

Moment sequences and difference equations

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ABSTRACT

We recall the definition and properties of a moment sequence and show that all real sequences whose Hankel matrices have finite rank (see definition in the sequel) satisfy a homogeneous linear equation with constant coefficients. Then we analyze the cases in which a difference equation with constant coefficients and suitably chosen initial conditions and having as an input a positive moment sequence has a solution that is a positive moment sequence. We give one general simple result and give many examples illustrating the theory. The main result states that the roots of the odd multiplicity of the characteristic equation must lie outside the support of the measure that produces the moment sequence that is in the input and the initial conditions suitably chosen.

Keywords: moment, moment sequence, random variable, distribution, difference equation

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1. Introduction

The purpose of this note is to examine the situation when there exists an overlap between a certain important class of real sequences and homogeneous difference equations with constant coefficients and also the cases when a difference equation with a positive moment (pm) sequence as input produces a positive moment sequence. It will turn out that firstly, all sequences having a finite rank of their so-called Hankel matrices satisfy a certain homogeneous difference equation with constant coefficients and secondly, that not every pm sequence produces a pm sequence in this way. This depends on whether the roots of the characteristic equation lie in the support of the measure that generates the input pm sequence and also on the fact that the initial conditions are suitably chosen. In this way, one can use the properties of the solutions of the difference equation with

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given pm input to test whether the roots of the characteristic equation lie in the support of the measure that produced the pm input sequence.

Another purpose of this note is to provide a probabilistic interpretation of the results presented in the excellent paper of Bennett [3] and thus present their sometimes much simpler proofs. We also aim to generalize some of these results and also provide a connection between moment sequences and difference equations.

The paper is organised as follows. In the next section, we recall the definition of a moment sequence and formulate sufficient conditions for a moment sequence to identify the measures that generate it. Then we analyse the set of positive moment (pm) sequences. We give several examples and formulate a series of operations on pm sequences that result in another pm sequence. Many results of this section are known and scattered throughout the literature; we recall them for the sake of completeness of the paper and provide, in cases when it is possible, the new proofs stemming mostly from probability. The following section presents new results on when a difference equation with constant coefficients and a pm input produces a pm sequence. The final section presents several important examples.

2. Moment sequences

Let us agree that whenever the domain of integration is not specified explicitly, it is assumed to coincide with the support of the measure with respect to which the integral is taken.

Let us start with the following definition of the moment sequence.

Let μ be a certain measure on the real line. Let μ^+ and μ^- be the elements of the Jordan-Hahn decomposition of μ , i.e., that μ^+ and μ^- are two positive measures such that $\mu(B) = \mu^+(B) - \mu^-(B)$ for every μ -measurable subset B of the real line. Let us denote $|\mu|$ the sum of the two positive measures μ^+ and μ^- . Let us consider a (signed in general) measure μ such that:

$$\sup_{n \geq 1} \int |x|^n |\mu| (dx) < \infty. \quad (1)$$

Then a sequence $\{m_n\}_{n \geq 0}$ of reals defined by:

$$m_n = \int x^n \mu(dx), \quad (2)$$

is called a moment sequence of the measure μ . Boas theorem (see [6]) states that every sequence is a moment sequence of some signed measure satisfying (1). In fact, Boas's theorem asserts some more precise knowledge of the signed measure μ . Namely, we have:

Theorem 2.1. [6] *Any real sequence of numbers $s = \{s_j\}_{j \geq 0}$ can be represented in the form*

$$s_n = \int_0^\infty x^n \mu(dx),$$

$n = 0, 1, \dots$ with

$$\int_0^\infty |\mu| (dx) < \infty.$$

Hence, for any real sequence $s := \{s_j\}_{j \geq 0}$ we can define the measure μ that generates this sequence according to (2). Any measure (not necessarily defined uniquely) generating sequence s will be called generating measure of this sequence, briefly $gm(s)$.

Further, together with generating measure let us define a sequence of Hankel matrices $\mathcal{H}_n \stackrel{df}{=} [s_{i+j}]_{i,j=0}^n$ and determinants of these matrices $D_n \stackrel{df}{=} \det \mathcal{H}_n$.

Definition 2.2. Let $\{a_i\}_{i \geq 0}$ be a real sequence. The sequence of Hankel matrices of this sequence i.e. $\{\mathcal{H}_n\}_{n \geq 0}$ will be called sequence of the Hankel matrices of the sequence (Hms). The sequence of determinants of the Hms, i.e., $\{D_n\}_{n \geq 0}$ is called Hankel transform of the sequence $\{a_n\}$.

