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Fault-tolerant locating-dominating sets on the infinite king grid

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ABSTRACT

Let G be a graph of a network system with vertices, $V(G)$, representing physical locations and edges, $E(G)$, representing informational connectivity. A *locating-dominating (LD)* set $S \subseteq V(G)$ is a subset of vertices representing detectors capable of sensing an "intruder" at precisely their location or at some unknown point in their open-neighborhood. An LD set must be capable of locating an intruder anywhere in the graph using this collection of detectors. We explore three types of fault-tolerant LD sets: *redundant LD sets*, which allow at most one detector to be removed or disabled, *error-detecting LD sets*, which allow at most one false negative, and *error-correcting LD sets*, which allow at most one error (false positive or false negative). In particular, we determine lower and upper bounds for the minimum density of these three fault-tolerant locating-dominating sets in the *infinite king grid*.

Keywords: domination, detection system, fault-tolerant, locating-dominating set, infinite king grid, density

2020 Mathematics Subject Classification: 05C69.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose G models some system or facility with vertices representing physical locations or pieces of hardware and edges representing their physical or informational connectivity. There may be some "intruder" or other undesirable event in the system, which must be automatically detected and located anywhere in the graph via detectors placed at some subset $S \subseteq V(G)$ of vertices.

A detector vertex $v \in S$ is associated with a set of *detection regions*, $R(v) \subseteq \mathscr{P}(V(G))$,

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representing the areas in which it can sense an intruder. A *detection system* is a set of detector vertices, $S \subseteq V(G)$, such that an intruder can be precisely located anywhere in the graph provided the detection signal (presence or absence of an intruder) for every detection region. In this paper, we will explore *locating-dominating (LD) sets*, which are a class of detection system whose detectors have two detection regions: $R(v) = \{\{v\}, N(v)\}$, where $N(v)$ denotes the open-neighborhood of v, $\{u \in V(G) : uv \in E(G)\}$. Other choices for $R(v)$ have been explored, such as $R(v) = \{N(v)\}$, yielding *open-locating-dominating (OLD) sets* [13, 17], or $R(v) = \{N[v]\}$—where $N[v]$ denotes the closed neighborhood of v, $N(v) \cup \{v\}$, yielding *identifying codes (ICs)* [12]. Jean and Lobstein [6] maintain a bibliography containing over 500 papers on these topics and related graphical parameters.

Conceptually, every LD detector vertex $v \in S$ will transmit a value of 0 denoting no intruder, 1 denoting an intruder in the open-neighborhood, or 2 denoting an intruder at v. The goal is then to select a sufficient subset of vertices $S \subseteq V(G)$ so that we can locate an intruder anywhere in the graph. In this paper, we are interested in going one step further: we would like the system to be able to function even in the event of an error, known as a *fault-tolerant LD (FTLD) set*. In particular, we consider three types of FTLD sets: *redundant LD (RED:LD) sets*, in which the system must be fully functioning even if at most one detector is removed or disabled, *error-detecting LD (DET:LD) sets*, where the system allows at most one false-negative (i.e. sending 0 instead of 1 or 2), and *error-correcting LD (ERR:LD) sets*, where the system allows any single transmission error (false-positive or false-negative).

The following theorems characterize the conditions for LD, RED:LD, DET:LD, and ERR:LD sets in arbitrary graphs.

Theorem 1.1 ([8])**.** *$S \subseteq V(G)$ is an LD set if and only if the following are true:*
  i. $\forall v \in V(G) - S$, $|N(v) \cap S| \geq 1$,
  ii. $\forall v, u \in V(G) - S$ *with* $v \neq u$, $|(N(v) \cap S) \triangle (N(u) \cap S)| \geq 1$.

Theorem 1.2 ([8])**.** *$S \subseteq V(G)$ is a RED:LD set if and only if the following are true:*
  i. $\forall v \in V(G)$, $|N[v] \cap S| \geq 2$,
  ii. $\forall v \in S$ *and* $\forall u \in V(G) - S$, $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{v\}| \geq 1$,
  iii. $\forall v, u \in V(G) - S$ *with* $u \neq v$, $|(N(v) \cap S) \triangle (N(u) \cap S)| \geq 2$.

Theorem 1.3 ([10])**.** *$S \subseteq V(G)$ is a DET:LD set if and only if the following are true:*
  i. $\forall v \in V(G)$, $|N[v] \cap S| \geq 2$,
  ii. $\forall v, u \in S$ *with* $u \neq v$, $|(N(v) \cap S) \triangle (N(u) \cap S)| \geq 1$,
  iii. $\forall v \in V(G) - S$ *and* $\forall u \in S$, $|(N(v) \cap S) - (N(u) \cap S)| \geq 2$ *or* $|(N(u) \cap S) - (N(v) \cap S)| \geq 1$,
  iv. $\forall v, u \in V(G) - S$ *with* $u \neq v$, $|(N(v) \cap S) - (N(u) \cap S)| \geq 2$ *or* $|(N(u) \cap S) - (N(v) \cap S)| \geq 2$.

Theorem 1.4 ([11])**.** *$S \subseteq V(G)$ is an ERR:LD set if and only if the following are true:*
  i. $\forall v \in V(G)$, $|N[v] \cap S| \geq 3$

ii. $\forall v, u \in S$ *with* $u \neq v$, $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{v, u\}| \geq 1$,
iii. $\forall v \in V(G) - S$ *and* $\forall u \in S$, $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{u\}| \geq 2$,
iv. $\forall v, u \in V(G) - S$ *with* $u \neq v$, $|(N(v) \cap S) \triangle (N(u) \cap S)| \geq 3$.

For an LD-based detection system, a vertex x is said to be *k-dominated* if $|N[v] \cap S| \geq k$, and a vertex pair (v, u) is said to be *k-distinguished* if $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{v, u\}| \geq k$. When discussing any particular type of detection system with a known characterization, we will also say that a vertex pair is simply "distinguished" if it satisfies said characterization. The properties given in Theorems 1.2 and 1.4 can be rewritten as the following, which make use of k-domination and k-distinguishing.

Corollary 1.5 ([8]). *$S \subseteq V(G)$ is a RED:LD set if and only if the following are true:*
i. $\forall v \in V(G)$, $|N[v] \cap S| \geq 2$,
ii. $\forall v, u \in V(G)$ *with* $u \neq v$, $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{v, u\}| \geq 2 - |\{v, u\} \cap S|$.

Corollary 1.6 ([11]). *$S \subseteq V(G)$ is an ERR:LD set if and only if the following are true:*
i. $\forall v \in V(G)$, $|N[v] \cap S| \geq 3$,
ii. $\forall v, u \in V(G)$ *with* $u \neq v$, $|((N(v) \cap S) \triangle (N(u) \cap S)) - \{v, u\}| \geq 3 - |\{v, u\} \cap S|$.

Given a graph, G, it is clear that any superset of a detection system on G is also a detection system on G, as the extra sensor data could simply be discarded. Thus, we are interested in the smallest (optimal) sets with the required properties. Jean and Seo [8, 10, 11] have studied such optimal sets for special classes of graphs such as trees, cubic graphs, and infinite grids. They also proved that these minimization problems on arbitrary graphs are NP-Complete. In this paper, we consider optimal sets for these three fault-tolerant LD set parameters in the *infinite king grid (K)*, an 8-regular graph inspired by a chess board, with edges denoting the typical eight moves of a king. Specifically, we explore upper and lower bounds for the smallest RED:LD, DET:LD, and ERR:LD sets on K.

The *density* of a subset $S \subseteq V(G)$ is the ratio of $|S|$ to $|V(G)|$, or more formally, $\limsup_{r \to \infty} \frac{|B_r(v) \cap S|}{|B_r(v)|}$ for some center point $v \in V(G)$ where $B_r(v) = \{u \in V(G) : d(u, v) \leq r\}$ is the ball of radius r about v. In this paper, we will only explore graphs which satisfy the "slow-growth" property [15] of $\lim_{r \to \infty} \frac{|B_{r+1}(v)|}{|B_r(v)|} = 1$, which guarantees that the density is invariant of the center point v, and that the density of a periodic tiling is equal to the density within any single tile. The notations RED:LD(G), DET:LD(G), and ERR:LD(G) denote the minimum density of such a set on G.

In Section 2, we discuss some of the previous results for optimal (minimum) fault-tolerant variants of LD, OLD, and IC sets. In Section 3, we introduce the concepts of "share arguments" and "discharging techniques," which are used later in Section 4 to determine bounds for RED:LD, DET:LD, and ERR:LD sets on the infinite king grid.

2. Previous results

The concept of LD sets was originally introduced by Slater [19, 20]. Slater [21] also explored "FTLD" sets (equivalent to DET:LD sets in this paper), and proved upper and lower bounds for optimal DET:LD sets in the *infinite square grid (SQ)*: $[\frac{12}{23}, \frac{3}{5}]$. Jean and Seo [8, 11] introduced two new fault-tolerant LD sets, RED:LD and ERR:LD sets, and fully characterized and established existence criteria for RED:LD, DET:LD, and ERR:LD sets; these characterizations are included in this paper as Theorems 1.2, 1.3, and 1.4, respectively. Note that characterizations of general fault-tolerant variants of distinguishing sets were established by Seo and Slater [18], though their general analysis does not cover LD set variants due to an assumption that each vertex has only one detection region. Jean and Seo [10] also proved upper and lower bounds for optimal FTLD sets in SQ: $[\frac{2}{5}, \frac{7}{16}]$ for RED:LD, $[\frac{12}{23}, \frac{7}{12}]$ for DET:LD (improved from [21]), and $[\frac{24}{37}, \frac{2}{3}]$ for ERR:LD.

A variety of graphical parameters have been studied on the infinite king grid [2, 3, 5, 14]; for instance, Charon et al. [1] proved IC(K) $= \frac{2}{9}$, Honkala and Laihonen [4] determined that LD(K) $= \frac{1}{5}$, Seo [16] found that $\frac{6}{25} \le$ OLD(K) $\le \frac{1}{4}$, and Jean and Seo [7] proved $\frac{3}{10} \le$ RED:OLD(K) $\le \frac{1}{3}$ and $\frac{3}{11} \le$ RED:IC(K) $\le \frac{1}{3}$.

Similar types of fault-tolerant open-locating-dominating sets on other infinite graphs have also been studied; for instance on the *infinite hexagonal grid (HEX)* and the *infinite triangular grid (TRI)*. Seo and Slater [18] proved that RED:OLD(HEX) $= \frac{2}{3}$, DET:OLD(HEX) $= \frac{6}{7}$, RED:OLD(SQ) $= \frac{1}{2}$, DET:OLD(SQ) $= \frac{3}{4}$, RED:OLD(TRI) $= \frac{3}{8}$, and DET:OLD(TRI) $\le \frac{5}{9}$. Later, Jean and Seo [9] proved that DET:OLD(TRI) $= \frac{1}{2}$.

3. Lower Bound Techniques

In this section, we explain the concepts of "share" and "discharging," which will be used in Section 4 to prove the lower bounds for FTLD sets in the infinite king grid. We also introduce several notations that will be used to shorten these proofs.

3.1. *Share*

Instead of finding a lowerbound for the optimal density directly, it is often easier to find an upperbound for the average "contribution" of each detector vertex to the domination of its neighbors. This "contribution" is measured by the *share* of a detector vertex, which was first introduced by Slater [21]. Let $S \subseteq V(G)$ be a set of LD detectors.

Definition 3.1. The *domination count* of a vertex $v \in V(G)$, denoted dom(v), is $|N[v] \cap S|$.

Definition 3.2. The *partial share* of a vertex $v \in V(G)$, denoted sh[v], is $\frac{1}{\text{dom}(v)}$.

Definition 3.3. The (total) *share* of a detector vertex $v \in S$, denoted sh(v), is $\sum_{u \in D} \text{sh}[u]$ where D is the set of vertices dominated by v.

