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Edge k-product cordial labeling of complete bipartite graph $K_{2,2n}$

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ABSTRACT

Recently, it was proved that if $K_{2,2n}$ admits an edge k-product cordial labeling, then either $k = 2n + 1$ or $k \geq 4n + 1$, and two related existence problems were posed. In this paper, we prove that $K_{2,2n}$ is edge $(2n + 1)$-product cordial if and only if $2n + 1$ is a prime. We further show that $K_{2,4}$ is edge k-product cordial if and only if $k = 5$ or $k \geq 9$, while $K_{2,6}$ is edge k-product cordial if and only if $k = 7$ or $k \geq 13$. These results completely answer one of the posed questions and provide further evidence for a general existence conjecture.

Keywords: edge k-product cordial labeling, shadow graph, star

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1. Introduction and general properties

A graph of order p and size q is called a (p, q)-graph. For $k \geq 2$, we let $\mathbb{Z}_k = \{0, 1, \dots, k-1\}$ be the standard complete residue system modulo k. Notation and concepts not defined in this paper are referred to the book [1].

Suppose $G = (V, E)$ is a (p, q)-graph without isolated vertex. Let $f : E \to \mathbb{Z}_k$ be an edge labeling of G. Let the vertex labeling $f^* : V \to \mathbb{Z}_k$ be defined by $f^*(v) \equiv \prod_{uv \in E} f(uv) \pmod{k}$, for every vertex v. An edge $e \in E$ and a vertex $v \in V$ are called an *i-edge* and *j-vertex* (under f), if $f(e) = i$ and $f^*(v) = j$, respectively, where $i, j \in \mathbb{Z}_k$. Let $e_f(i)$ and $v_f(j)$ denote the number of i-edges and the number of j-vertices, respectively.

For $k \geq 2$, an edge labeling $f : E \to \mathbb{Z}_k$ of $G = (V, E)$ is said to be an *edge k-product cordial labeling of G* if $|e_f(i_1) - e_f(i_2)| \leq 1$ and $|v_f(j_1) - v_f(j_2)| \leq 1$ for $i_1, i_2, j_1, j_2 \in \mathbb{Z}_k$. G is *edge k-product cordial* if G admits an edge k-product cordial labeling.

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We first introduce some necessary notation and concepts. Suppose $f : E \to \mathbb{Z}_k$ is an edge labeling of $G = (V, E)$. In the following, we always let E_0 be the set of all 0-edges under f and let G_0 be the subgraph of G induced by the edges in E_0, i.e., $G_0 = G[E_0]$. Clearly $e_f(0) = |E_0|$ and $v_f(0) \geq |V(G_0)|$. We shall keep these notation throughout the paper.

Let G be a graph and G' be a copy of G. The *shadow graph* of a graph G, denoted as $D(G)$, is a graph obtained by joining each vertex v of G to the neighbors of the vertex v' in G' which corresponds to v. This concept was first introduced by Sampathkumar [2] in 1973. In this paper, we consider the shadow graph of a star $K_{1,n}$ for $n \geq 1$.

Let the vertex and the edge sets of the star $K_{1,n}$ be $\{u, u_1, \dots, u_n\}$ and $\{uu_i \mid 1 \leq i \leq n\}$, respectively. Then we let $\{u, u'\} \cup \{u_i, u_i' \mid 1 \leq i \leq n\}$ and $\{uu_i, uu_i', u'u_i, u'u_i' \mid 1 \leq i \leq n\}$ be the vertex and edge sets of the graph $D(K_{1,n})$, respectively. Thus, $D(K_{1,n})$ is a $(2n + 2, 4n)$-graph. Note that $D(K_{1,n}) \cong K_{2,2n}$. So in the coming sections, we shall use $K_{2,2n}$ instead of $D(K_{1,n})$.

In [3], there are two questions related to the theorem below.

Theorem 1.1. [3, Theorem 2.1] *Let $n \geq 1$ and $k \geq 2$. Suppose $D(K_{1,n})$ admits an edge k-product cordial labeling f, then $k \geq 4n + 1$ or $k = 2n + 1$.*

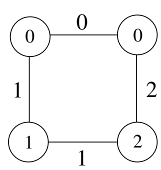
Question 1.2. [3, Question 2.5] *Is $D(K_{1,n})$ edge $(2n + 1)$-product cordial for $n \geq 3$?*

Question 1.3. [3, Question 2.6] *Is $D(K_{1,n})$ edge k-product cordial for $n \geq 2$ when $k \geq 4n + 1$?*

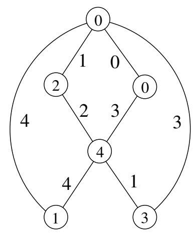
In this paper, we will answer the first question completely and investigate the second question for $n = 2, 3$.

2. Edge $(2n + 1)$-product cordiality of $K_{2,2n}$

In [3], there is an edge $(2n + 1)$-product cordial labeling for each of $K_{2,2n}$ when $n = 1, 2$. For completeness, these labelings are reproduced in Figures 1(a) and 1(b). All edge labels are indicated next to the respective edges and the induced vertex labels are indicated on the vertices, respectively. It is straight forward to verify the correctness.



(a) Edge 3-product cordial labeling $K_{2,2}$



(b) Edge 5-product cordial labeling of $K_{2,4}$

Fig. 1.