Let us note, that there can be many real sequences having the same Hankel transform. The in-depth analysis of the relationship between a real sequence and its Hankel transform was done e.g. in [13](chapter 5). For our purposes it will be enough, to notice that following the Theorem of Kronecker for a sequence $\{a_n\}_{n \geq 0}$ with the sequence of Hms $\{\mathcal{H}_n\}_{n \geq 0}$ and Hankel transform $\{D_n\}_{n \geq 0}$ to satisfy condition

$$\sup_{n \geq 1} \text{rank}(\mathcal{H}_n) = r < \infty, \quad (3)$$

it is necessary and sufficient that $D_{r-1} \neq 0$ and $\forall n \geq r : D_n = 0$.

Sequences s satisfying condition (3) will be called of finite rank briefly fR-sequences.

Remark 2.3. Notice that, if the support of measure μ_X is finite say consisting of points $\{\alpha_i\}_{i=1}^m$ with masses $\{p_i\}_{i=1}^m$, then each moment m_n has the following form :

$$m_n = EX^n = \sum_{i=1}^m p_i \alpha_i^n,$$

and thus the sequence of Hms has the following form

$$[EX^{i+j}]_{i,j=0}^n = \sum_{k=1}^m p_k [\alpha_k^{i+j}]_{i,j=0}^n.$$

Now notice, that each matrix of the form $[\alpha_k^{i+j}]_{i,j=0,\dots,n}$ has rank 1, hence for every $n \geq m$ matrix $[EX^{i+j}]_{i,j=0,\dots,n}$ has at most rank m . Thus the sequences of such, finitely supported measures, have their Hankel transforms consisting of zeros for elements with indices greater than m . In other words, we have shown that all moment sequences of finitely supported measure are fR-sequences.

Proposition 2.4. Let s be a real sequence. Assume that its $gm(s)$ is finitely supported and the cardinality of the support is r , then there exist r constants $b_0, \dots, b_{r-1}$ such that for $m \geq 0$:

$$\sum_{k=0}^{r-1} b_k s_{k+m} = s_{r+m}. \quad (4)$$

In other words all elements of sequence s satisfy a homogeneous linear equation with constant coefficients (see the definition and properties below in Section 3).

Proof. Assume that the support of $gm(s)$, that we will denote by μ , consists of points $x_1, \dots, x_r \in \mathbb{R}$. Let coefficients b_i , $i = 0, \dots, r-1$ be defined by the relationship.

$$\prod_{j=1}^r (x - x_j) = x^r - \sum_{j=0}^{r-1} b_j x^j.$$

Now notice that the measure v defined by:

$$v(dx) = \prod_{j=1}^r (x - x_j) \mu(dx),$$

is the zero measure. Hence, in particular, we have $\forall m \geq 0$: $\int x^m v(dx) = 0$. But this means that Eq. (4) is satisfied by the sequence s . $\square$

The converse statement also holds. Namely, we have the following theorem that has been formulated and proved in [13](Chapter 5 Thm. 2.3). This theorem can be traced back to Kronecker (from 1881).

Theorem 2.5. *Rank of Hankel transform of a real sequence s is finite iff its generating measure $\mu(dx)$ satisfies the following condition: There exist a polynomial $Q(x)$ such that the measure $v(dx) = Q(x)\mu(dx)$ satisfies the following condition $\forall m \geq 0$:*

$$\int x^m v(dx) = 0. \quad (5)$$

More precisely, let the rank in question be r , then there exist r numbers $b_0, \dots, b_{r-1}$ such that for $m \geq 0$:

$$\sum_{k=0}^{r-1} b_k s_{k+m} = s_{r+m}.$$

In other words there exists a homogeneous difference equation of order r such that all elements of s with indices greater than $r-2$ are defined by this equation with initial conditions $s_0, \dots, s_{r-1}$. Moreover, coefficients $\{b_i\}_{i=0}^{r-1}$ are defined by the elements $s_r, \dots, s_{2r-1}$ of the sequence s . In other words the sequence is uniquely defined by its first $2r$ values. Besides, denoting $Q(x) = x^r - \sum_{j=0}^{r-1} b_j x^j$ and $v(dx) = Q(x)\mu(dx)$ we see that the condition (5) is satisfied.