For LD detectors, $\mathrm{dom}(v) = |N[v] \cap S|$ and $\mathrm{sh}(v) = \sum_{u \in N[v]} \mathrm{sh}[u]$. If S is a *dominating set*—that is, $\forall v \in V(G)$, $\mathrm{dom}(v) \geq 1$—it is clear that for a finite graph, $\sum_{x \in S} \mathrm{sh}(x) = |V(G)|$ because each vertex which is k-dominated has partial share $\frac{1}{k}$ and this fraction is repeated k times due to having k dominators. Thus, the average share will be equal to $\frac{|V(G)|}{|S|}$, the inverse density of S in $V(G)$. When G has the slow-growth property, this equivalence between the average share and the inverse density extends even to infinite graphs [15].

Definition 3.4. For some $A \subseteq V(G)$, let $\mathrm{sh}[A]$ denote the *block share* of A, $\sum_{x \in A} \mathrm{sh}[x]$.

For example, $\mathrm{sh}[\{v_1, v_2\}] = \mathrm{sh}[v_1] + \mathrm{sh}[v_2]$, though we often shorten this to $\mathrm{sh}[v_1 v_2]$.

Notation 3.5. Let D_k denote the set of vertices which are exactly k-dominated, $\{v \in V(G) : \mathrm{dom}(v) = k\}$, and D_{k+} denote the set of vertices which are at least k-dominated, $\cup_{j \geq k} D_j$.

Notation 3.6. Let σ_A be $\sum_{x \in A} \frac{1}{x}$ where A is a finite sequence of single-character symbols denoting natural numbers (e.g. $\sigma_{123} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$ or $\sigma_{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ where $a, b, c \in \mathbb{N}$).

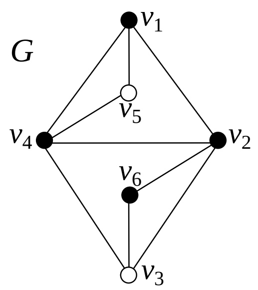


Fig. 1. Example RED:LD set

Consider the graph G in Figure 1, where shaded vertices represent detectors, elements of RED:LD set $S = \{v_1, v_2, v_4, v_6\}$. We observe that $\{v_5, v_6\} \subseteq D_2$, $\{v_1, v_3, v_4\} \subseteq D_3$, and $v_2 \in D_4$, thus $\mathrm{sh}[v_5] = \mathrm{sh}[v_6] = \frac{1}{2}$, $\mathrm{sh}[v_1] = \mathrm{sh}[v_3] = \mathrm{sh}[v_4] = \frac{1}{3}$, and $\mathrm{sh}[v_2] = \frac{1}{4}$. Therefore, $\mathrm{sh}(v_1) = \mathrm{sh}[N[v_1]] = \mathrm{sh}[v_1 v_2 v_4 v_5] = \sigma_{3432} = \frac{17}{12}$ and $\mathrm{sh}(v_2) = \mathrm{sh}[N[v_2]] = \mathrm{sh}[v_1 v_2 v_3 v_4 v_6] = \sigma_{34332} = \frac{7}{4}$. Similarly, we can show that $\mathrm{sh}(v_4) = \frac{7}{4}$ and $\mathrm{sh}(v_6) = \frac{13}{12}$. We see that the sum of shares for the dominating set S is $\frac{17}{12} + \frac{7}{4} + \frac{7}{4} + \frac{13}{12} = 6 = |V(G)|$, as expected, and the average is indeed $\frac{6}{4} = \frac{3}{2}$, the inverse density. Although it is not shown, using $S = V(G)$ in the example graph from Figure 1 would be an ERR:LD set, and using $S = V(G) - \{v_3\}$ would be a DET:LD set.

3.2. Discharging

In proving a maximum (target) value for the average share of all detector vertices, we would ideally like to prove a maximum value for the share of any individual detector vertex, as this would necessarily be a maximum for the average as well. However, in doing so we might run into "problem cases" where the share of a detector vertex exceeds our target. In these cases we must prove that said problem case also induces one or more other detector vertices with sufficiently low share that the average will still be bounded by the target. To prove this bound for the average share, we apply a *discharging argument*, in which we attempt to distribute the excess share of each problematic detector vertex to other detectors with share values below the target. So long as the final result after discharging ensures that each detector has a share value no more than the target, the bound on the average share is proven. This operation is sound because the sum of shares, and therefore the average share, remains the same. The full proof of this soundness is presented in [15].

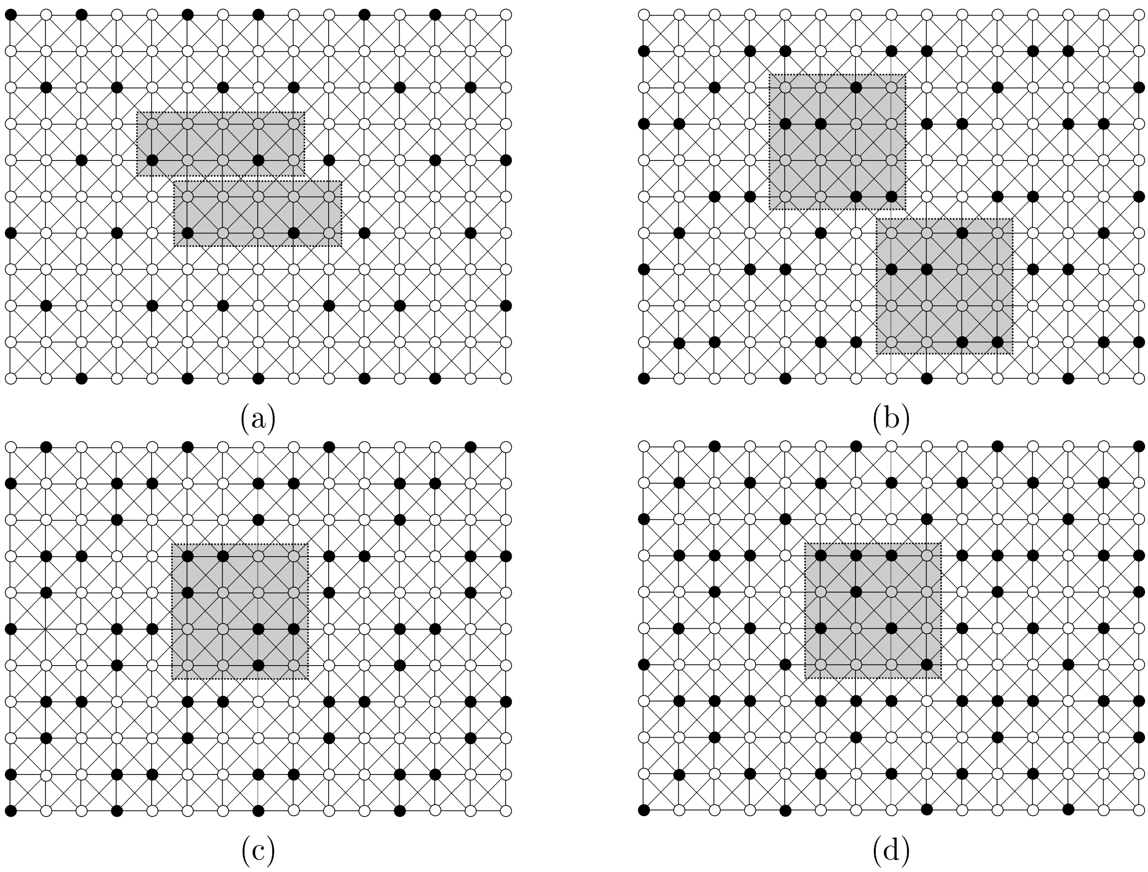
When discussing problem cases, we consider having a "center" vertex, say x, which is assumed to have $\mathrm{sh}(x) > t$ where t is the target. In order for the discharging argument to be valid, we require that $\forall u \in S$, $\widehat{sh}(u) \leq t$, where $\widehat{sh}(u)$ denotes the share value of u after discharging. Naturally, the precise definition of $\widehat{sh}(u)$ depends on the specific method used to perform discharging; in this paper, we allow discharging only to neighbors. For any $v \in N(x) \cap S$ with $\max \mathrm{sh}(v) < t$—where $\max \mathrm{sh}(v)$ denotes the maximum possible share value of v in the current context of which vertices are known to be detectors or non-detectors—v can accept at most $t - \max \mathrm{sh}(v)$ discharge in total. However, there may exist multiple problematic vertices adjacent to v which could attempt to discharge to v, and we must ensure that the total transfer to v does not exceed $t - \max \mathrm{sh}(v)$. To guarantee this, we will conservatively limit the transfer from x to $\frac{1}{k(v)}(t - \max \mathrm{sh}(v))$ where $k(v) \geq 1$ denotes an upper bound on the number of problematic vertices which could simultaneously attempt to discharge to v, including x itself. Thus, the discharging arguments in this paper will use $\widehat{sh}(x) = \mathrm{sh}(x) - \sum_{v \in N(x) \cap S} \frac{1}{k(v)} \max(t - \max \mathrm{sh}(v), 0)$. This ensures that each discharging target, v, which is implied to have $\max \mathrm{sh}(v) < t$, receives at most $\frac{1}{k(v)}(t - \max \mathrm{sh}(v))$ discharge from at most $k(v)$ problematic neighbors, for a total of at most $t - \max \mathrm{sh}(v)$, implying $\widehat{sh}(v) \leq \max \mathrm{sh}(v) + (t - \max \mathrm{sh}(v)) = t$.

This strategy differs from other discharging approaches which have been used in that it works backwards: instead of determining the excess share of a problem vertex and dividing it equally to its neighbors, we determine for each neighbor the amount of discharge it can receive from the problem vertex. Instead of uniform outgoing discharge from the problem case, we have uniform incoming discharge to the non-problem neighbors. This technique is perhaps more powerful for problem cases with asymmetric shares; that is, where the maximum shares of discharge targets are not equal.

Algorithm 1 formalizes the logic behind the discharging technique we have described. Suppose we have some vertex $x \in S$ with $\mathrm{sh}(x) > t$; this problem case can be resolved if Algorithm 1 returns *true*, given arguments x and t. Note that although Algorithm 1 examines all $v \in N(x) \cap S$ to calculate $\widehat{sh}(x)$, this is not strictly necessary as $\widehat{sh}(x) \leq p$, implying $p \leq t$ could be used as an early exit condition.

Algorithm 1 Problem case resolution algorithm

function DISCHARGERESOLVES(x, t) ▷ we assume $x \in S$ with $\mathrm{sh}(x) > t$
  $p \leftarrow \mathrm{sh}(x)$
  for all $v \in N(x) \cap S$ **do** ▷ for any discharge target v
    $s \leftarrow \max \mathrm{sh}(v)$ ▷ find the highest share v can have
    if $s < t$ **then** ▷ check if discharge to v is possible
      $k \leftarrow \max|\{u \in N(v) \cap S : \mathrm{sh}(u) > t\}|$ ▷ max number of discharge sources
      $p \leftarrow p - (t - s)/k$ ▷ safely discharge from x to v
    end if
  end for
  return $p \le t$ ▷ p is now $\widehat{sh}(x)$; check if we resolved
end function



(a) (b) (c) (d)

Fig. 2. Our best constructions of LD (a), RED:LD (b), DET:LD (c), and ERR:LD (d) sets on the King's grid. Shaded vertices denote detectors.

4. Bounds for Fault-Tolerant LD sets on K

In this section we will determine upper and lower bounds for the optimal (minimum) densities of RED:LD, DET:LD, and ERR:LD sets on the infinite king grid, K. Firstly, for the upper bound, we have constructed LD, RED:LD, DET:LD, and ERR:LD sets on the infinite king grid with densities $\frac{1}{5}$, $\frac{5}{16}$, $\frac{6}{16}$, and $\frac{7}{16}$, respectively, as shown in Figure 2 (note that Figure 2 (a) was found to be isomorphic to a previous solution by Honkala and Laihonen [4]). One may verify that each of these constructions are valid detection systems using Theorems 1.1, 1.2, 1.3, and 1.4, respectively.