Remark 2.1. In Theorem 1.1, suppose $K_{2,2n}$ admits an edge $(2n + 1)$-product cordial labeling f. Note that $K_{2,2n}$ is a $(4n, 2n+2)$-graph. From the definition of $(2n+1)$-product cordial labeling, we have $e_f(i) \geq \left\lfloor \frac{4n}{2n+1} \right\rfloor = 1$ and $v_f(j) \leq \left\lceil \frac{2n+2}{2n+1} \right\rceil = 2$, for $0 \leq i, j \leq 2n$. Thus, $e_f(0) \geq 1$ and this implies that $v_f(0) \geq 2$. Therefore, $v_f(0) = 2$. Since there are only $2n+2$ vertices, $v_f(j) = 1$ for $1 \leq j \leq 2n$.

If $e_f(0) \geq 2$, i.e., $|E_0| \geq 2$, then G_0 is either acyclic or contains a cycle of order at least 4. This implies that $v_f(0) \geq 3$ which is a contradiction. Thus, we have $e_f(0) = 1$, $v_f(0) = 2$ and $v_f(j) = 1$ for $1 \leq j \leq 2n$. Since each edge is incident to either u or u', by symmetry, we may assume that $f(uu'_n) = 0$.

Before studying the edge k-product cordiality of $K_{2,2n}$, we recall some structure about the group of units modulo k.

Let $U(\mathbb{Z}_k)$ be the set of all units in $\mathbb{Z}_k$, i.e., $U(\mathbb{Z}_k) = \{i \in \mathbb{Z}_k \mid \gcd(i,k) = 1\}$. It is known that $U(\mathbb{Z}_k)$ is a multiplicative Abelian group. Suppose $k = \prod_{j=1}^{r} p_j^{k_j}$ is the prime factorization, where $p_1, \dots, p_r$ are distinct primes. It is well-known that

$$U(\mathbb{Z}_k) \cong U(\mathbb{Z}_{p_1^{k_1}}) \times \cdots \times U(\mathbb{Z}_{p_r^{k_r}}).$$

Moreover, $U(\mathbb{Z}_k)$ is a cyclic group if and only if $k = 2, 4, p^j, 2p^j$, where p is an odd prime and $j \geq 1$. We shall use $\mathfrak{C}_n$ to denote the multiplicative cyclic group of order n. In fact, $U(\mathbb{Z}_2) \cong \mathfrak{C}_1$, $U(\mathbb{Z}_4) \cong \mathfrak{C}_2$, $U(\mathbb{Z}_{p^i}) \cong U(\mathbb{Z}_{2p^i}) \cong \mathfrak{C}_{\phi(p^i)} = \mathfrak{C}_{p^i - p^{i-1}}$, where p is an odd prime, $i \geq 1$ and ϕ is the Euler function. Moreover, $U(\mathbb{Z}_{2^j}) \cong \mathfrak{C}_2 \times \mathfrak{C}_{2^{j-2}}$, where $j \geq 3$. Thus, the order of $U(\mathbb{Z}_k)$ is always even when $k \geq 3$.

Theorem 2.2. *Suppose $2n+1$ is a prime for $n \geq 1$. $K_{2,2n}$ is edge $(2n+1)$-product cordial.*

Proof. We only need to consider $n \geq 3$. First we construct the following table.

Table 1. Entries in the table are exponents of a primitive root

	1	2	$\cdots$	n-1	n	n+1	$\cdots$	2n-1
	1	2	$\cdots$	n-1	n+1	n+2	$\cdots$	2n
Column sum of the above array	2	4	$\cdots$	2n-2	2n+1	2n+3	$\cdots$	4n-1
Take modulo 2n	2	4	$\cdots$	2n-2	1	3	$\cdots$	2n-1

Note that entries in the last row of the table are the final exponents of the induced column products of a primitive root.

Let ρ be the primitive root of modulo $2n+1$, i.e., ρ is a generator of the cyclic group $U(\mathbb{Z}_{2n+1}) \cong \mathfrak{C}_{2n}$. Define $f : E \to \mathbb{Z}_{2n+1}$ by

1) $f(uu_i) = \rho^i$, $1 \leq i \leq n$ (the first n entries of the first row of Table 1);
2) $f(uu'_i) = \rho^{n+i}$, $1 \leq i \leq n-1$ (the last $n-1$ entries of the first row of Table 1) and $f(uu'_n) = 0$;
3) $f(u'u_i) = \rho^i$, $1 \leq i \leq n-1$ (the first $n-1$ entries of the second row of Table 1) and $f(u'u_n) = \rho^{n+1}$ (the first n-th entry of the second row of Table 1);

4) $f(u'u'_i) = \rho^{n+1+i}$, $1 \le i \le n-1$ (the last $n-1$ entries of the second row of Table 1) and $f(u'u'_n) = \rho^{2n}$.

Note that $\rho^{2n} \equiv 1 \pmod{2n+1}$ and $\rho^n \equiv -1 \equiv 2n \pmod{2n+1}$. Now $e_f(0) = 1$, $e_f(\rho^n) = e_f(2n) = 1$, and $e_f(i) = 2$, where $1 \le i \le 2n-1$.