Remark 2.6. Notice that if we assume that $gm(s)$ is identifiable by moments, then the condition (5) means that the measure $Q(x)\mu(dx)$ is the zero measure and iff polynomial $Q(x)$ has all different and real roots, then from Proposition 2.4 it follows that the measure is finitely supported and their support consists of the root of the polynomial Q .

From now on we will be interested in the cases when measure μ is a positive measure, that is $\mu^- = 0$. The exhaustive study of conditions guaranteeing identifiability by moments of the positive measure was done recently in [11].

Let $\text{supp } \mu$ denote its support and then let us define a sequence of numbers defined in the following way:

$$m_n = \int_{\text{supp } \mu} x^n \mu(dx),$$

for $n \geq 0$. We will call such sequence a p(positive)m(oment) sequence. It is easily seen that by multiplying a pm sequence by a positive number we get another pm sequence. Hence the set of pm sequences forms a cone $\mathcal{M}$ in the space of all sequences. To simplify notation we will denote for simplicity integrals over $\text{supp } \mu$ by simply by $\int$. Obviously, we have $m_{2i} \geq 0$, for $i = 0, 1, 2, \dots$. Using Cauchy-Schwarz inequality, we immediately have:

$$m_n^2 = \left(\int x^{n-j+j} \mu(dx) \right)^2 \leq m_{2(n-j)} m_{2j},$$

for all $0 \leq j \leq n$. Remembering that function $|x|^\alpha$ is convex for all $\alpha > 1$ and after applying Jensen's inequality we get for all $0 < \alpha < \beta$

$$\left(\int |x|^\alpha \mu(dx) \right)^{\beta/\alpha} \leq m_0^{\beta/\alpha-1} \int |x|^\beta \mu(dx).$$

Consequently, for the case $m_0 = 1$, we get for all $j \geq n$

$$m_{2n}^{1/(2n)} \leq m_{2j}^{1/(2j)}.$$

Additionally, if $\text{supp } \mu \subset [0, \infty)$ then obviously we also have seen that $m_n^{1/n}$ constitutes a non-decreasing sequence.

Sometimes, switching to the so-called random variables is more straightforward and intuitive. Namely, it is known that for every positive such measure μ that $\int \mu(dx) = 1$ one can define another measurable space $(\Omega, \mathcal{B}, P)$ (so-called probability space) and measurable mapping $X : \Omega \rightarrow \mathbb{R}$ such that:

$$P(X^{-1}(B)) = \mu(B),$$

for every measurable subset B of the real line. By the way, measure μ is called the distribution of the random variable X and we obviously have:

$$\int_{\Omega} f(X(\omega)) P(d\omega) = \int f(x) \mu(dx),$$

for every real, integrable (mod μ) function f on the real line. The distribution of any random variable is called sometimes a probability measure on the real line. Traditionally, the integral over Ω with respect to the probability measure P is simply denoted by E (-expectations). So we will interchangeably use the notation $Ef(X) = \int f(x) \mu_X(dx)$, where μ_X denotes the distribution of the random variable X .

We have important result called Hamburger moment criterion.

Theorem 2.7. *A real sequence is a pm sequence iff the sequence of its Hms is positive semi-definite.*

Proof. The necessity is simple. Let X be a random variable with a moment sequence $\{a_n\}$, i.e., $a_k = EX^k$, $k \geq 0$. Then for any real sequence $\{\alpha_n\}$ then for any N we have $E \left(\sum_{j=0}^N \alpha_j X^j \right)^2 \geq 0$. But this quantity can be written in the following way:

$$\mathbf{f}_N^T (E\mathbf{X}_N\mathbf{X}_N^T) \mathbf{f}_N \geq 0,$$

where $\mathbf{f}_N^T = (\alpha_0, \dots, \alpha_N)$, $\mathbf{X}_N^T = (1, X, \dots, X^N)$ and $\mathbf{x}^T$ denotes transposition of the vector $\mathbf{x}$. Now, this inequality means that matrices $\{E\mathbf{X}_N\mathbf{X}_N^T\}_{N \leq 0}$ are nonnegative definite, hence their major determinants must be nonnegative.

To prove the converse statement is more complicated. That is we will skip it, since it is not the main topic of the paper. $\square$

Remark 2.8. Let us recall that from the matrix theory it follows that:

- 1) the symmetric matrix is positive definite iff all its leading principal minors are positive,
- 2) the symmetric matrix is positive semi-definite iff all its principal minors are non-negative.