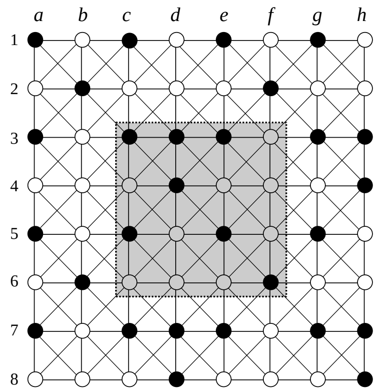


Fig. 3. Example periodic ERR:LD set on K

As an example of verifying these upper bound solutions, we will show Figure 2 (d) is an ERR:LD set using the labeling from Figure 3. First, we must verify that each vertex $u \in V(\mathrm{K})$ is at least 3-dominated. Because S is formed from a periodic tiling of the shaded region of Figure 3, we can assume without loss of generality that u lies within the tile. This simplifies the verification of u being at least 3-dominated from an infinite number of values of u to only 16 equivalence classes, one for each vertex in the tile. We can exhaustively check that each such u is at least 3-dominated, so S satisfies property i) of Theorem 1.4. Next, we will check that every vertex pair $u, v \in V(G)$ with $u \neq v$ is 3-distinguished. If $d(u,v) \geq 3$, then clearly the pair is 6-distinguished because each is at least 3-dominated and no single detector can dominate both of them; thus, we need only consider when $d(u,v) \leq 2$. Similar to the verification of 3-domination, we may assume without loss of generality that u lies within the tile, meaning we need only consider the finite choices of v with $d(u,v) \leq 2$; this is precisely the region of K shown in Figure 3.

As the number of cases for (u,v) is finite, it can be shown by exhaustion that every vertex pair (u,v) is 3-distinguished, implying S is an ERR:LD set on K. For example, take the detector pair (v_{c3}, v_{d3}): aside from themselves, they are dominated by v_{b2}, v_{e3}, and v_{d4}; however v_{d4} is common to both. Thus, v_{c3} and v_{d3} are 2-distinguished; they need only be 1-distinguished, so they are distinguished. As another example, take the detector pair (v_{e5}, v_{d7}): they have no common detectors, and are clearly 5-distinguished, so they are distinguished. Consider the non-detector pair (v_{f4}, v_{f5}); we see they are 3-distinguished by v_{e3}, v_{g3}, and v_{f6}, and therefore meet the minimum requirements to be distinguished. We also see the non-detector pair (v_{f5}, v_{g6}) are 3-distinguished by v_{e5}, v_{g7}, and v_{h7}, and are thus distinguished. Consider the mixed pair (v_{c5}, v_{d5}); we find they are 2-distinguished by v_{b6} and v_{e5}, and thus satisfy the minimum requirements to be distinguished. As another example, we see the mixed pair (v_{d2}, v_{d4}) is 4-distinguished by v_{c1}, v_{e1}, v_{c5}, and v_{e5}, and thus is distinguished. As all pairs can be shown to be distinguished by exhaustion, S is an ERR:LD set. Thus, we have an upper bound for the optimal density: $\mathrm{ERR{:}LD}(\mathrm{K}) \leq \frac{7}{16}$.

Using this same technique of examining only a single tile of the periodic construction and all vertices within distance 2 by exhaustion, we can verify that the tilings in (a), (b), and (c) from Figure 2 satisfy the requirements of Theorems 1.1, 1.2, and 1.3, respectively, so we have the following upper bounds for the optimal densities: RED:LD(K) $\leq \frac{5}{16}$ and DET:LD(K) $\leq \frac{6}{16}$. Figure 2 shows an upper bound of $\frac{1}{5}$ for LD(K), which has already been proven to be optimal [4]; it is included for comparison purposes.

With upper bounds for each parameter now established, we will proceed to explore lower bounds for these sets. If S is a RED:LD set, then Theorem 1.2 guarantees every $v \in V(\mathrm{K})$ must be 2-dominated, so $\mathrm{sh}[v] \leq \frac{1}{2}$; thus, $\forall x \in S$, $\mathrm{sh}(x) \leq 9 \times \frac{1}{2} = \frac{9}{2}$, giving us a simple lower bound: RED:LD(K) $\geq \frac{2}{9}$. Similarly, by Theorems 1.3 and 1.4, we find that DET:LD(K) $\geq \frac{2}{9}$ and ERR:LD(K) $\geq \frac{3}{9} = \frac{1}{3}$. However, these simple lower bounds can be greatly improved by applying the discharging technique we have described. In what follows, we will use this technique to improve these bounds for RED:LD and ERR:LD to $\frac{3}{11}$ and $\frac{15}{38}$, respectively.

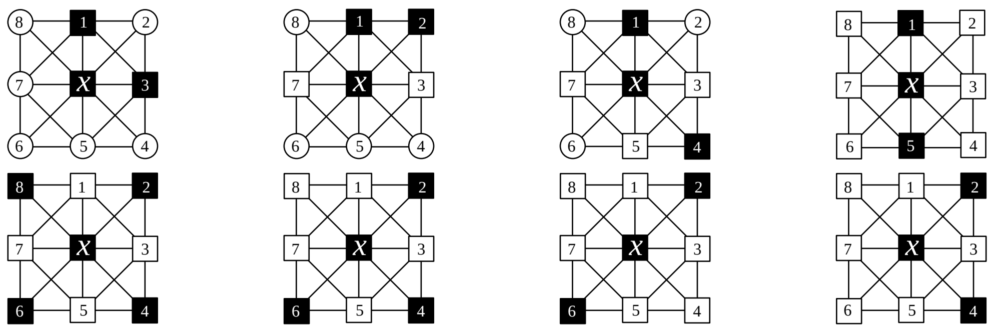


Fig. 4. Configurations around a detector vertex x

Theorem 4.1. *Let $S \subseteq V(G)$ be an ERR:LD set for the king's grid. Then, the average share of all vertices in S is at most $\frac{38}{15}$.*

Proof. Let $S \subseteq V(G)$ be an ERR:LD set on K, let the notations *sh*, *dom*, and *k*-dominated implicitly use S, and let a vertex pair being *distinguished* denote satisfying the relevant pair property of Theorem 1.4. Let $x \in S$ be a detector vertex; Theorem 1.4 yields that $|N[x] \cap S| \geq 3$. There are eight non-isomorphic ways to at least 3-dominate x, as shown in Figure 4. Square vertices represent detectors (shaded) or non-detectors (non-shaded), and round vertices denote unknowns. Note that vertices are labeled with numbers such as k, but will be referenced in the text as v_k.

Case L. First, we consider the partial shares of vertices $A = \{x, v_1, v_2, v_3\}$. We require $|A \cap D_{4+}| \geq 2$ for the vertices in A to be distinguished. Suppose v_2 is a non-detector. Let $d = \min_{w \in \{x, v_1, v_3\}} \mathrm{dom}(w)$. If $d = 3$ then $\mathrm{dom}(v_2) \geq 5$ to be distinguished; if $d = 4$ then $\mathrm{dom}(v_2) \geq 4$ to be distinguished; otherwise $d \geq 5$ and $\mathrm{dom}(v_2) \geq 3$. Thus, $\mathrm{sh}[xv_1v_2v_3] \leq \max\{\sigma_{3445}, \sigma_{3445}, \sigma_{4444}, \sigma_{5553}\} = \frac{31}{30}$. Otherwise, v_2 is a detector; in order to distinguish it from the other three detectors, x, v_1, and v_3, we need $\mathrm{sh}[xv_1v_2v_3] \leq \sigma_{4555} < \frac{31}{30}$. In either case, $\mathrm{sh}[xv_1v_2v_3] \leq \frac{31}{30}$.

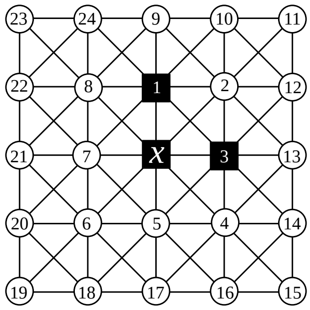


Fig. 5. L configuration.

Next, we consider the partial shares of v_4 and v_5. If both are detectors, then $\mathrm{sh}[v_4v_5] \leq \sigma_{45}$ to be distinguished. If both are non-detectors, then $\mathrm{sh}[v_4v_5] \leq \sigma_{34}$ to be distinguished. Otherwise, we let v_1 be unknown to allow symmetry between v_4 and $v5$; then by symmetry let $\{v_4, v_5\} \cap S = \{v_4\}$; then $\mathrm{sh}[v_4v_5] \leq \max\{\sigma_{35}, \sigma_{44}\}$. Thus, $\mathrm{sh}[v_4v_5] \leq \sigma_{34}$, and by symmetry, $\mathrm{sh}[v_7v_8] \leq \sigma_{34}$ as well. Therefore, $\mathrm{sh}(x) = \mathrm{sh}[xv_1v_2v_3] + \mathrm{sh}[v_4v_5] + \mathrm{sh}[v_6] + \mathrm{sh}[v_7v_8] \leq \sigma_{3445} + \sigma_{34} + \sigma_3 + \sigma_{34} = \frac{38}{15}$, and we are done.

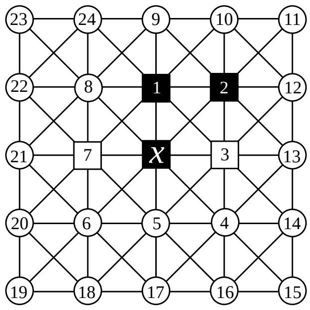


Fig. 6. Triangle configuration.

Case Triangle. Next, we consider the case where x is dominated by two neighbors in a "triangle" shape, as in Figure 6. We will assume that $\{v_3, v_7\} \cap S = \varnothing$, as otherwise we fall into the previous case.

We consider $\mathrm{sh}[xv_1v_2v_3]$, $\mathrm{sh}[v_7v_8]$, and $\mathrm{sh}[v_4v_5v_6]$. First, if $\mathrm{dom}(v_3) = 3$, then $\{x, v_1, v_2\} \subseteq D_{5+}$ to distinguish from v_3, so $\mathrm{sh}[xv_1v_2v_3] \leq \sigma_{3555}$; if $\mathrm{dom}(v_3) = 4$, then $\mathrm{sh}[xv_1v_2v_3] \leq \sigma_{4444}$ to be distinguished; otherwise $\mathrm{dom}(v_3) \geq 5$, in which case $\mathrm{sh}[xv_1v_2v_3] \leq \sigma_{3445}$. Therefore, $\mathrm{sh}[xv_1v_2v_3] \leq \sigma_{3445}$. If $v_8 \in S$, then $\mathrm{sh}[v_7v_8] \leq \max\{\sigma_{35}, \sigma_{44}\} < \sigma_{34}$; otherwise, $\mathrm{sh}[v_7v_8] \leq \sigma_{34}$. If $\{v_4, v_5, v_6\} \cap D_{4+} \neq \varnothing$, then $\mathrm{sh}(x) = \mathrm{sh}[xv_1v_2v_3] + \mathrm{sh}[v_4v_5v_6] + \mathrm{sh}[v_7v_8] \leq \sigma_{3445} + \sigma_{334} + \sigma_{34} = \frac{38}{15}$ and we will be done.

Otherwise, we assume $\{v_4, v_5, v_6\} \subseteq D_3$; we consider the following sub-cases depending on whether these three vertices are detectors or not.

Subcase 1. If $\{v_4, v_5, v_6\} \cap S = \varnothing$, then v_5 needs two additional detectors, but at least one of them will be in $N(v_4)$; thus, v_4 and v_5 are not distinguished, a contradiction.

Subcase 2. If $\{v_4, v_5, v_6\} \cap S = \{v_5\}$, then v_4 and v_6 are not distinguished, a contradiction.