From the last row of Table 1, we have $f^*(u_j) = \rho^{2j}$, $1 \le j \le n-1$; $f^*(u_n) = \rho^{2n+1} = \rho$; $f^*(u'_j) = \rho^{2j+1}$, $1 \le j \le n-1$; and $f^*(u'_n) = f^*(u) = 0$. Now

$$\begin{aligned} f^*(u') &= \left(\prod_{i=1}^{n-1} \rho^i\right)\left(\prod_{i=n+1}^{2n} \rho^i\right)\rho^{2n} \\ &= \left(\prod_{i=1}^{n-1} \rho^i\right)\left(\prod_{j=1}^{n-1} \rho^{2n-j}\right)\rho^{2n}\rho^{2n} && \text{(write } i = 2n-j\text{)} \\ &= \left(\prod_{i=1}^{n-1} \rho^i\rho^{2n-i}\right)\rho^{2n}\rho^{2n} = 1. && (\rho^{2n} = 1 \text{ in } \mathbb{Z}_{2n+1}) \end{aligned}$$

Now, the values of $f^*(u') = 1 = \rho^0$, $f^*(u_n) = \rho$ and $f^*(u_j) = \rho^{2j}$, $f^*(u'_j) = \rho^{2j+1}$ for $1 \le j \le n-1$ run through $\rho^0, \rho^1, \dots, \rho^{2n-1}$ once, i.e., $\mathbb{Z}_{2n+1} \setminus \{0\}$ once. Combining with $f^*(u'_n) = f^*(u) = 0$, we have $v_f(j) = 1$, $1 \le j \le 2n$ and $v_f(0) = 2$. Thus f is an edge $(2n+1)$-product cordial labeling of $K_{2,2n}$. □

Example 2.3. Let $n = 3$ so that $k = 2n+1 = 7$. It is known that 3 is a primitive root modulo 7. Namely, $3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$ in $\mathbb{Z}_7$. According to the proof above, we have the following edge 7-product cordial labeling for $K_{2,2n}$ (Figure 2).

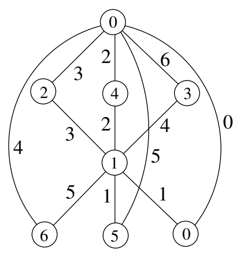


Fig. 2. An edge 7-product cordial labeling for $K_{2,6}$

Instead of drawing a figure, we shall present the values of a labeling f for $K_{2,6}$ by the following table.

$f(uu_1)$	$f(uu_2)$	$f(uu_3)$	$f(uu'_1)$	$f(uu'_2)$	$f(uu'_3)$	$f^*(u)$
$f(u'u_1)$	$f(u'u_2)$	$f(u'u_3)$	$f(u'u'_1)$	$f(u'u'_2)$	$f(u'u'_3)$	$f^*(u')$
$f^*(u_1)$	$f^*(u_2)$	$f^*(u_3)$	$f^*(u'_1)$	$f^*(u'_2)$	$f^*(u'_3)$	$\mathbb{Z}_k$

=

3	2	6	4	5	0	0
3	2	4	5	1	1	1
2	4	3	6	5	0	$\mathbb{Z}_7$

We shall use a similar presentation for a general $n \ge 3$, namely

$f(uu_1)$	$\cdots$	$f(uu_n)$	$f(uu'_1)$	$\cdots$	$f(uu'_n)$	$f^*(u)$
$f(u'u_1)$	$\cdots$	$f(u'u_n)$	$f(u'u'_1)$	$\cdots$	$f(u'u'_n)$	$f^*(u')$
$f^*(u_1)$	$\cdots$	$f^*(u_3)$	$f^*(u'_1)$	$\cdots$	$f^*(u'_3)$	$\mathbb{Z}_k$

Theorem 2.4. *If $k = 2n+1$ is a composite, then $K_{2,2n}$ is not edge k-product cordial.*

Proof. Let $k = ab$, where $3 \le a \le b < k$ and a is a prime. Suppose $K_{2,2n}$ admits an edge k-product cordial labeling f. We keep the notation defined in Remark 2.1.

Since $1 \le e_f(i) \le 2$ for $1 \le i \le 2n$ and $\sum\limits_{i=1}^{2n} e_f(i) = 4n-1$, there is only one i such that $e_f(i) = 1$. Let A be the set of all ja-edges for some $1 \le j \le b-1$ and B be the set of all la-vertices for some $1 \le l \le b-1$. By Remark 2.1 and the above result, we have $2(b-1) \ge |A| \ge 2(b-2)+1$ and $|B| = b-1$.

Suppose $u' \in B$. There are $b-2$ vertices of degree 2 in B. Let v be one such degree 2 vertex. Therefore, v is incident with at most two edges in A. Thus, there are at most $2(b-2)$ edges incident with vertices in $B \setminus \{u'\}$. Consequently, there is at least one edge in A whose end vertex is not of degree 2 which is impossible

Suppose $u' \notin B$. Thus, all $b-1$ vertices in B are of degree 2. Moreover, each vertex in B is incident to only one edge in A. Therefore, $|A| = |B| = b-1$. It contradicts $|A| \ge 2b-3$. $\square$

We have the following theorem immediately.