Remark 2.9. Consequently, we can notice that if the sequence of Hankel transforms of a sequence is positive, then the sequence of Hms is positive definite, consequently, the sequence is a pm sequence.

Remark 2.10. On the other hand, we have sequences having non-negative Hankel transforms that are not positive moment sequences. One of them is the following sequence $\{1, 1, 1, 1, 0, 0, \dots\}$. One can see that the Hankel transform of this sequence is $\{1, 0, 0, 1, 0, \dots\}$. The sequence $\{1, 1, 1, 1, 0, 0, \dots\}$ cannot be a pm sequence since $EX^3 = 1$ and $EX^4 = 0!$

Recently Berg and Szwarz in [5] have made a contribution that clarifies the case of non-negativity of the sequence of Hankel transform of a sequence s and the existence of a finitely supported positive measure generating s . Namely, they proved the following result:

Theorem 2.11. [5] *Let $s = \{s_i\}_{i=0}^\infty$ be such a real sequence that its sequence of Hankel transforms $\{D_j\}_{j=0}^\infty$ satisfy the following condition: $\exists r > 0 : D_j > 0$ for $j \leq r - 1$ and $D_j = 0$ for $j \geq r$, then there exists a positive measure μ supported on exactly r points such that s is a pm sequence and $gm(s) = \mu$.*

Following the paper by Bennett [3], we have yet another way of deciding if a given sequence is a moment sequence. Namely, we have the following criterion (compare [1]).

Definition 2.12. Let I be a segment of a real line. A given sequence $\{m_j\}_{j \geq 0}$ is a pm sequence on I iff for every polynomial $\sum_{k=0}^n c_k x^k$ that is nonnegative on I we have

$$\sum_{k=0}^n c_k m_k \geq 0.$$

Remark 2.13. It is well-known that if the cardinality of the support of the measure μ is infinite, then the sequence of Hankel transforms of the moment sequence of the measure μ is strictly positive.

Now, having random variables we can simply utter some rules concerning moment sequences. Most of them were formulated in the paper of Bennett [3]. Some of them had quite complicated proofs. Due to probabilistic interpretation, we can substantially simplify these proofs. Assertions 6. 7. and 8. of the Proposition below seem to be unknown to Bennett. Before we formulate the results let us introduce the following notion

$$h_{i,n}(\chi) = \binom{n}{i} \int_0^1 \theta^i (1-\theta)^{n-i} \chi(d\theta), \quad (6)$$

where with χ being some probability measure on $[0, 1]$. We call numbers $\{h_{i,n}(\chi)\}_{n \geq 0, 0 \leq i \leq n}$ Hausdorff means. We have:

Proposition 2.14. *Let $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be two pm sequences. Then, so are the following sequences:*

1. $\{\alpha a_n + \beta b_n\}_{n \geq 0}$, for $\alpha, \beta \geq 0$.
2. $\left\{ \sum_{i=0}^n h_{i,n}(\chi) \alpha^i a_i \beta^{n-i} b_{n-i} \right\}_{n \geq 0}$.
3. $\{a_n b_n\}_{n \geq 0}$, $\{a_{kn}\}_{n \geq 0}$, $k \in \mathbb{N}$, $c_n = \begin{cases} a_{2k} & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$, $k = 0, 1, \dots$.
4. If a pm sequence $\{a_n\}_{n \geq 0}$ is nonnegative, then also pm is the following sequence:

$$b'_n = \begin{cases} 0 & \text{if } n = 2k+1 \\ a_k & \text{if } n = 2k \end{cases}.$$
5. Suppose $\{a_n\}_{n \geq 0}$ is a pm sequence, then for all $k \geq 1$ the following sequences $\{a_{2k+n}\}_{n \geq 0}$ are the pm sequences.
6. Suppose that for a given pm sequence $\{a_n\}_{n \geq 0}$ we have $a_0 = 1$ and $a_2 = a_1^2$, then $\forall n \geq 0 : a_n = a_1^n$.
7. Suppose that a sequence $\{b_n\}_{n \geq 0}$ is a pm sequence and
 - i) suppose further that $b_0 = 1$, and $b_{4m+2} = b_{2m+1}^2$ for some $m \geq 0$. Then $\forall n \geq 0 :$