Subcase 3. If $\{v_4, v_5, v_6\} \cap S = \{v_6\}$, then v_5 needs one additional detector, $w \in S$; if $w \in N(v_4)$ then $\{v_4, v_5\}$ are not distinguished, a contradiction; otherwise $w = v_{18}$ and

$\{v_5, v_6\}$ are not distinguished, a contradiction.

Similarly, if $\{v_4, v_5, v_6\} \cap S = \{v_4\}$ then we reach a contradiction.

Subcase 4. If $\{v_4, v_5, v_6\} \cap S = \{v_4, v_5\}$ then v_4 and v_5 are not distinguished, a contradiction; similarly, if $\{v_4, v_5, v_6\} \cap S = \{v_5, v_6\}$ then v_5 and v_6 are not distinguished, contradiction.

Subcase 5. If $\{v_4, v_5, v_6\} \cap S = \{v_4, v_6\}$, consider the third dominator for v_6.

If $v_{17} \in S$ or $v_{18} \in S$ is the third dominator, we get $\mathrm{dom}(v_5) \geq 4$, a contradiction. If $v_{20} \in S$ or $v_{21} \in S$ is the third dominator, then we need $v_8 \in S$ or $v_{22} \in S$ to distinguish v_7 from v_6, which in turn will make $\mathrm{dom}(v_7) \geq 5$ and $\mathrm{dom}(v_8) \geq 4$; thus $\mathrm{sh}(x) = \mathrm{sh}[xv_1v_2v_3] + \mathrm{sh}[v_4v_5v_6] + \mathrm{sh}[v_7v_8] \leq \sigma_{3445} + \sigma_{333} + \sigma_{45} < \frac{38}{15}$. Otherwise, $v_{19} \in S$ is the third dominator. Then we need $v_8 \in S$ or $v_{22} \in S$ to distinguish v_7 from v_5, which will make $\mathrm{dom}(v_7) \geq 4$ and $\mathrm{dom}(v_8) \geq 4$; thus $\mathrm{sh}(x) = \mathrm{sh}[xv_1v_2v_3] + \mathrm{sh}[v_4v_5v_6] + \mathrm{sh}[v_7v_8] \leq \sigma_{3445} + \sigma_{333} + \sigma_{44} = \frac{38}{15}$, completing the proof of the triangle case.

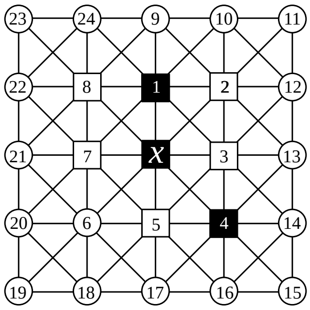


Fig. 7. Bent configuration.

Case Bent. Consider the case where x is dominated by two neighbors in a "bent" shape, as in Figure 7. We will assume that $\{v_2, v_3, v_5, v_7, v_8\} \cap S = \varnothing$, as otherwise we fall into previous cases.

First, we consider when $v_6 \in S$, then $\mathrm{dom}(x) = 4$. We observe that $\{v_3, v_5, v_7\} \subseteq D_{4+}$ to be distinguished from x. We now consider how to 4-dominate v_7. Suppose $v_{20} \notin S$, then $\{v_{21}, v_{22}\} \cap S \neq \varnothing$, and we observe that $\mathrm{dom}(v_8) \geq 5$ to be distinguished from v_7, so $\mathrm{sh}(x) \leq \frac{38}{15}$ and we are done. Thus, we assume that $v_{20} \in S$. If $\{v_{21}, v_{22}\} \cap S \neq \varnothing$ then again $\mathrm{dom}(v_8) \geq 5$ to be distinguished from v_7 and we are done; so we assume $\{v_{21}, v_{22}\} \cap S = \varnothing$. By symmetry, we can assume that $\{v_{12}, v_{13}, v_{14}\} \cap S = \{v_{14}\}$. If $\{v_2, v_8\} \subseteq D_{4+}$ we are done, so without loss of generality we assume $\mathrm{dom}(v_8) = 3$. If $v_{23} \in S$ then v_1 and v_2 cannot be distinguished, a contradiction. Otherwise $v_{24} \in S$ or $v_9 \in S$; in either case, v_1 and v_8 cannot be distinguished, a contradiction. Therefore, $\mathrm{sh}(x) \leq \frac{38}{15}$.

Finally, we consider when $v_6 \notin S$. We observe that $\mathrm{dom}(v_3) \geq 5$ to be distinguished from x, which in turn requires $\mathrm{dom}(v_2) \geq 4$ to distinguish it from v_3; thus, $\mathrm{sh}[xv_2v_3] \leq \sigma_{345}$. Next, we consider $\mathrm{sh}[v_1v_7v_8]$ based on $\mathrm{dom}(v_7)$. If v_7 is 5-dominated, then we need $\mathrm{dom}(v_8) \geq 6$ to distinguish it from v_7. If v_7 is 4-dominated with $\{v_{20}, v_{21}\}$ or $\{v_{20}, v_{22}\}$, we need $\mathrm{dom}(v_8) \geq 5$ to distinguish from v_7; otherwise, v_7 is 4-dominated with $\{v_{21}, v_{22}\}$ and we need $\mathrm{dom}(v_8) \geq 7$ to distinguish from v_7. If v_7 is 3-dominated with v_{21} or v_{22}, we need $\mathrm{dom}(v_8) \geq 6$ to distinguish from v_7; otherwise, v_7 is 3-dominated with v_{21} and we need $\mathrm{dom}(v_8) \geq 4$, which in turn requires $\mathrm{dom}(v_1) \geq 4$. From the three cases of $\mathrm{dom}(v_7)$, we have $\mathrm{sh}[v_1v_7v_8] \leq \sigma_{344}$. If $\{v_4, v_5, v_6\} \cap D_{4+} \neq \varnothing$ then we will be done because $\mathrm{sh}(x) = \mathrm{sh}[xv_2v_3] + \mathrm{sh}[v_4v_5v_6] + \mathrm{sh}[v_1v_7v_8] \leq \sigma_{345} + \sigma_{334} + \sigma_{344} \leq \frac{38}{15}$. Otherwise, we assume $\{v_4, v_5, v_6\} \subseteq D_3$. Then, we observe $\{v_{15}, v_{16}, v_{17}\} \cap S = \varnothing$ because v_4 is already 3-dominated, which in turn requires $v_{18} \in S$ to 3-dominate v_5. We find that v_5 and v_6 are not distinguished, a contradiction, completing the proof for Case Bent.

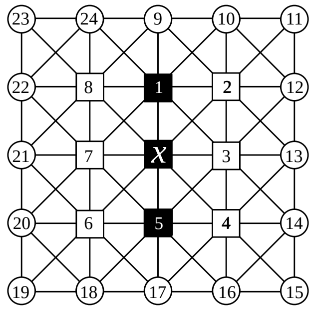


Fig. 8. Vertical configuration.

Case Vertical. Consider the case where when x is dominated in a "vertical" shape, as in Figure 8. We can assume that $\{v_2, v_3, v_4, v_6, v_7, v_8\} \cap S = \varnothing$, as otherwise we fall into previous cases. We observe that $\mathrm{dom}(v_3) \geq 5$ to be distinguished from x. Further, $\{v_2, v_4\} \subseteq D_{4+}$ to be distinguished from v_3, thus $\mathrm{sh}[v_2v_3v_4] \leq \sigma_{454}$. By symmetry, we have $\mathrm{sh}[v_6v_7v_8] \leq \sigma_{454}$, as well. Therefore, $\mathrm{sh}(x) = \mathrm{sh}[xv_1v_5] + \mathrm{sh}[v_2v_3v_4] + \mathrm{sh}[v_7v_8v_9] \leq \sigma_{333} + \sigma_{454} + \sigma_{454} \leq \frac{38}{15}$, and we are done.

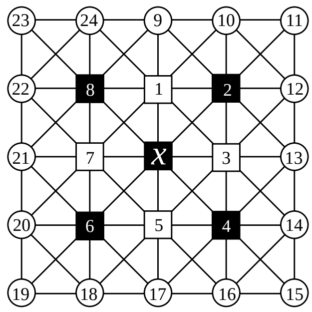


Fig. 9. X configuration.

Case X. Consider the case where x is dominated by four neighbors in a "X" shape, as in Figure 9. We can assume that $\{v_1, v_3, v_5, v_7\} \cap S = \varnothing$, as otherwise we fall into previous cases.

Suppose $\{v_1, v_3, v_5, v_7\} \subseteq D_{4+}$, then $\mathrm{sh}(x) \leq \frac{38}{15}$ and we are done; otherwise, without loss of generality we assume $\mathrm{dom}(v_7) = 3$. Then $\{v_1, v_5\} \subseteq D_{4+}$ to be distinguished from v_7. Suppose $\mathrm{dom}(v_3) \geq 4$. If $v_{14} \notin S$, then $\mathrm{dom}(v_2) \geq 4$ to be distinguished from v_3 and we are done; thus, we assume with symmetry that $\{v_{12}, v_{14}\} \subseteq S$. If $v_{13} \in S$, then $\mathrm{dom}(v_2) \geq 4$ and we are done, so we assume $v_{13} \notin S$. If $\mathrm{dom}(v_2) \geq 4$ we are done, so we assume $\{v_9, v_{10}, v_{11}\} \cap S = \varnothing$, requiring $v_{24} \in S$ to 3-dominate v_1. If $\mathrm{dom}(v_8) \geq 4$ we are done, otherwise we find v_1 and v_8 are not distinguished, a contradiction. Now, we can assume $\{v_3, v_7\} \subseteq D_3$ and examine $\mathrm{dom}(v_1)$ and $\mathrm{dom}(v_5)$. Without loss of generality we can assume $\mathrm{dom}(v_1) \le \mathrm{dom}(v_5)$; then $(\mathrm{dom}(v_1), \mathrm{dom}(v_5)) \in \{(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$ and consider the following four sub-cases: where $\mathrm{dom}(v_5) = 6$, and the other three possibilities.

First, if $\mathrm{dom}(v_5) = 6$, then it requires $\{v_{16}, v_{17}, v_{18}\} \subseteq S$, which implies that $\{v_4, v_6\} \subseteq D_4$, so $\mathrm{sh}(x) = \mathrm{sh}[xv_2v_3v_7v_8] + \mathrm{sh}[v_1v_5] + \mathrm{sh}[v_4v_6] \le \sigma_{53333} + \sigma_{46} + \sigma_{44} \le \frac{38}{15}$. Second, assume $\mathrm{dom}(v_1) = \mathrm{dom}(v_5) = 4$. If $v_9 \in S$ to 4-dominate v_1, then $\{v_2, v_8\} \subseteq D_{4+}$ to distinguish from v_1. Otherwise, $v_{24} \in S$ or $v_{10} \in S$, then $\{v_2, v_8\} \cap D_{4+} \neq \varnothing$ to distinguish from v_1. Hence, $\mathrm{sh}[v_1v_2v_8] \le \sigma_{344}$ and by symmetry, we also have $\mathrm{sh}[v_4v_5v_6] \le \sigma_{344}$. Therefore, $\mathrm{sh}(x) = \mathrm{sh}[xv_3v_7] + \mathrm{sh}[v_1v_2v_8] + \mathrm{sh}[v_4v_5v_6] \le \sigma_{533} + \sigma_{344} + \sigma_{344} = \frac{38}{15}$ and we are done.