Theorem 2.5. *$K_{2,2n}$ is edge $(2n+1)$-product cordial if and only if $2n+1$ is a prime.*

3. Edge k-product cordiality of $K_{2,4}$

Let us consider the graph $K_{2,4}$. We know that $K_{2,4}$ is edge 5-product cordial in [3]. So we consider the first case listed in Theorem 1.1, i.e., $k \ge 4n+1 = 9$.

Recall that, we shall show the edge labels and vertex labels of a labeling f by a table. So one will see from the table that the edge labels are distinct as well as the vertex labels are also distinct. This means that $e_f(i)$ is either 0 or 1 and $v_f(j)$ is also either 0 or 1. Hence, this implies that f is a required labeling.

Lemma 3.1. *If $U(\mathbb{Z}_k)$ contains an element of order at least 8, then $K_{2,4}$ is edge k-product cordial.*

Proof. Let ρ be an element in $U(\mathbb{Z}_k)$ of order $m \ge 8$. We define an edge labeling f for $K_{2,4}$ as follows (Table 2).

Table 2. An edge k-product cordial labeling for $K_{2,4}$

1	ρ	ρ^{-1}	ρ^4	ρ^4
ρ^{-2}	ρ^2	ρ^3	ρ^{-3}	1
ρ^{-2}	ρ^3	ρ^2	ρ	$\mathbb{Z}_k$

Since $m \ge 8$, $1, \rho$, $\rho^{-1} = \rho^{m-1}$, ρ^4, $\rho^{-2} = \rho^{m-2}$, ρ^2, $\rho^{-3} = \rho^{m-3}$ are distinct in $U(\mathbb{Z}_k)$. Similarly, ρ^{-2}, 1, ρ, ρ^2, ρ^3, ρ^4 are distinct in $U(\mathbb{Z}_k)$. Hence, f is an edge k-product cordial labeling for $K_{2,4}$. $\square$

Example 3.2. Let $k = 13$. It is known that 2 is a primitive root modulo 13. Namely, $2^2 = 4$, $2^3 = 8$, $2^4 = 3$, $2^5 = 6$, $2^6 = 12$, $2^7 = 11$, $2^8 = 9$, $2^9 = 5 = 2^{-3}$, $2^{10} = 10 = 2^{-2}$, $2^{11} = 7 = 2^{-1}$, $2^{12} = 1$ in $\mathbb{Z}_{13}$. According to Table 2, we have the following 13-product cordial labeling for $K_{2,4}$.

1	2	7	3	3
10	4	8	5	1
10	8	4	2	$\mathbb{Z}_{13}$

Corollary 3.3. *Suppose $k = 2^j$, $j \geq 4$. $K_{2,4}$ is edge k-product cordial.*

Proof. For $j = 4$, i.e., $k = 16$, we have the following labeling.

3	13	1	15	9
2	4	8	6	0
6	4	8	10	$\mathbb{Z}_{16}$

Consider $j \geq 5$. Since $U(\mathbb{Z}_{2^j}) \cong \mathbb{Z}_2 \times Z_{2^{j-2}}$, there is an element $\rho \in U(\mathbb{Z}_{2^j})$ of order 8. By Lemma 3.1, we have the corollary. $\square$

Example 3.4. Let $k = 32$. Thus, $\phi(32) = 16$ and $U(\mathbb{Z}_{32}) \cong \mathfrak{C}_2 \times \mathfrak{C}_8$. 3 is an element in $U(\mathbb{Z}_{32})$ of order 8. Namely, $3^2 = 9$, $3^3 = 27$, $3^4 = 17$, $3^5 = 3^{-3} = 19$, $3^6 = 3^{-2} = 25$, $3^7 = 3^{-1} = 11$, $3^8 = 1$ in $\mathbb{Z}_{32}$. According to Table 2, we have the labeling

1	3	11	17	17
25	9	27	19	1
25	27	9	13	$\mathbb{Z}_{32}$

Corollary 3.5. *Let p be an odd prime. Suppose $k = p^j$ or $k = 2p^j$ for $j \geq 1$. If $k \geq 9$, then $K_{2,4}$ is edge k-product cordial.*

Proof. Now, $U(\mathbb{Z}_k)$ is a cyclic group of order $\phi(k) = p^{j-1}(p-1)$.

1) If $p \geq 11$, then $\phi(k) \geq 10$. By Lemma 3.1, we have the result.

2) Suppose $p = 7$. Since $k \geq 9$, $k = 7^j$ for $j \geq 2$ or $k = 2 \times 7^i$ for $i \geq 1$. If $k = 7^j$ with $j \geq 2$, then $\phi(k) = 7^{j-1} \times 6$. We may choose ρ in $U(\mathbb{Z}_k)$ of order 14. If $k = 2 \times 7^i$ with $i \geq 2$, then it is similar to the previous case. The remaining case is $k = 14$. We shall consider this case later.

3) Suppose $p = 5$. If $k = 5^j$ with $j \geq 2$, then $\phi(k) = 5^{j-1} \times 4$. We may choose ρ in $U(\mathbb{Z}_k)$ of order 10. If $k = 2 \times 5^i$ with $i \geq 2$, then it is similar to the previous case. The remaining case is $k = 10$. We shall consider this case later.

4) Suppose $p = 3$. Suppose $k = 3^j$ with $j \geq 2$. Then $\phi(k) = 3^{j-1} \times 2 \geq 18$ if $j \geq 3$. We may choose ρ in $U(\mathbb{Z}_k)$ of order 18. The remaining case is $k = 9$.