$$b_n = b_{2m+1}^{n/(2m+1)}.$$
 - ii) suppose further that $b_0 = 1$ and $b_{4m} = b_{2m}^2$ for some $m \geq 1$. Let us set $p = (b_1 + b_{2m}^{1/(2m)}) / (2b_{2m}^{1/(2m)})$. Then for all $n \geq 0 : b_{2n} = b_{2m}^{n/m}$ and $b_{2n+1} = b_{2m}^{(2n+1)/(2m)} p - b_{2m}^{(2n+1)/(2m)} (1-p)$.
8. Suppose that a sequence $\{b_n\}_{n \geq 0}$ is a pm sequence then for all $n \geq 1 :$

$$b_{2n-2} b_{2n+2} \geq b_{2n}^2.$$

Further, if the measure generating sequence $\{b_n\}$ has infinite support, then the sequence $\{b_{2n}\}$ is log-convex. In particular the following sequence $\{b_{2n+2}/b_{2n}\}_{n \geq 0}$ is non-decreasing.

Proof.

1. Let us define the following matrices $A_n = [a_{i+j}]_{i,j=0,1,\dots,n}$ and $B_n = [b_{i+j}]_{i,j=0,1,\dots,n}$. Since the sequences $\{\det A_n\}_{n \geq 0}$ and $\{\det B_n\}_{n \geq 0}$ are nonnegative by assumption, then so the matrices $\{A_n\}$ and $\{B_n\}$ are nonnegative defined hence so are their convex combinations, that is the sequence $\{\det[\alpha A_n + \beta B_n]\}_{n \geq 0}$ is non-negative.
2. Obviously, the sequence $\{E(\alpha\theta X + \beta(1-\theta)Y)^n\}$ where independent random variables X and Y and independent also of the random variable θ are such that $a_n = EX^n$ and $b_n = EY^n$ for all $n \geq 0$. But we have

$$E(\alpha\theta X + \beta(1-\theta)Y)^n = \sum_{i=0}^n h_{i,n}(\chi) \alpha^i a_i \beta^{n-i} b_{n-i}.$$

3. If two independent random variables X and Y are such that $EX^n = a_n$ and $EY^n = b_n$, the $E(XY)^n = EX^n EY^n$ is also a pm sequence. Notice, that $a_{kn} = E(X^k)^n$. Note, that if X is such a random variable that $a_n = EX^n$, then $(-1)^n a_n = E(-X)^n$. Now, we apply assertion 1 with $\alpha = \beta = 1$, $b_n = (-1)^n a_n$.
4. First notice that if a pm sequence $\{a_n\}_{n \geq 0}$ is nonnegative, then the supporting measure must be concentrated on $[0, \infty)$. We apply assertion 1. with $\alpha = \beta = 1$, $a_n = E(\sqrt{|X|})^n$, $b_n = (-1)^n a_n$ for some random variable X . Now the sequence $\{a_n + b_n\}$ contains zeros for odd indices and $2E|X|^k$ for $n = 2k$. But we can always divide elements of a pm sequence by a positive number. See also [7], p. 40.
5. Let us notice that if for some k we have $a_{2k} = 0$, then the supporting measure of X must be equal to zero, i.e., $\mu(B) = 0$ for all Borel sets B . Consequently, we would have $a_0 = 0$ and the statement assertion would be true. So let us assume that $a_0 > 0$ and consequently that $a_{2k} > 0$ for all $k \geq 0$. Let X be such a random variable that $EX^n = a_n/a_0$. Let $\mu(\cdot)$ denote the distribution of X and let us denote by ν_k the probability measure $x^{2k} d\mu(x)/a_{2k}$. Let us denote Y_k the random variable that has distribution ν_k . Then $EY_k^n = \frac{1}{a_{2k}} \int y^n y^{2k} d\mu(y) = \frac{1}{a_{2k}} EX^{2k+n} = a_{2k+n}/a_{2k}$. hence $\{a_{2k+n}/a_{2k}\}_{n \geq 0}$ is a pm sequence and by assertion 1. also $\{a_{2k+n}\}_{n \geq 0}$ is a pm sequence.
6. Let X denote a random variable whose moments are α_n , that is $EX^n = \alpha_n$. We have $\text{var}(X) = EX^2 - (EX)^2 = \rho^2 - \rho^2 = 0$. But this equality means that the distribution of X is a one-point distribution, i.e. $P(X = \rho) = 1$.
7. For the proof see Lemma 2.2 of [19].
8. Since by the fact that $\{b_n\}_{n \geq 0}$ is a pm sequence, then all central minors of the matrix $[b_{i+j}]_{i,j \geq 0}$ must be nonnegative and moreover, in particular, we must have