Next, assume $\mathrm{dom}(v_1) = 4$ and $\mathrm{dom}(v_5) = 5$. If $v_9 \in S$, then we need $\{v_2, v_8\} \subseteq D_{4+}$ for v_1 to be distinguished from v_2 and v_8, and we get $\mathrm{sh}(x) = \mathrm{sh}[xv_3v_7] + \mathrm{sh}[v_1v_2v_8] + \mathrm{sh}[v_4v_5v_6] \le \sigma_{533} + \sigma_{444} + \sigma_{353} = \frac{149}{60} < \frac{38}{15}$, so we are done; thus, we assume $v_9 \notin S$. We need $v_{10} \in S$ or $v_{24} \in S$ to 4-dominate v_1; by symmetry, let $v_{24} \in S$, which requires $v_{11} \in S$ to 3-dominate v_2 and $v_{23} \in S$ to distinguish v_8 from v_1. If $v_{17} \in S$, then we need $v_{16} \in S$ or $v_{18} \in S$ to 5-dominate v_5; by symmetry, let $v_{16} \in S$, then $\mathrm{dom}(v_4) \ge 5$ to be distinguished from v_5 and we are done. Thus, we assume $v_{17} \notin S$, requiring $\{v_{16}, v_{18}\} \subseteq S$ to 5-dominate v_5. If $v_{15} \in S$ or $v_{19} \in S$, then $\mathrm{sh}(x) = \mathrm{sh}[xv_3v_7] + \mathrm{sh}[v_1v_2v_8] + \mathrm{sh}[v_4v_5v_6] \le \sigma_{533} + \sigma_{434} + \sigma_{453} \le \frac{38}{15}$, and we are done. Therefore, we assume $v_{15} \notin S$ and $v_{19} \notin S$, which leads us to the first problem configuration, which will be handled separately.

Lastly, assume $\mathrm{dom}(v_1) = 5$ and $\mathrm{dom}(v_5) = 5$. Similarly to the previous case, if $v_{17} \in S$, then $\mathrm{sh}(x) \le \frac{38}{15}$ and we are done, so with symmetry we assume $v_{17} \notin S$ and $v_9 \notin S$. Therefore, we require $\{v_{10}, v_{24}, v_{16}, v_{18}\} \subseteq S$ to 5-dominate v_1 and v_5. If $\{v_{11}, v_{15}, v_{19}, v_{23}\} \cap S \neq \varnothing$, then $\mathrm{sh}(x) = \mathrm{sh}[xv_3v_7] + \mathrm{sh}[v_1v_5] + \mathrm{sh}[v_2v_4v_6v_8] \le \sigma_{533} + \sigma_{55} + \sigma_{3334} \le \frac{38}{15}$, and we are done. Thus, we assume $\{v_{11}, v_{15}, v_{19}, v_{13}\} \cap S = \varnothing$, which leads to the second problem configuration, which will be handled separately.

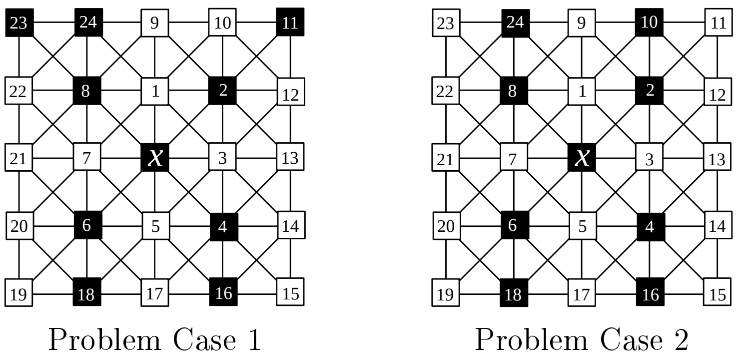


Fig. 10. X configuration problem cases

In either of the Problem Cases, as shown in Figure 10, we see that $\mathrm{sh}(x) \le \max\left\{\frac{77}{30}, \frac{13}{5}\right\} = \frac{13}{5}$. To handle these problems, we consider the maximum share of v_4, which will be the same in either configuration. We observe that we need $\mathrm{dom}(v_{17}) \ge 6$ to distinguish from v_5, $\mathrm{dom}(v_{13}) \ge 4$ to distinguish from v_3, and $\mathrm{sh}[v_{14}v_{15}] \le \sigma_{34}$ to distinguish v_{14} from v_{15}. Thus $\mathrm{sh}(x) = \mathrm{sh}[xv_3v_4v_5v_{17}] + \mathrm{sh}[v_{13}v_{14}v_{15}v_{16}] = \sigma_{53356} + \sigma_{4343} = \frac{12}{5} < \frac{38}{15}$. Additionally, the only detector vertex adjacent to v_4 other than x is v_{16}, which previous cases have shown to have share at most $\frac{38}{15}$. Thus, v_4 can safely accept $\frac{38}{15} - \frac{12}{5} = \frac{2}{15}$ from x. Therefore, $\widehat{sh}(x) \leq \frac{13}{5} - \frac{2}{15} = \frac{37}{15} < \frac{38}{15}$ and we are done.

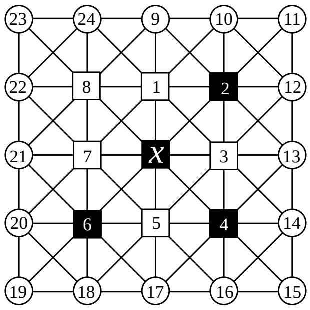


Fig. 11. Lambda configuration

Case Lambda. Consider the case where x is dominated by three neighbors in a "lambda" shape, as in Figure 11. We will assume that $\{v_1, v_3, v_5, v_7, v_8\} \cap S = \varnothing$, as otherwise we fall into previous cases.

Firstly, we observe that $\{v_3, v_5\} \subseteq D_{4+}$ to be distinguish from x; so $\mathrm{sh}[xv_3v_5] \leq \sigma_{444}$. Suppose $\{v_{20}, v_{21}\} \cap S \neq \varnothing$, then we find $\mathrm{sh}[v_6v_7] \leq \sigma_{35}$ to distinguish $\{v_6, v_7\}$. By symmetry, $\{v_9, v_{10}\} \cap S \neq \varnothing$ implies $\mathrm{sh}[v_1v_2] \leq \sigma_{35}$. Therefore, if $\{v_{20}, v_{21}\} \cap S \neq \varnothing$ and $\{v_9, v_{10}\} \cap S \neq \varnothing$, then $\mathrm{sh}(x) \leq \sigma_{444} + \sigma_{35} + \sigma_{35} + \sigma_{33} < \frac{38}{15}$ and we are done; otherwise by symmetry we assume that $\{v_9, v_{10}\} \cap S = \varnothing$, requiring $v_{24} \in S$ to 3-dominate v_1. We then see that $\mathrm{dom}(v_8) \geq 4$ to be distinguished from v_1. Suppose $\{v_{20}, v_{21}\} \cap S \neq \varnothing$ then $\mathrm{sh}[v_6v_7] \leq \sigma_{35}$ and we are done; thus, we assume $\{v_{20}, v_{21}\} \cap S = \varnothing$, requiring $\{v_{22}, v_{23}\} \subseteq S$ to 4-dominate v_8. If $\{v_{12}, v_{13}\} \cap S \neq \varnothing$, then $\mathrm{sh}[v_2v_3] \leq \sigma_{35}$ and we are done; thus, by symmetry, we assume that $\{v_{12}, v_{13}, v_{17}, v_{18}\} \cap S = \varnothing$, requiring $v_{11} \in S$ and $v_{19} \in S$ to 3-dominate v_2 and v_6, respectively. Additionally, we require $v_{14} \in S$ and $v_{16} \in S$ to 4-dominate v_3 and v_5, respectively. If $v_{15} \in S$ then $\mathrm{dom}(v_4) \geq 5$ and we are done, so we assume $v_{15} \notin S$, leading to the configuration shown in Figure 12.

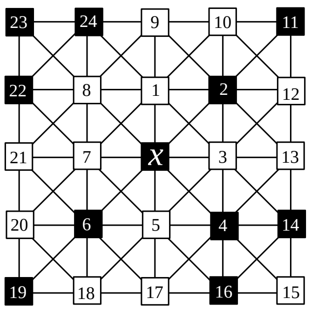


Fig. 12. Lambda configuration problem case.

To handle this problem case, we look at the share of v_4. We observe that we need $\mathrm{dom}(v_{15}) \geq 4$ and $\mathrm{dom}(v_{13}) \geq 5$ to be distinguished from v_4 and v_3, respectively. Therefore, $\mathrm{sh}(v_4) \leq \frac{2}{5} + \frac{5}{4} + \frac{2}{3} = \frac{139}{60}$. Additionally, v_4 is only adjacent to v_{14} and v_{16}, which preceding Cases have proven to have share at most $\frac{38}{15}$. Thus, v_4 has no neighboring detectors with share exceeding $\frac{38}{15}$ other than x itself, so it can safely accept $\frac{38}{15} - \frac{139}{60} = \frac{13}{60}$ discharge from x. Therefore, $\widehat{sh}(x) \leq \frac{31}{12} - \frac{13}{60} = \frac{71}{30} < \frac{38}{15}$ and we are done.

Case Diagonal. Consider the case where x is dominated by two neighbors in a "diagonal" shape, as in Figure 13. We can assume that $\{v_1, v_3, v_4, v_5, v_7, v_8\} \cap S = \varnothing$, as otherwise we fall into previous cases. As this case is perhaps the most complicated, we will begin by proving two smaller claims:

Claim 1. $|\{w \in A : \mathrm{dom}(w) = 3\}| \leq 1$ for $A = \{v_1, v_3\}$ or $A = \{v_5, v_7\}$.

Proof. By symmetry, we need only examine one event: suppose that $\mathrm{dom}(v_1) = \mathrm{dom}(v_3) = 3$; then v_1 and v_3 are not distinguished, a contradiction. □

Claim 2. If $\mathrm{dom}(v_1) \geq k$ or $\mathrm{dom}(v_3) \geq k$ in the diagonal configuration, then $\mathrm{dom}(v_2) \geq k$.

Proof. By Theorem 1.4, we know $k \leq 3$ is always satisfied, and vertices v_1 and v_3 can be at most 5-dominated, so we consider two cases: $k = 4$ and $k = 5$.

For $k = 4$, without loss of generality assume to the contrary that $\mathrm{dom}(v_3) \geq 4$ but $\mathrm{dom}(v_2) \leq 3$. We see that v_2 is already dominated by itself and x, and we consider the third dominator for v_2. If $\{v_9, v_{10}, v_{11}\} \cap S \neq \varnothing$, then $\mathrm{dom}(v_3) < 4$, a contradiction; thus, we assume $\{v_9, v_{10}, v_{11}\} \cap S = \varnothing$. We see that v_2 and v_3 are not distinguished, a contradiction.

For $k = 5$, without loss of generality, assume to the contrary that $\mathrm{dom}(v_3) \geq 5$ but $\mathrm{dom}(v_2) \leq 4$. We require $\{v_{12}, v_{13}, v_{14}\} \subseteq S$ to 5-dominate v_3; thus $\mathrm{dom}(v_2) \geq 4$, so $\mathrm{dom}(v_2) = 4$ and we observe that v_2 and v_3 are not distinguished, a contradiction. □

By symmetry, Claim 2 also applies to $\mathrm{dom}(v_5)$ and $\mathrm{dom}(v_7)$ influencing $\mathrm{dom}(v_6)$.

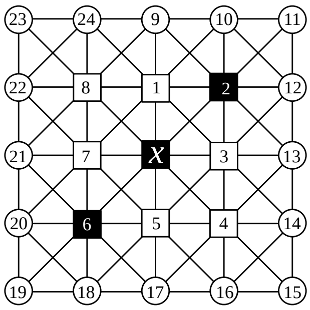
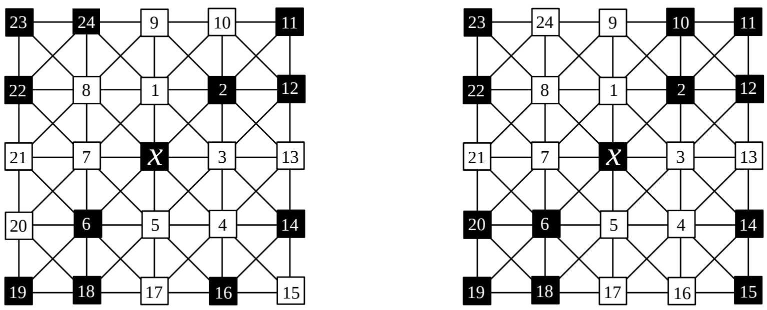


Fig. 13. Diagonal configuration.