   Suppose $k = 2 \times 3^i$ with $i \geq 1$. Since $k \geq 9$, $i \geq 2$. If $i \geq 3$, then it is similar to the previous case. The remaining case is $k = 18$.

Thus, the remaining cases are $k = 9, 10, 14, 18$. For each k, we have the following labelings.

2	4	1	8	1
6	5	7	3	0
3	2	7	6	$\mathbb{Z}_9$

3	7	1	9	9
4	5	6	2	0
2	5	6	8	$\mathbb{Z}_{10}$

3	9	1	13	1
2	7	12	4	0
6	7	12	10	$\mathbb{Z}_{14}$

5	13	1	17	7
2	3	16	6	0
10	3	16	12	$\mathbb{Z}_{18}$

Hence, we have the corollary. $\square$

Lemma 3.6. *Suppose $k \geq 12$ is a composite number, and k is neither p^j nor $2p^j$ for some prime p and $j \geq 1$. $K_{2,4}$ is edge k-product cordial.*

Proof. Let $k = ab$ with $\gcd(a, b) = 1$. Suppose $a = 2$. Since $k = 2b$ is neither p^j nor $2p^j$, b has at least two distinct odd prime factors, say p, q. Let i be the larger integer such that p^i is a factor of b. Rewrite $k = (2p^i)(b/p^i)$, then $\gcd(2p^i, b/p^i) = 1$, $2p^i \geq 3$ and b/p^i is odd.

So, now we may assume $k = ab$, $\gcd(a, b) = 1$, $a \geq 3$ and b is odd. Since $\phi(a)$ and $\phi(b)$ are even, $|U(\mathbb{Z}_{ab})| = \phi(ab) = \phi(a)\phi(b)$ is a multiple of 4. Note that $\{1, -1\} \subset U(\mathbb{Z}_{ab})$.

Claim 1: Suppose $\alpha \in U(\mathbb{Z}_{ab}) \setminus \{1, -1\}$, then $\alpha \neq -\alpha$.

If not, then we have $2\alpha \equiv 0 \pmod{ab}$. Since $\gcd(\alpha, ab) = 1$, we have $2 \equiv 0 \pmod{ab}$ which is impossible.

1) Suppose $|U(\mathbb{Z}_{ab})| \geq 8$. We may choose $\alpha, \beta, \gamma \in U(\mathbb{Z}_{ab})$ such that 1, -1, α, $-\alpha$, β, $-\beta$, γ, $-\gamma$ are distinct. Define f by Table 3

Table 3. An edge ab-product cordial labeing

1	-1	α	$-\alpha$	α^2
a	$-b$	β	$-\gamma$	0
a	b	$\alpha\beta$	$\alpha\gamma$	$\mathbb{Z}_{ab}$

Since $a, b \notin U(\mathbb{Z}_{ab})$, $\gcd(a, b) = 1$ and $\{1, -1, \alpha, -\alpha, \beta, -\gamma\} \subset U(\mathbb{Z}_{ab})$, all edge labels are distinct. Since α, β, γ are distinct elements in the group $U(\mathbb{Z}_{ab})$, $\alpha^2, \alpha\beta, \alpha\gamma$ are distinct. Thus, all six vertex labels are distinct. Hence, f is an edge ab-product cordial labeling for $K_{2,4}$.

2) Suppose $|U(\mathbb{Z}_{ab})| = 4$, then $\phi(a) = \phi(b) = 2$. Since b is odd, $b = 3$. Since $\gcd(a, b) = 1$ and $\phi(a) = 2$, we have $a = 4$. Thus $k = 12$. Define f by

1	5	7	11	1
2	3	4	6	0
2	3	4	6	$\mathbb{Z}_{12}$

Hence, f is an edge 12-product cordial labeling for $K_{2,4}$.

$\square$

Remark 3.7. From the proof of Lemma 3.6, we have checked that the edge labels (or the vertex labels) are distinct. In general, there are two kinds of labels, say U and N, where U is the list of labels in $U(\mathbb{Z}_k)$ and N is the list of labels not in $U(\mathbb{Z}_k)$. Thus, it is suffice to show that each list contains distinct elements.

Example 3.8. Let $k = 35$. Now $U(\mathbb{Z}_{35}) \cong U(\mathbb{Z}_5) \times U(\mathbb{Z}_7) \cong \mathfrak{C}_4 \times \mathfrak{C}_6$. We use the following two methods to demonstrate the proof of Lemma 3.1 and the proof of Lemma 3.6.