$$b_{2n-2}b_{2n+2} \geq b_{2n}^2, \quad (7)$$

for all $n \geq 1$. Now, if the measure that produces the sequence $\{b_n\}$ has infinite support, then $\forall n \geq 0 : b_{2n} > 0$. Hence,

$$\log b_{2(n-1)} + \log b_{2(n+1)} \geq 2 \log b_{2n},$$

it proves log-convexity. On the other hand, we can easily deduce from (7) that $b_{2n+2}/b_{2n} \geq b_{2n}/b_{2n-2}$ since $b_{2n} > 0$. □

Remark 2.15. Let us consider Hausdorff means $\{h_{i,n}(\chi)\}$. We have $h_{i,n}(\chi) \geq 0$ and moreover $\forall n \geq 0 : \sum_{i=0}^n h_{i,n}(\chi) = 1$. Hence, for any number sequence $\{\gamma_n\}_{n \geq 0}$ the sums

$$\sum_{i=0}^n h_{i,n}(\chi) \gamma_i,$$

are the kind of averages.

Let us give some examples.

- i) Take measure χ to be a one-point probability measure concentrated at $t \in (0, 1)$. Then we have $h_{i,n}(\chi) = \binom{n}{i} t^i (1-t)^{n-i}$. Let us take $b_n = c$, and $\alpha = \beta = 1$, then from assertion 2. it follows that $\left\{ c \sum_{j=0}^n \binom{n}{j} t^j a_j (1-t)^{n-j} \right\}$ is a pm sequence provided $\{a_j\}$ is. Further taking $\alpha = 1/2, t = 1/2$, we see that the so-called binomial transform of the sequence $\{a_j\}$, i.e., $\left\{ \sum_{j=0}^n \binom{n}{j} a_j \right\}$ is also a pm sequence. If we take $\alpha = -\beta = -1 = -2t$, then also the following sequence $\left\{ \sum_{j=0}^n (-1)^j \binom{n}{j} a_j \right\}$ which is called inverse binomial transform is a pm sequence.

- ii) Take $\chi(dx) = \beta(1-x)^{\beta-1}$, for some $\beta \geq 0$. Then

$$h_{i,n}(\chi) = \beta \binom{n}{i} \frac{\Gamma(i+1)\Gamma(n-i+\beta)}{\Gamma(n+\beta+1)} = \frac{\beta n! \Gamma(n-i+\beta)}{\Gamma(n+\beta)(n-i)!}.$$

In particular, if $\beta = 1$ we have $h_{i,n}(\chi) = \frac{1}{n+1}$. Hence, following assertion 2. Proposition 2.14 we get

$$\frac{1}{n+1} \sum_{i=0}^n a_i b_{n-i}, \tag{8}$$

is a pm sequence provided $\{a_i\}$ and $\{b_n\}$ are.

Given some two sequences $\{a_i\}_{i \geq 0}$ and $\{b_i\}_{i \geq 0}$ the following sequence $\left\{ \sum_{j=0}^i a_j b_{i-j} \right\}_{i \geq 0}$ is called the *convolution* of the sequences $\{a_i\}_{i \geq 0}$ and $\{b_i\}_{i \geq 0}$. Hence we see that sequence of arithmetic means of the *convolution* of two pm sequences is also a pm sequence.

More particular cases and their applications can be found in [3] (formulae (31)–(34)).

As a corollary from the interesting results of Layman (see [10]) we have the following result.