Resuming with the diagonal case expansion, suppose $\{v_9, v_{10}\} \cap S = \varnothing$, then $\mathrm{dom}(v_1) = 3$ with $v_{24} \in S$. Then Claim 1 yields that $\mathrm{dom}(v_3) \geq 4$, and Claim 2 yields that $\mathrm{dom}(v_2) \geq 4$. Additionally, $\{v_5, v_7\} \cap D_{4+} \neq \varnothing$, implying $\mathrm{dom}(v_6) \geq 4$. We also require $\mathrm{dom}(v_8) \geq 4$ to be distinguished from v_1. We now have five 4-dominated vertices; if we have any more or if any of them is at least 5-dominated then we are done. Thus, we will assume $\{v_1, v_4\} \subseteq D_3$, $\{v_2, v_3, v_6, v_8\} \subseteq D_4$, and $(\mathrm{dom}(v_5), \mathrm{dom}(v_7)) \in \{(3,4), (4,3)\}$. If $\{v_{17}, v_{18}\} \subseteq S$ then v_5 and v_6 are not distinguished, a contradiction. If $\{v_{17}, v_{18}\} \cap S = \varnothing$, then v_4 and v_5 are not distinguished, a contradiction. Therefore, we assume $|\{v_{17}, v_{18}\} \cap S| = 1$; then $\mathrm{dom}(v_5) = 4$ to be distinguished from v_6, requiring $\mathrm{dom}(v_7) = 3$. If $v_{20} \in S$ then v_6 and v_7 are not distinguished, a contradiction, so we assume $v_{20} \notin S$; similarly, we can assume

$v_{21} \notin S$, requiring $\{v_{22}, v_{23}\} \subseteq S$. If $\{v_{12}, v_{13}\} \subseteq S$ then v_2 and v_3 are not distinguished, a contradiction. If $\{v_{12}, v_{13}\} \cap S = \varnothing$ then $\text{dom}(v_2) = 3$, a contradiction. Therefore, we assume $|\{v_{12}, v_{13}\} \cap S| = 1$, providing symmetry with $\{v_{17}, v_{18}\}$. If $v_{18} \notin S$ then v_4 and v_5 are not distinguished, a contradiction, so we assume $v_{18} \in S$, requiring $v_{17} \notin S$; by symmetry $v_{12} \in S$ and $v_{13} \notin S$. Then we require $\{v_{16}, v_{19}\} \subseteq S$ and $\{v_{11}, v_{14}\} \subseteq S$. Finally, $v_{15} \notin S$ because $\text{dom}(v_4) = 3$. This gives us Problem Case 1, which will be handled separately.

Otherwise, $\{v_9, v_{10}\} \cap S \neq \varnothing$, and by symmetry $\forall A \in \{\{v_{12}, v_{13}\}, \{v_{17}, v_{18}\}, \{v_{20}, v_{21}\}\}$, $A \cap S \neq \varnothing$. Consider the corner $\{v_1, v_2, v_3\}$: we now know that $\text{sh}[v_1v_2] \leq \max\{\sigma_{35}, \sigma_{44}\}$, and similarly for $\text{sh}[v_2v_3]$. Thus, $\text{sh}[v_1v_2v_3] \leq \max\{\sigma_{345}, \sigma_{444}\} = \sigma_{345}$. Suppose $\text{sh}[v_1v_2v_3] \leq \sigma_{444}$, then by symmetry $\text{sh}[v_5v_6v_7] \leq \sigma_{345}$ and we are done. So we can assume $\exists w \in \{v_1, v_2, v_3\}$ such that $\text{dom}(w) = 3$, and by Claims 1 and 2, w is unique. Suppose $w = v_2$, then by Claim 2 we reach a contradiction; without loss of generality let $w = v_1$; then $\text{dom}(v_2) \geq 5$ to distinguish from v_1 because $\{v_9, v_{10}\} \cap S \neq \varnothing$. If $\text{dom}(v_3) \geq 5$ then we are done, so we assume $\text{dom}(v_3) = 4$. If $\text{dom}(v_2) \geq 6$ we are done, so we assume $\text{dom}(v_2) = 5$. With symmetry we can also assume that $\text{dom}(v_6) = 5$, that $(\text{dom}(v_5), \text{dom}(v_7)) \in \{(3, 4), (4, 3)\}$, and that $\text{dom}(v_4) = \text{dom}(v_8) = 3$. If $v_9 \in S$ or $v_{24} \in S$ then v_1 and v_8 are not distinguished, a contradiction; thus, $v_9 \notin S$ and $v_{24} \notin S$, requiring $v_{10} \in S$. If $\text{dom}(v_7) = 3$ then by symmetry $v_{21} \notin S$ and $v_{22} \notin S$, but we see that v_8 cannot be 3-dominated, a contradiction; therefore $\text{dom}(v_7) = 4$, meaning $\text{dom}(v_5) = 3$. By symmetry with $\{v_9, v_{10}, v_{24}\}$, we can assume $\{v_{16}, v_{17}, v_{18}\} \cap S = \{v_{18}\}$. If $v_{21} \in S$ then $v_{20} \in S$ to distinguish v_7 from v_8, however this results in v_6 and v_7 not being distinguished, a contradiction; thus, we assume $v_{21} \notin S$. Then to 5-dominate v_6 and 3-dominate v_8 we require $\{v_{19}, v_{20}, v_{22}, v_{23}\} \subseteq S$. By symmetry, we can also assume that $v_{13} \notin S$ and $\{v_{11}, v_{12}, v_{14}, v_{15}\} \subseteq S$. This results in Problem Case 2, which will be handled separately.



Problem Case 1 Problem Case 2

Fig. 14. Diagonal configuration problem cases

To handle the first "diagonal" problem case, we consider the share of v_6. We observe that we need $\text{dom}(v_{17}) \geq 5$, $\text{dom}(v_{20}) \geq 4$, and $\text{dom}(v_{21}) \geq 4$ to distinguish from v_5, v_6, and v_7, respectively. We also see that $\{v_{18}, v_{19}\} \cap D_{4+} \neq \varnothing$ to distinguish from each other. Therefore, $\text{sh}(v_6) \leq \frac{1}{5} + \frac{5}{4} + \frac{3}{3} = \frac{49}{20}$. Additionally, the only detectors adjacent to v_6 other than x are v_{18} and v_{19}, which preceding Cases have proven to have share at most $\frac{38}{15}$. Thus, v_6 has no neighboring detectors with share exceeding $\frac{38}{15}$ other than x itself, so it can safely accept $\frac{38}{15} - \frac{49}{20} = \frac{1}{12}$ discharge from x. Therefore, $\widehat{sh}(x) \leq \frac{31}{12} - \frac{1}{12} = \frac{5}{2} < \frac{38}{15}$

and we are done.

To handle the other "diagonal" problem case, we consider the share of v_6. We observe that we need $\mathrm{dom}(v_{17}) \geq 4$ and $\mathrm{dom}(v_{21}) \geq 5$ to distinguish from v_7 and v_5, respectively. We also see that $|\{v_{18}, v_{19}, v_{20}\} \cap D_{5+}| \geq 2$ to be distinguished. Therefore, $\mathrm{sh}(v_6) \leq \frac{4}{5} + \frac{3}{4} + \frac{2}{3} = \frac{133}{60}$. Additionally, the only detectors adjacent to v_6 other than x are v_{18}, v_{19} and v_{20}, which preceding Cases have proven to have share at most $\frac{38}{15}$. Thus, v_6 has no neighboring detectors with share exceeding $\frac{38}{15}$ other than x itself, so it can accept $\frac{38}{15} - \frac{133}{60} = \frac{19}{60}$ discharge from x. Therefore, $\widehat{sh}(x) \leq \frac{77}{30} - \frac{19}{60} = \frac{9}{4} < \frac{38}{15}$, completing the diagonal configuration.

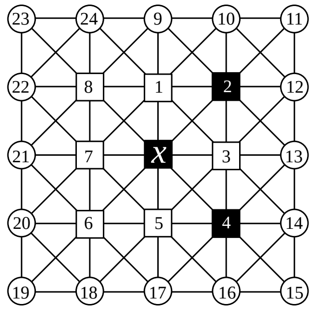


Fig. 15. Wedge configuration.

Case Wedge. As the last case, we consider when x is dominated by two neighbors in a "wedge" shape, as in Figure 15. We can assume that $\{v_1, v_3, v_5, v_6, v_7, v_8\} \cap S = \varnothing$, as otherwise we fall into previous cases.

First, we observe that $\mathrm{dom}(v_3) \geq 5$ to be distinguished from x, and $\mathrm{dom}(v_7) \in \{3, 4\}$. If $\mathrm{dom}(v_7) = 4$, then $\mathrm{dom}(v_6) \geq 5$ is required to be distinguished from v_7; by symmetry, $\mathrm{dom}(v_8) \geq 5$, so $\mathrm{sh}(x) \leq \frac{3}{5} + \frac{1}{4} + \frac{5}{3} \leq \frac{38}{15}$ and we are done. If v_7 is 3-dominated with $\{v_{21}, v_{22}\} \subseteq S$, then we observe that $\mathrm{dom}(v_8) \geq 6$ to be distinguished from v_7, requiring $\{v_9, v_{23}, v_{24}\} \subseteq S$; thus, $\mathrm{dom}(v_1) \geq 4$. We also find that $\mathrm{dom}(v_6) \geq 4$ to be distinguished from v_7. Therefore, $\mathrm{sh}(x) \leq \frac{1}{6} + \frac{1}{5} + \frac{2}{4} + \frac{5}{3} = \frac{38}{15}$ and we are done. By symmetry, we get the same results if v_7 is 3-dominated with $\{v_{20}, v_{21}\} \subseteq S$.

Therefore, we assume v_7 is 3-dominated with $\{v_{20}, v_{22}\} \subseteq S$. We observe that $\mathrm{dom}(v_8) \geq 4$ to be distinguished from v_7; by symmetry, this is also true for v_6. Suppose that $v_{19} \notin S$. Then $\{v_{17}, v_{18}\} \subseteq S$ to 4-dominate v_6, and the only way to distinguish v_5 and v_6 is to have $v_{16} \in S$, so $\mathrm{dom}(v_5) = 5$ and $\mathrm{dom}(v_4) \geq 4$. Therefore, $\mathrm{sh}(x) \leq \frac{2}{5} + \frac{3}{4} + \frac{4}{3} = \frac{149}{60} < \frac{38}{15}$ and we are done; otherwise, we assume $v_{19} \in S$, and by symmetry we also assume $v_{23} \in S$. Next, suppose that $\{v_{18}, v_{24}\} \cap S = \varnothing$, then $v_{17} \in S$ and $v_9 \in S$ to 4-dominate v_6 and v_8, respectively. We observe that we need $\mathrm{dom}(v_2) \geq 5$ to be distinguished from v_1, and by symmetry $\mathrm{dom}(v_4) \geq 5$ as well; thus, $\mathrm{sh}(x) \leq \frac{3}{5} + \frac{2}{4} + \frac{4}{3} = \frac{73}{30} < \frac{38}{15}$ and we are done. Otherwise we can assume that $\{v_{18}, v_{24}\} \cap S \neq \varnothing$, and without loss of generality let $v_{18} \in S$; we now consider the two cases depending on whether v_{24} is a detector.