Method 1: Since $U(\mathbb{Z}_{35}) \cong \mathfrak{C}_4 \times \mathfrak{C}_6$, there is an element in $U(\mathbb{Z}_{35})$ whose order is lcm$\{4,6\}$=12. Actually, the order of 2 in $U(\mathbb{Z}_{35})$ is 12. Namely $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^9 = 22$, $2^{10} = 9$, $2^{11} = 18$ (all arithmetics are taken in $\mathbb{Z}_{35}$ to label the edges of the graph $K_{2,4}$). According to Table 2, we have

1	2	18	16	16
9	4	8	22	1
9	8	4	2	$\mathbb{Z}_{35}$

Method 2: We may choose $\alpha = 2$, $\beta = 3$, $\gamma = 4$. According to Table 3 we have

1	-1	2	-2	4
5	-7	3	-4	0
5	7	6	8	$\mathbb{Z}_{35}$

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1	34	2	33	4
5	28	3	31	0
5	7	6	8	$\mathbb{Z}_{35}$

Theorem 3.9. *$K_{2,4}$ is edge k-product cordial if and only if $k = 5$ or $k \geq 9$.*

Proof. The necessity part follows from Theorem 1.1. Combining Theorem 2.2, and Lemma 3.6, Corollaries 3.3 and 3.5, we have the sufficient part. □

4. Edge k-Product Cordiality of $K_{2,6}$

Let us consider the graph $K_{2,6}$. We know that $K_{2,6}$ is edge 7-product cordial in the above section. So we consider the first case listed in Theorem 1.1, i.e., $k \geq 4n + 1 = 13$.

Lemma 4.1. *If $U(\mathbb{Z}_k)$ contains an element of order at least 12, then $K_{2,6}$ is edge k-product cordial.*

Proof. Let ρ be an element in $U(\mathbb{Z}_k)$ of order $m \geq 12$. We define an edge labeling f for $K_{2,6}$ as follows.

1	ρ^{-1}	ρ^4	ρ^{-4}	ρ^5	ρ^{-5}	ρ^{-1}
ρ^{-2}	ρ	ρ^{-3}	ρ^6	ρ^3	ρ^2	ρ^7
ρ^{-2}	1	ρ	ρ^2	ρ^8	ρ^{-3}	$\mathbb{Z}_k$

The exponents of the twelve edge labels form the set of consecutive integers, $\{-5, -4, \dots, 6\}$. Thus, they are distinct in $U(\mathbb{Z}_k)$.

Since $m \geq 12$, $\rho^{-3} = \rho^{m-3}$, $\rho^{-2} = \rho^{m-2}$, $\rho^{-1} = \rho^{m-1}$, 1, ρ, ρ^2, ρ^7, ρ^8 are distinct. Thus, f is an edge k-product cordial labeling for $K_{2,6}$. □

Corollary 4.2. *Suppose $k = 2^j$, $j \geq 4$. $K_{2,6}$ is edge k-product cordial.*

Proof. Consider $j = 4, 5$, i.e., $k = 16$ and 32. We have the following labelings, respectively.

1	3	9	11	7	15	1
2	4	6	8	13	12	0
2	12	6	8	11	4	$\mathbb{Z}_{16}$

1	17	3	25	19	11	11
27	9	2	4	6	8	0
27	25	6	4	18	24	$\mathbb{Z}_{32}$

Consider $j \geq 6$. Since $U(\mathbb{Z}_{2^j}) \cong \mathbb{Z}_2 \times Z_{2^{j-2}}$, there is an element $\rho \in U(\mathbb{Z}_{2^j})$ of order 16. By Lemma 4.1, we have the corollary. □

Corollary 4.3. *Let p be an odd prime. Suppose $k = p^j$ or $k = 2p^j$ for $j \geq 1$. If $k \geq 13$, then $K_{2,6}$ is edge k-product cordial.*

Proof. Now, $U(\mathbb{Z}_k)$ is a cyclic group of order $\phi(k) = p^{j-1}(p-1)$. If $U(\mathbb{Z}_k)$ contains an element of order $m \geq 12$, then by Lemma 4.1, we have an edge k-product labeling.

Thus, we only need to deal with those k such that $\phi(k) \leq 11$, i.e., $p^{j-1}(p-1) \leq 11$. Since $\phi(k)$ is even, first we need to find out the values of p such that $p^{j-1}(p-1) \leq 10$.

Suppose $j = 1$, then $p - 1 \leq 10$ implies that $p = 3, 5, 7, 11$. Since $k \geq 13$, we have $k = 14, 22$.

Suppose $j \geq 2$. We have $3(p-1) \leq p^{j-1}(p-1) \leq 10$ and $2p^{j-1} \leq p^{j-1}(p-1) \leq 10$. The only solution is $p = 3$ and $j = 2$. Hence $k = 18$. Tables below define an edge k-product cordial labeling for $K_{2,6}$, where $k = 14, 18, 22$, respectively.

1	3	5	9	11	13	13
2	4	7	10	8	6	0
2	12	7	6	4	8	$\mathbb{Z}_{14}$

1	5	7	11	13	17	17
2	3	4	6	8	9	0
2	15	10	12	14	9	$\mathbb{Z}_{18}$

1	3	5	7	9	13	9
2	4	21	19	17	11	0
2	12	17	1	21	11	$\mathbb{Z}_{22}$

□

Lemma 4.4. *Let $k = 2^r 3^s 5^t 7^q$, where $r, s, t, q \geq 0$. If every element in $U(\mathbb{Z}_k)$ is of order at most 9, then the followings hold.*

(a) *q and t are not both positive;*

(b) *if $t = 1$, then $s = 0, 1$;*

(c) *$0 \leq q \leq 1$, $0 \leq t \leq 1$, $0 \leq s \leq 2$ and $0 \leq r \leq 5$.*

Proof. First we know that

$$U(\mathbb{Z}_k) \cong U(\mathbb{Z}_{2^r}) \times (\mathfrak{C}_{2\times 3^{s-1}}) \times (\mathfrak{C}_{4\times 5^{t-1}}) \times (\mathfrak{C}_{6\times 7^{q-1}}).$$

Note that, if the exponent of the prime is zero, then the corresponding subgroup does not appear in the above formula. If $q \geq 1$ and $t \geq 1$, then $U(\mathbb{Z}_k)$ contains an element with order at least $\operatorname{lcm}\{4,6\} = 12$. We have (a).

