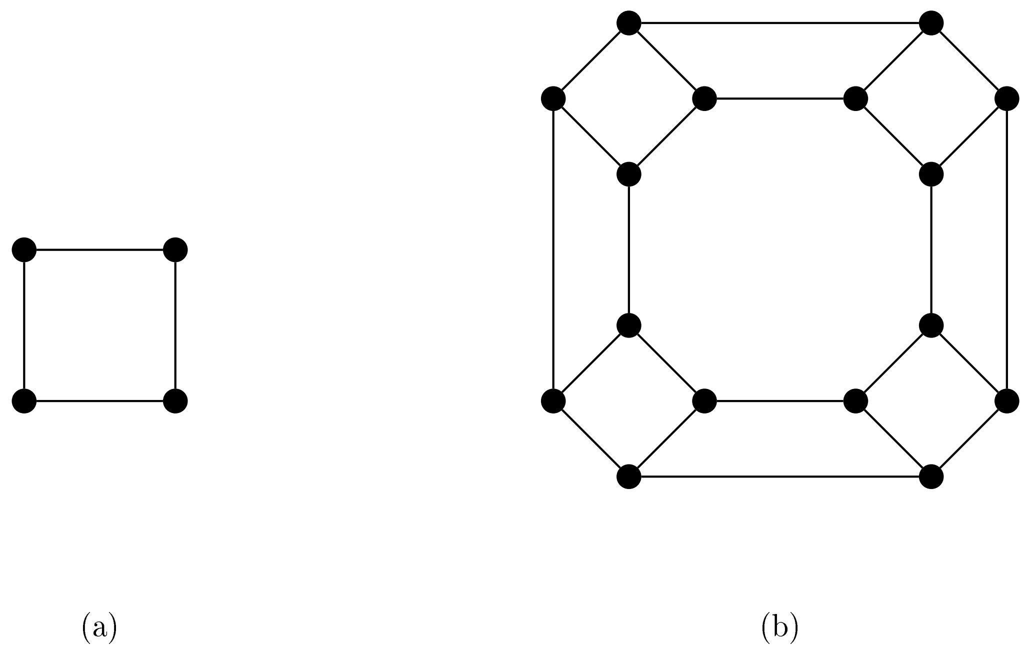


Fig. 5. (a) Cycle of order 4 (b) Cubic inflation of cycle

5. Cubic Inflation

For the notation for cubic inflation, we follow [2]. Let G be a graph without vertices of degree one. First, we replace each vertex $v \in V(G)$ by a cycle C_v of length $2deg(v)$, and then replace every edge uv of G by two edges joining C_u and C_v. The resulting graph is called the cubic inflation of G. Order of $CI(G)$ is $\sum\limits_{v \in V(G)} 2\deg(v)$. Figure 5 illustrates cycle of order 4 and cubic inflation of cycle.

Theorem 5.1. *Let $G = C_n$ be a cycle graph on n vertices. Then*

$$\chi_c(CI(G)) = \begin{cases} 3 & n \equiv 0 \pmod 3, \\ 2 & \textit{otherwise}. \end{cases} \tag{30}$$

Proof. Let $G = C_n$ be a cycle graph on n vertices where $V(G) = \{v_1, v_2, \cdots, v_n\}$. After applying cubic inflation, each vertex v_i is replaced by the cycle C_4 where the vertices are labeled as $v_i^1, v_i^2, v_i^3, v_i^4$. Also the edges are defined as follows: for each $1 \leq i \leq n-1$, the vertex v_i^1 is adjacent to v_{i+1}^4, and v_i^2 is adjacent to v_{i+1}^3. In addition, v_n^1 is adjacent to v_1^4, and v_n^2 is adjacent to v_1^3. We know that $\chi_c(CI(G)) \leq 3$.

Case 1. $n \equiv 0 \pmod 3$. Let $c : V(CI(G)) \to [3]$ be a coloring defined as follows.

$$c(v_i^j) = \begin{cases} 1 & i - j \equiv 1 \pmod 3, \\ 2 & i - j \equiv 2 \pmod 3, \\ 3 & i - j \equiv 0 \pmod 3. \end{cases} \tag{31}$$

This coloring ensures that all three colors appear in the neighborhood of each vertex, thereby satisfying the coupon coloring condition.

Case 2. $n \not\equiv 0 \pmod 3$. Suppose that a 3-coupon coloring $c : V(CI(G)) \to [3]$ exists. Consider the vertex v_1^4, whose neighbors are v_1^3, v_1^1, and v_n^1. Since c is a 3-coupon coloring, $c(v_1^3) \in \{1, 2, 3\}$. Without loss of generality, assume that $c(v_1^1) = 1$ and $c(v_1^3) = 2$. Then it follows that $c(v_n^1) = 3$, which in turn implies that $c(v_n^3) = 1$. Continuing like this we have 2 cases.

Case 2.1. $n \equiv 1 \pmod 3$.

$$c(v_i^1) = \begin{cases} 1 & i \equiv 2 \pmod 3, \\ 2 & i \equiv 0 \pmod 3, \\ 3 & i \equiv 1 \pmod 3, \end{cases} \tag{32}$$

$$c(v_i^3) = \begin{cases} 1 & i \equiv 1 \pmod 3, \\ 2 & i \equiv 2 \pmod 3, \\ 3 & i \equiv 0 \pmod 3. \end{cases} \tag{33}$$

The vertex v_2^4 becomes a bad vertex because its neighborhood does not contain all three colors.