If $v_{24} \notin S$, then $v_9 \in S$ to 4-dominate v_8 and we observe that $\mathrm{dom}(v_2) \geq 5$ to be distinguished from v_1. If $v_{17} \in S$ then $\mathrm{dom}(v_6) = 5$ and $\mathrm{dom}(v_5) \geq 4$, resulting in $\mathrm{sh}(x) \leq \frac{3}{5} + \frac{2}{4} + \frac{4}{3} = \frac{73}{30} < \frac{38}{15}$ and we are done; otherwise we assume $v_{17} \notin S$. If $\{v_1, v_4, v_5\} \cap D_{4+} \neq \varnothing$ then $\mathrm{sh}(x) \leq \frac{2}{5} + \frac{3}{4} + \frac{4}{3} = \frac{149}{60} < \frac{38}{15}$ and we are done; thus, we

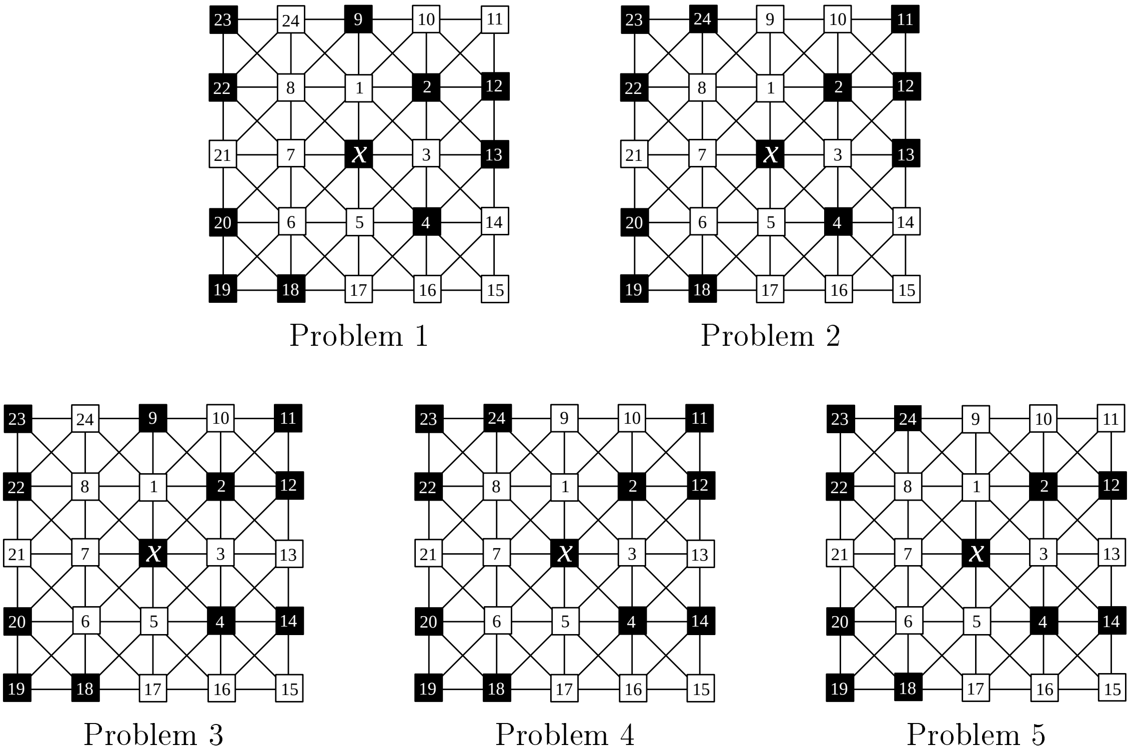


Fig. 16. Wedge configuration problem cases.

can assume that $\{v_1, v_4, v_5\} \subseteq D_3$. Therefore, $v_{10} \notin S$ and $v_{16} \notin S$ because v_1 and v_5 are 3-dominated, respectively. Suppose $v_{13} \in S$; then $\{v_{14}, v_{15}\} \cap S = \varnothing$ because v_4 is 3-dominated, so we require $v_{12} \in S$ to 5-dominate v_3. If $v_{11} \in S$ then $\mathrm{dom}(v_2) = 6$, resulting in $\mathrm{sh}(x) \leq \frac{1}{6} + \frac{1}{5} + \frac{2}{4} + \frac{5}{3} = \frac{38}{15}$ and we are done; otherwise $v_{11} \notin S$ and we arrive at Problem Case 1, which will be handled separately. Otherwise, we consider $v_{13} \notin S$; then $\{v_{12}, v_{14}\} \subseteq S$ to 5-dominate v_3. This requires $v_{15} \notin S$ because v_4 is 3-dominated, and $v_{11} \in S$ to 5-dominate v_2; this leads to Problem Case 3, which will be handled separately.

We now consider when $v_{24} \in S$. Suppose that $v_9 \in S$; then $\mathrm{dom}(v_8) = 5$ and $\mathrm{dom}(v_1) \geq 4$. If $\mathrm{dom}(v_2) \geq 4$ then $\mathrm{sh}(x) \leq \frac{2}{5} + \frac{3}{4} + \frac{4}{3} = \frac{149}{60} < \frac{38}{15}$ and we are done; otherwise, we assume that $\mathrm{dom}(v_2) = 3$. Therefore, $\{v_{10}, v_{11}, v_{12}, v_{13}\} \cap S = \varnothing$, however this results in v_1 and v_2 not being distinguished, a contradiction. Therefore, $v_9 \notin S$, and by symmetry $v_{17} \notin S$ as well. By applying the same argumentation, we can also prove that $v_{10} \notin S$, and by symmetry $v_{16} \notin S$ as well. Suppose $v_{13} \in S$. If $\mathrm{dom}(v_4) = 3$ then $\{v_{14}, v_{15}\} \cap S = \varnothing$, we require $v_{12} \in S$ to 5-dominate v_3 and $v_{11} \in S$ to distinguish v_2 and v_3; this leads to Problem Case 2 and will be handled separately. Otherwise, $\mathrm{dom}(v_4) \geq 4$, and by symmetry $\mathrm{dom}(v_2) \geq 4$ as well, so $\mathrm{sh}(x) \leq \frac{1}{5} + \frac{4}{4} + \frac{4}{3} = \frac{38}{15}$ and we are done. Therefore, we can assume $v_{13} \notin S$; then $\{v_{12}, v_{14}\} \subseteq S$ to 5-dominate v_3. If $\{v_{11}, v_{15}\} \subseteq S$ then $\{v_2, v_4\} \subseteq D_{4+}$, so we are done; otherwise, without loss of generality we assume $v_{15} \notin S$. If $v_{11} \in S$ then we arrive at Problem Case 4, otherwise we arrive at Problem Case 5; both of these will be handled separately later.

Wedge Problem Case 1.

To handle Wedge Problem Case 1 configuration, we consider the maximum share of

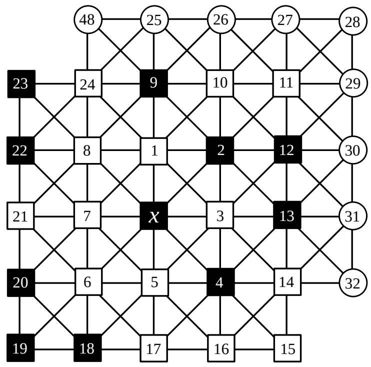


Fig. 17. Problem case 1, expanded.

v_2; we will show that $\mathrm{sh}(v_2) \le \frac{37}{15}$. We observe that $\mathrm{dom}(v_{13}) \ge 5$ and $\mathrm{dom}(v_{10}) \ge 4$ to distinguish from v_3 and v_1, respectively. If $\mathrm{dom}(v_{10}) \ge 5$ then $\mathrm{sh}(v_2) \le \frac{4}{5} + \frac{5}{3} = \frac{37}{15}$ and we are done; otherwise we assume $\mathrm{dom}(v_{10}) = 4$. If $\{v_9, v_{11}\} \cap D_{4+} \ne \varnothing$ then we are done; so we assume $\{v_9, v_{11}\} \subseteq D_3$. If $v_{26} \in S$ or $v_{27} \in S$ to 4-dominate v_{10}, then v_{10} and v_{11} are not distinguished, a contradiction. Thus, $v_{25} \in S$, $v_{26} \notin S$, and $v_{27} \notin S$, but we find that v_9 and v_{10} are not distinguished, a contradiction; therefore, $\mathrm{sh}(v_2) \le \frac{37}{15}$. Additionally, v_2 is only adjacent to three detectors other than x, v_9, v_{12}, and v_{13}. From previous cases, we know that $\mathrm{sh}(v_{12}) \le \frac{38}{15}$ and $\mathrm{sh}(v_{13}) \le \frac{38}{15}$; however, $\mathrm{sh}(v_9)$ might exceed $\frac{38}{15}$. Therefore, we conservatively allow v_2 to accept discharge from two sources—x and potentially v_9—up to $\frac{1}{2}\left[\frac{38}{15} - \frac{37}{15}\right] = \frac{1}{30}$ from each. Thus, $\widehat{sh}(x) = \frac{77}{30} - \frac{1}{30} = \frac{38}{15}$ and we are done.

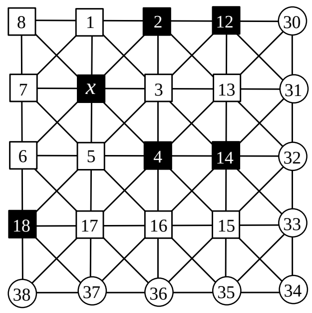


Fig. 18. Problems 3–5 common region.

Wedge Problem Case 2.

To handle this situation, we look at the share of v_2. We observe that $\mathrm{sh}[v_{10}v_{11}] \le \sigma_{35}$ to distinguish v_{10} and v_{11}. Therefore, $\mathrm{sh}(v_2) \le \frac{3}{5} + \frac{2}{4} + \frac{4}{3} = \frac{73}{30}$. Additionally, v_2 is only adjacent to v_{11}, v_{12}, v_{13}, which the "L" and "Triangle" cases have proven to have share at most $\frac{38}{15}$. Thus, v_2 has no neighboring detectors with share exceeding $\frac{38}{15}$ other than x itself, so it can accept $\frac{38}{15} - \frac{73}{30} = \frac{1}{10}$ discharge from x. Therefore, $\widehat{sh}(x) \le \frac{77}{30} - \frac{1}{10} = \frac{37}{15} < \frac{38}{15}$ and we are done.

Wedge Problem Cases 3, 4, and 5.

Consider Problem Cases 3–5, as shown in Figure 16; to handle these cases, we will show that $\mathrm{sh}(v_4) \le \frac{49}{20}$. We observe that $\mathrm{dom}(v_{13}) \ge 6$ to distinguish from v_3, $\mathrm{dom}(v_{17}) \ge 4$ to distinguish from v_5. If $\{v_{35}, v_{36}\} \cap S \ne \varnothing$ then we require $\mathrm{sh}[v_{15}v_{16}] \le \max\{\sigma_{36}, \sigma_{45}\} = \sigma_{36}$ to distinguish v_{15} from v_{16}, so $\mathrm{sh}(v_4) \le \frac{49}{20}$ and we are done; thus, we assume $\{v_{35}, v_{36}\} \cap S =$

$\varnothing$, meaning $\mathrm{dom}(v_{16}) = 3$, so $\mathrm{dom}(v_{15}) \geq 4$. If $\mathrm{dom}(v_{14}) \geq 4$ then we will be done, so we assume $\mathrm{dom}(v_{14}) = 3$. If $v_{30} \notin S$ or $v_{33} \in S$ then we contradict that $\mathrm{dom}(v_{13}) \geq 6$, so we assume $v_{30} \in S$ and $v_{33} \notin S$. Thus, we require $\{v_{32}, v_{34}\} \subseteq S$ to 4-dominate v_{15}, and $v_{31} \notin S$ because $\mathrm{dom}(v_{14}) = 3$. We see that v_{14} and v_{15} are not distinguished, a contradiction. Therefore, we have proven that $\mathrm{sh}(x_4) \leq \frac{49}{20}$. Additionally, the only detector adjacent to v_4 other than x is v_{14}, which previous cases have shown to have share no more than $\frac{38}{15}$; so v_4 can accept $\frac{38}{15} - \frac{49}{20} = \frac{1}{12}$ discharge from x. Consider Problem Cases 3 and 4: $\mathrm{sh}(x) \leq \max\left\{\frac{157}{60}, \frac{77}{30}\right\} = \frac{157}{60}$. Then $\widehat{sh}(x) \leq \frac{157}{60} - \frac{1}{12} = \frac{38}{15}$ and we are done. Consider Problem Case 5: $\mathrm{sh}(x) = \frac{27}{10}$. By symmetry, we can also discharge up to $\frac{1}{12}$ from x into v_2, so $\widehat{sh}(x) \leq \frac{27}{10} - 2 \times \frac{1}{12} = \frac{38}{15}$, completing the proof of Theorem 4.1. $\square$

Theorem 4.2. *Let $S \subseteq V(G)$ be a RED:LD set on K. Then, the average share of all vertices in S is at most $\frac{11}{3}$.*

Proof. Let $S \subseteq V(G)$ be a RED:LD set on K, let the notations *sh*, *dom*, and k-dominated implicitly use S, and let a vertex pair being *distinguished* denote satisfying the relevant pair property of Theorem 1.2. Let $x \in S$ be a detector vertex; Theorem 1.2 yields that $|N[x] \cap S| \geq 2$. There are two non-isomorphic ways to 2-dominate x, either "vertically" or "diagonally."