If $t = 1$ and $s \geq 2$, then $U(\mathbb{Z}_k)$ contains an element with order $\operatorname{lcm}\{4,6\} = 12$. We have (b).

1) If $q \geq 2$, then $U(\mathbb{Z}_k)$ contains an element of order at least 42.
2) If $t \geq 2$, then $U(\mathbb{Z}_k)$ contains an element of order 20.
3) If $s \geq 3$, then $U(\mathbb{Z}_k)$ contains an element of order 18.
4) If $r \geq 6$, then $U(\mathbb{Z}_k)$ contains an element of order 16.

So we have (c). □

Theorem 4.5. *Suppose $k \geq 13$ is a composite number, and k is neither p^j nor $2p^j$ for some prime p and $j \geq 1$. $K_{2,6}$ is edge k-product cordial.*

Proof. Similar to the proof of Lemma 3.6, we may let $k = ab$, where $\gcd(a,b) = 1$, $a \geq 3$ and $b \geq 3$. Moreover the order of $U(\mathbb{Z}_k)$ is a 4 multiple.

Suppose ρ is an element in $U(\mathbb{Z}_k)$ of order $m \geq 10$. We define an edge labeling f as follows.

1	ρ	ρ^{-1}	ρ^2	ρ^{-2}	ρ^5	ρ^5
a	ρ^3	ρ^4	ρ^{-3}	b	ρ^{-4}	0
a	ρ^4	ρ^3	ρ^{-1}	$b\rho^{-2}$	ρ	$\mathbb{Z}_{ab}$

The exponents of the ten edge labels in the above table, which are in $U(\mathbb{Z}_k)$, form the set of consecutive integers, $\{-4, -3, \dots, 5\}$. So they are distinct. Clearly, a, b are distinct. Hence, all edge labels are distinct. Similarly, $\rho, \rho^{-1} = \rho^{m-1}, \rho^3, \rho^4, \rho^5$ are distinct elements in $U(\mathbb{Z}_k)$. Also $0, a, b\rho^{-2}$ are distinct elements not in $U(\mathbb{Z}_k)$. If not, suppose $b\rho^{-2} \equiv a \pmod{ab}$ (or $b\rho^{-2} \equiv 0 \pmod{ab}$, resp.). Thus, $b^2\rho^{-2} \equiv ab \equiv 0 \pmod{ab}$ (or $b \equiv 0 \pmod{ab}$), respectively). This implies $b^2 \equiv 0 \pmod{ab}$ (or $b \equiv 0 \pmod{ab}$, respectively) which is impossible. Thus, all vertex labels are distinct. Hence f is an edge k-product cordial labeling for $K_{2,6}$.

Thus, we assume that every element in the group $U(\mathbb{Z}_k)$ is of order at most 9. If k has a prime factor p with $p \geq 11$, then $U(\mathbb{Z}_k)$ contains an element of order at least 10. Therefore, we only need to consider $k = 2^r 3^s 5^t 7^q$, where $r, s, t, q \geq 0$. By Lemma 3.7, we have $0 \leq q \leq 1$, $0 \leq t \leq 1$, $0 \leq s \leq 2$ and $0 \leq r \leq 5$. For easy discussion, let us consider k being odd or even.

a) Suppose k is odd. If $s = 0$, then under the assumption, it is not a case. So $s = 1, 2$. Suppose $s = 2$. By Lemma 3.7, $k = 3^2 \times 7 = 63$ only. Suppose $s = 1$, then $k = 15, 21$.

b) Suppose k is even. If $r = 1$, then under the assumption, at least two of s, t, q are positive. However, t and q cannot be both positive. Thus, $k = 30, 42, 126$.

   If $5 \geq r \geq 2$, then $U(\mathbb{Z}_{2^r}) \cong \mathfrak{C}_2 \times \mathfrak{C}_{2^{r-2}}$. Under the assumption, at least one of s, t, q is positive. But $(q,t) = (1,1)$ and $(s,t) = (2,1)$ are excluded.

i) Suppose $r = 4, 5$. If $q = 1$ or $s = 2$, then $\mathfrak{C}_{2^{r-2}} \times \mathfrak{C}_6$ contains an element of order 12. Thus, we may assume $q = 0$ and $s = 0, 1$. If $t = 0$, then under the assumption $k = 48, 96$. If $t = 1$, then $k = 80, 160, 240, 480$.

ii) Suppose $r = 3$. Then $k = 56, 168, 504, 40, 120, 24, 72$.

iii) Suppose $r = 2$. Then $k = 28, 84, 252, 20, 60, 12$ (not a case)$, 36$.

Let us summarize the above cases.

Case 1. $k =$ 21, 28, 36, 42, 56, 63, 72, 84, 126, 168, 252, 504.

Case 2. $k =$ 80, 160, 240, 480.