Case 2.2. $n \equiv 2 \pmod 3$.

$$c(v_i^1) = \begin{cases} 1 & i \equiv 0 \pmod 3, \\ 2 & i \equiv 1 \pmod 3, \\ 3 & i \equiv 2 \pmod 3, \end{cases} \tag{34}$$

$$c(v_i^3) = \begin{cases} 1 & i \equiv 2 \pmod 3, \\ 2 & i \equiv 0 \pmod 3, \\ 3 & i \equiv 1 \pmod 3. \end{cases} \tag{35}$$

The vertex v_1^2 becomes a bad vertex because its neighborhood does not contain all three colors. Thus when $n \not\equiv 0 \pmod 3$, $\chi_c(CI(G)) \leq 2$. Define a coloring $c : V(CI(G)) \to [2]$ as follows

$$c(v_i^j) = \begin{cases} 1 & j = 1, 2, \\ 2 & j = 3, 4. \end{cases} \tag{36}$$

Clearly, c is a coupon coloring. □

Theorem 5.2. *Let $G = W_n$ be a wheel graph on n vertices. Then $\chi_c(CI(G)) = 2$.*

Proof. Let $G = W_n$ be a wheel graph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$, where v_1 serves as the universal vertex. After applying cubic inflation, each vertex $v_i, i = 2, 3, \cdots, n$ is replaced by the cycle C_6 with vertices labeled as $v_i^1, v_i^2, \cdots, v_i^6$ and the vertex v_1 is replaced by cycle $C_{2(n-1)}$ with vertices labeled as $v_1^1, v_1^2, \cdots, v_1^{2(n-1)}$. In $CI(G)$ the vertices v_1^{2i-1} is adjacent to v_{i+1}^1 and v_1^{2i} is adjacent to v_{i+1}^2 where $i = 1, 2, \cdots, n-1$. Also, for every $i = 2, 3, \ldots, n-1$, the vertex v_i^3 is adjacent to v_{i+1}^6, while v_i^4 is adjacent to v_{i+1}^5. Also the vertices v_n^3 and v_n^4 are adjacent to v_2^6 and v_2^5 respectively. Since $\delta(CI(G)) = 3$ we have $\chi_c(CI(G)) \leq 3$. Assume there exists a 3−coupon coloring $c : V(CI(G)) \to [3]$. Consider the vertex v_2^5 which is adjacent to v_2^4, v_2^6 and v_n^4. Suppose $c(v_2^4) = a$, $a \in \{1, 2, 3\}$. Without loss of generality, assume that $c(v_2^4) = 1$, $c(v_2^6) = 2$ and $c(v_n^4) = 3$. Now consider the vertex v_n^3, we get $c(v_n^2) = 1$. The neighboring vertices of the vertex v_1^{2n-2} are v_n^2, v_1^1 and v_1^{2n-1}. The vertex v_1^1 will receive either color 2 or 3. If v_1^1 is assigned the color 2, then v_2^1 becomes bad vertex since it will have two neighbors with the same color 2 and this will result in lack of three required colors in the neighborhood of v_2^1. Hence, we must have $c(v_1^1) = 3$ and $c(v_1^{2n-1}) = 2$. Finally considering the vertex v_n^1, we obtain $c(v_n^6) = 3$ which causes v_n^5 to become a bad vertex. Thus, $\chi_c(CI(G)) \leq 2$. It now suffices to show that $\chi_c(CI(G)) \geq 2$ by defining a 2-coupon coloring of the graph $CI(G)$. We define $c : V(CI(G)) \to [2]$ as follows. Set $c(v_1^i) = 2$ for all i and for each $j = 2, 3, \cdots, n$ $c(v_j^1) = c(v_j^2) = 1$, $c(v_j^3) = c(v_j^6) = 2$ and $c(v_j^5) = c(v_j^4) = 1$. Consider the vertices on larger cycle replacing the universal vertex that is v_1^i, $1 \leq i \leq 2(n-1)$, these vertices has the color 2 among themselves. So each vertex will have color 2 in its neighborhood. Also v_1^{2i-1} is adjacent to v_{i+1}^1 and v_1^{2i} is adjacent to v_{i+1}^2 and by the coloring pattern $c(v_j^1) = c(v_j^2) = 1$ for any j and hence any arbitrary $v_1^i, 1 \leq i \leq 2(n-1)$ is adjacent to vertices which has color 1. Now for $2 \leq l \leq 5$, consider the vertices $v_j^l, 2 \leq j \leq n$ and these vertices will receive the required two colors from their adjacent vertices in the same cycle. For v_j^1 and v_j^6 these vertices are connected to vertices having color 2 and they receive the color 1 from the adjacent vertices in the same cycle. It can be easily seen that c is a coupon coloring. Thus $\chi_c(CI(G)) = 2$. □

6. Conclusion and future work

In this paper, we explored coupon coloring of inflated graphs and introduced new graph operations derived from inflation and analyzed their impact on coupon coloring. This can also be extended to other classes of graphs to study how these operations behave under varying structural properties.

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