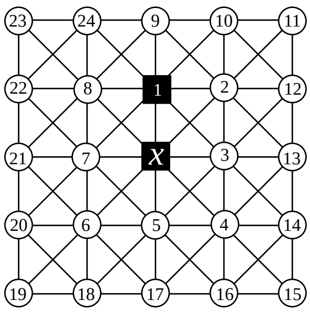


Fig. 19. Vertical configuration.

Case Vertical. Consider the "vertical" configuration given in Figure 19. We begin by examining the vertex pair (v_2, v_3). If $|\{v_2, v_3\} \cap S| = 2$ then $\{x, v_1, v_2, v_3\} \subseteq D_{4+}$, so $\mathrm{sh}(x) \leq \frac{4}{4} + \frac{5}{2} < \frac{11}{3}$ and we are done. Next, suppose $|\{v_2, v_3\} \cap S| = 1$, then $\{x, v_1, v_2, v_3\} \subseteq D_{3+}$. If $\{v_7, v_8\} \cap S \neq \varnothing$ then $\mathrm{sh}(x) \leq \frac{2}{4} + \frac{2}{3} + \frac{5}{2} = \frac{11}{3}$ and we are done, so we assume $\{v_7, v_8\} \cap S = \varnothing$. We observe that if $\{v_7, v_8\} \subseteq D_2$ then they will not be distinguished, so $\mathrm{dom}(v_7) \geq 3$ or $\mathrm{dom}(v_8) \geq 3$ and we are done. Lastly, suppose $|\{v_2, v_3\} \cap S| = 0$; by symmetry we can assume that $\{v_7, v_8\} \cap S = \varnothing$ as well. We require $\mathrm{sh}[v_2 v_3] \leq \max\{\sigma_{33}, \sigma_{24}\}$ to be distinguished; by symmetry, this is also true for $\mathrm{sh}[v_7 v_8]$. If $\mathrm{dom}(v_2) = 2$, then $\{x, v_1\} \subseteq D_{3+}$ and $\mathrm{dom}(v_3) \geq 4$ to distinguish from v_2; thus $\mathrm{sh}(x) \leq \sigma_{24} + \sigma_{24} + \sigma_{33} + \sigma_{222} = \frac{11}{3}$ and we are done. Similarly, we are also done if $\mathrm{dom}(v_3) = 2$. Therefore, we can assume $\{v_2, v_3, v_7, v_8\} \subseteq D_{3+}$. If $\mathrm{dom}(x) \geq 3$ we are done, so $\{v_4, v_5, v_6\} \cap S = \varnothing$. If $\{v_4, v_5, v_6\} \cap D_{3+} \neq \varnothing$ then we will be done, so we assume $\{v_4, v_5, v_6\} \subseteq D_2$. Vertex v_5 must be 2-dominated by exactly one of v_{16}, v_{17}, or v_{18}. If $v_{16} \in S$ or $v_{17} \in S$ then v_4 and

v_5 will not be distinguished, a contradiction; otherwise $v_{18} \in S$, but then v_5 and v_6 are not distinguished, a contradiction.

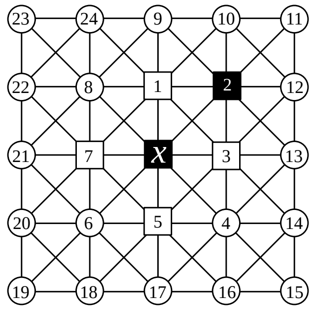


Fig. 20. Diagonal configuration.

Case Diagonal. Consider the other case for 2-dominating x, the "diagonal" configuration show in Figure 20. We can assume that $\{v_1, v_3, v_5, v_7\} \cap S = \varnothing$, as otherwise we fall into the previous case.

We observe that $\mathrm{sh}[v_1v_3] \le \max\{\sigma_{33}, \sigma_{24}\}$ to be distinguished. Suppose $\mathrm{dom}(v_1) = 2$, then $\mathrm{dom}(v_3) \ge 4$ to be distinguished. Additionally, $\{x, v_2\} \subseteq D_{3+}$ to be distinguished from v_1. If $v_6 \in S$ then $\mathrm{sh}[v_5v_7] \le \sigma_{24}$ and we are done; so we assume $v_6 \notin S$. Suppose $v_8 \in S$, then $\mathrm{dom}(v_7) \ge 4$ to distinguish from v_1 and we are done; so we assume $v_8 \notin S$. Thus, $v_4 \in S$ to 3-dominate x. If $\{v_4, v_5\} \subseteq D_2$ then they are not distinguished, so $\mathrm{sh}[v_4v_5] \le \sigma_{23}$ and we are done. Therefore, we can assume that $\{v_1, v_3\} \subseteq D_{3+}$. If $v_6 \in S$ then we can assume $\{v_5, v_7\} \subseteq D_{3+}$, and $\mathrm{dom}(x) \ge 3$, so we are done; thus, we assume $v_6 \notin S$. Next, we consider the vertex pair (v_4, v_8). Suppose $\{v_4, v_8\} \subseteq S$. If $\{v_4, v_5\} \subseteq D_2$ they will not be distinguished, so $\mathrm{sh}[v_4v_5] \le \sigma_{23}$, and by symmetry $\mathrm{sh}[v_7v_8] \le \sigma_{23}$, and we are done. Next, we assume $|\{v_4, v_8\} \cap S| = 1$, and without loss of generality let $v_4 \in S$. We see that $\mathrm{sh}[v_4v_5] \le \sigma_{23}$ to be distinguished. If $\{v_6, v_7, v_8\} \cap D_{3+} \ne \varnothing$ we will be done, so we assume $\{v_6, v_7, v_8\} \subseteq D_2$. Any way to 2-dominate v_7 will result in either (v_7, v_8) or (v_6, v_7) not being distinguished, a contradiction.

Finally, we assume that $\{v_4, v_8\} \cap S = \varnothing$. Suppose $v_{17} \in S$ then $\mathrm{dom}(v_4) \ge 3$ and $\mathrm{dom}(v_6) \ge 3$ to distinguish from v_5. If $\mathrm{dom}(v_5) \ge 3$ we are done, so we assume $\mathrm{dom}(v_5) = 2$, so $\{v_{16}, v_{17}\} \cap S = \varnothing$. Thus, $\mathrm{dom}(v_4) \ge 4$ and $\mathrm{dom}(v_6) \ge 4$ to distinguish from v_5, and we are done; so we assume $v_{17} \notin S$, and by symmetry $v_{21} \notin S$. If $\mathrm{dom}(v_5) \ge 3$, then $\{v_{16}, v_{18}\} \subseteq S$; we require $\mathrm{dom}(v_4) \ge 3$ and $\mathrm{dom}(v_6) \ge 3$ to distinguish from v_5, and we are done, so we can assume $\mathrm{dom}(v_5) = 2$, and by symmetry $\mathrm{dom}(v_7) = 2$. Suppose $v_{16} \in S$, then $v_{18} \notin S$. We require $\mathrm{dom}(v_4) \ge 4$ to distinguish from v_5. If $v_{22} \in S$ as well, then $v_{20} \notin S$ and we find that $\mathrm{dom}(v_8) \ge 4$ to distinguish from v_7, and we are done; thus, we assume $v_{22} \notin S$, so $v_{20} \in S$ to 2-dominate v_7. We find that v_6 and v_7 are not distinguished, a contradiction; therefore, we assume $v_{16} \notin S$, and by symmetry $v_{22} \notin S$. Thus, $\{v_{18}, v_{20}\} \subseteq S$, and $\mathrm{dom}(v_6) = 4$ with $v_{19} \in S$ to distinguish from v_5. Suppose $\{v_{12}, v_{13}\} \cap S \ne \varnothing$; then $\mathrm{dom}(v_2) \ge 3$. If $\mathrm{dom}(v_2) \ge 4$ we are done, so we assume $\mathrm{dom}(v_2) = 3$; thus, $\{v_9, v_{10}, v_{11}\} \cap S = \varnothing$. We require $v_{24} \in S$ to 3-dominate v_1 and $\mathrm{dom}(v_8) \ge 3$ to distinguish from v_1, so we are done. Therefore, we can assume

$\{v_{12}, v_{13}\} \cap S = \varnothing$. Then $v_{14} \in S$ to 3-dominate v_3 and $\mathrm{dom}(v_4) \geq 3$ to distinguish from v_3, and by symmetry we find $\mathrm{dom}(v_8) \geq 3$ as well, completing the proof. □

5. Summary and future directions

We have now explored upper and lower bounds for the optimal densities of three types of fault-tolerant LD sets—RED:LD, DET:LD, and ERR:LD—on the infinite king grid. Table 1 summarizes the bounds proven in this paper, as well as the tight bound for the base (non-fault-tolerant) LD set proven by Honkala and Laihonen [4] for comparison.

Table 1. Summary of results

Parameter	Bounds	Sources
LD(K)	$\frac{1}{5}$	[4]
RED:LD(K)	$\left[\frac{3}{11}, \frac{5}{16}\right]$	Theorem 4.2 and Figure 2 (b)
DET:LD(K)	$\left[\frac{3}{11}, \frac{3}{8}\right]$	Theorem 4.2 and Figure 2 (c)
ERR:LD(K)	$\left[\frac{15}{38}, \frac{7}{16}\right]$	Theorem 4.1 and Figure 2 (d)

We note that the lowerbound of $\frac{3}{11}$ for DET:LD(K) reuses the result of Theorem 4.2 for RED:LD(K) due to the fact that every DET:LD set is also a RED:LD set, implying $\text{RED:LD}(G) \leq \text{DET:LD}(G)$ for any graph G. This fact can be verified by observing that the conditions of Theorem 1.3 imply the conditions of Theorem 1.2.

By computer-assisted search, we believe the same discharging technique used in the proofs of Theorems 4.1 and 4.2 can be used to prove the following stronger bound on DET:LD(K); however, we have not constructed a formal proof of this fact and leave it as future work.

Conjecture 5.1. $\frac{15}{44} \leq \mathit{DET{:}LD}(K)$.

Even using the same discharging technique introduced in this paper, it may be possible to strengthen these lowerbounds in future research. For example, our discharging argument only requires $k(v)$ to be an upperbound on the number of neighbors of v which simultaneously have share exceeding t. However, our current applications of this technique use a conservative upper bound by expanding each neighbor of v separately rather than simultaneously. Essentially, we have used $k'(v) = |\{u \in N(v) \cap S : \max \mathrm{sh}(u) > t\}|$ rather than the possibly tighter $k(v) = \max|\{u \in N(v) \cap S : \mathrm{sh}(u) > t\}| \leq k'(v)$ presented in Algorithm 1. This dramatically reduces casing complexity, but may result in a worse lowerbound than otherwise could be proven using this technique.

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