Case 3. $k = 40, 48, 60, 96, 120$.

Case 4. $k =$ 15, 20, 24, 30.

For Case 1, one may check that $U(\mathbb{Z}_k)$ contains a subgroup isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_6$. Without loss of generality, we may let the generators of this subgroup be -1 and α, where the order of α is 6. That is, $C_2 \times \mathfrak{C}_6 = \{\pm\alpha^i \mid 0 \leq i \leq 5\}$. We define an edge labeling f as follows.

1	-1	α	$-\alpha$	α^3	$-\alpha^3$	$-\alpha^2$
α^4	α^5	$-\alpha^2$	$-\alpha^5$	α^2	$-\alpha^4$	$-\alpha^4$
α^4	$-\alpha^5$	$-\alpha^3$	1	α^5	α	$\mathbb{Z}_k$

Clearly, all edge labels are distinct and all vertex labels are also distinct.

For Case 2, one may check that $U(\mathbb{Z}_k)$ contains a subgroup isomorphic to $\mathfrak{C}_4 \times \mathfrak{C}_4$. Without loss of generality, we may let the generators of this subgroup be β and γ, where the order of β and γ are 4. That is, $\mathfrak{C}_4 \times \mathfrak{C}_4 = \{\beta^i\gamma^j \mid 0 \leq i, j \leq 3\}$. We define an edge labeling f as follows.

$\beta^2\gamma$	γ	γ^2	β	$\beta\gamma$	$\beta\gamma^2$	$\beta\gamma^3$
1	β^3	β^2	$\beta^2\gamma^3$	$\beta^3\gamma$	$\beta^3\gamma^2$	$\beta\gamma^2$
$\beta^2\gamma$	$\beta^3\gamma$	$\beta^2\gamma^2$	$\beta^3\gamma^3$	γ^2	1	$\mathbb{Z}_k$

Clearly, all edge labels are distinct and all vertex labels are also distinct.

For Case 3, $U(\mathbb{Z}_k)$ contains a subgroup isomorphic to $\mathfrak{C}_2 \times \mathfrak{C}_2 \times \mathfrak{C}_4$. Without loss of generality, we may let this subgroup be $\{\pm\delta^i\epsilon^j \mid 0 \leq i \leq 1, 0 \leq j \leq 3\}$, where the order of δ is 2 and the order of ϵ is 4. We define an edge labeling f as follows.

1	-1	ϵ	ϵ^3	$-\epsilon$	$-\epsilon^3$	-1
$\delta\epsilon^3$	$\delta\epsilon$	ϵ^2	$-\epsilon^2$	$-\delta$	δ	1
$\delta\epsilon^3$	$-\delta\epsilon$	ϵ^3	$-\epsilon$	$\delta\epsilon$	$-\delta\epsilon^3$	$\mathbb{Z}_k$

For Case 4, we have

1	2	4	7	8	11	8
14	13	9	6	3	5	0
14	11	6	12	9	10	$\mathbb{Z}_{15}$

1	3	7	9	11	13	7
19	17	6	2	4	5	0
19	11	2	18	4	5	$\mathbb{Z}_{20}$

1	5	7	17	19	23	23
8	3	6	10	16	21	0
8	15	18	2	16	3	$\mathbb{Z}_{24}$

1	7	11	13	17	19	13
2	23	29	3	4	5	0
2	11	19	9	8	5	$\mathbb{Z}_{30}$

Hence we have the theorem. □

Combining all results in this section and Theorem 1.1, we have

Theorem 4.6. *$K_{2,6}$ is edge k-product cordial if and only if $k = 7$ or $k \geq 13$.*

5. Discussion on further study

From the above proof, in order to find an edge k-product cordial labeling for $K_{2,2n}$ with $k \geq 4n+1$, it is sufficient to choose $4n$ distinct nonzero residues and obtain $2n+2$ distinct row and column products.

If the order of $U(\mathbb{Z}_k)$ is at least $2n$, then we can fill $2n$ elements of $U(\mathbb{Z}_k)$ in the first row. Now, the first row product is an element in $U(\mathbb{Z}_k)$. Next, we arrange another $2n$ elements in the second row such that all column products are distinct nonzero elements (see Lemma 3.1). If we can do that, then it is easy to swap some pairs of elements within the same column such that all the products are distinct. If k is not a prime, then we choose other $2n$ elements for the second row such that the product of these elements is zero (see the proof of Theorem 4.5).

However, it would be more difficult when the order of $U(\mathbb{Z}_k)$ is less than $2n$. Let us show an example. Suppose $n = 4$ and $k = 18 > 4n+1$. Then $U(\mathbb{Z}_{18}) = \{1, 5, 7, 11, 13, 17\}$. We have

1	5	7	17	13	11	8	4	4
2	14	9	10	16	15	3	12	0
2	16	9	8	10	3	6	12	$\mathbb{Z}_{18}$

Hence, we have an edge 18-product cordial labeling for $K_{2,8}$.

Finally, we propose the following conjecture.

Conjecture 5.1. *For $n \geq 4$, $K_{2,2n}$ is edge k-product cordial if and only if $k = 2n+1$ is a prime or $k \geq 4n+1$.*

Conflicts of Interest

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The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Author Contributions

All the authors contributed equally. All authors have read and agreed to the published version of the manuscript.

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