Lemma 2.16. *The following two pm sequences $\{a_n\}_{n \geq 0}$ and $\left\{ \sum_{j=0}^n \binom{n}{j} (-a_1)^j a_{n-j} \right\}_{n \geq 0}$ have the same Hankel transforms provided $a_0 = 1$.*

Proof. Let us denote elements of the second sequence by $\hat{a}_n$. First if $a_1 = 0$ then the assertion is true. So let us assume that $a_1 \neq 0$. Secondly notice $\det[a_{i+j}]_{i,j=0,1} = a_0 a_2 - a_1^2$, while $\det[\hat{a}_{i+j}]_{i,j=0,1} = a_0(a_2 - a_1^2)$. Hence, to have equality of the two determinants we have to assume $a_0 = 1$. Let us notice that we have

$$\sum_{j=0}^n \binom{n}{j} (-a_1)^j a_{n-j} = (-a_1)^n \sum_{j=0}^n \binom{n}{j} a_{n-j} / (-a_1)^{n-j}. \quad (9)$$

Now notice that $\sum_{j=0}^n \binom{n}{j} a_{n-j} / (-a_1)^{n-j}$ is the binomial transform of the sequence $\{a_j / (-a_1)^j\}_{j \geq 0}$. By the Layman's Theorem 1, these two sequences have the same Hankel transforms. Now, the sequence $\{a_j / (-a_1)^j\}_{j \geq 0}$ has the Hankel transform equal to $\{\det[a_{i+j}]_{0 \leq i,j \leq n} / (-a_1)^{2n}\}$. Hence the sequence $\left\{ \sum_{j=0}^n \binom{n}{j} a_{n-j} / (-a_1)^{n-j} \right\}$ has the same Hankel transform. Now by (9) we deduce that the sequence $\left\{ \sum_{j=0}^n \binom{n}{j} (-a_1)^j a_{n-j} \right\}_{n \geq 0}$ has the Hankel transform equal to $\{(-a_1)^{2n} \det[a_{i+j}]_{0 \leq i,j \leq n} / (-a_1)^{2n}\} = \{\det[a_{i+j}]_{0 \leq i,j \leq n}\}$. $\square$

Remark 2.17. From the above-mentioned lemma it follows that the random variables X and $X - EX$ have the same Hankel transforms. In other words, the sequence of (ordinary) moments of a random variable and the sequence of its central moments have the same Hankel transforms.

As before majority of the assertions of the Proposition below are known. We provide their simple, probabilistic proofs.

Proposition 2.18. *The following sequences are pm sequences:*

- 1) $\forall a \in \mathbb{R} : \{a^n\}$,
- 2) $\{n!\}_{n \geq 0}$,
- 3) $\begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ (2k-1)!! & \text{if } n = 2k \end{cases}, k = 1, 2, \dots$,
- 4) Catalan numbers i.e. $\left\{ \binom{2n}{n} / (n+1) \right\}_{n \geq 0}$,
- 5) $\forall k > -1 : \{1/(n+1)^{k+1}\}_{n \geq 0}$,

- 6) $\forall \alpha \geq 0 : \left\{ (\alpha)^{(n)} \right\}_{n \geq 0}$, where $(\alpha)^{(n)} = \alpha(\alpha+1) \dots (\alpha+n-1)$ is the so called raising factorial of α ,
- 7) $\forall \alpha, \beta > 0 : \left\{ \frac{(\alpha)^{(n)}}{(\alpha+\beta)^{(n)}} \right\}_{n \geq 0}$, consequently $\forall \alpha > 0 : \left\{ (\alpha)^{(n)} / n! \right\}_{n \geq 0}$,
- 8) $\{F_{n+1}\}_{n \geq 0}$, $\{F_{n+3}\}_{n \geq 0}$, $\{F_{n+5}\}_{n \geq 0}$, $\dots$, $\{F_{2n+2}\}_{n \geq 0}$, $\{F_{n+1}/(n+1)\}_{n \geq 0}$, $\{F_{2n+2}/(n+1)\}_{n \geq 0}$, $\{(F_{n+2}-1)/(n+1)\}_{n \geq 0}$, $\{(F_{2n+1}-1)/(n+1)\}_{n \geq 1}$, for any natural k $\{1/F_{2k+n}\}_{n \geq 0}$, where F_n denotes n -th Fibonacci number.
- 9) $\forall \lambda \geq 0 : \left\{ \sum_{j=0}^n \lambda^j \{n_j\} \right\}_{n \geq 0}$ where $\{n_j\}$ denotes Stirling number of the second kind.
- 10) $\{B_n\}_{n \geq 0}$, $\{B_{n+1}\}_{n \geq 0}$ where B_n is Bell number.

Proof.

- 1) $\{a^n\}_{n \geq 0}$ is the moment sequence of a constant random variable.
- 2) $n! = \int_0^\infty x^n \exp(-x) dx$.
- 3) $\begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ (2k-1)!! & \text{if } n = 2k \end{cases} = \int_{-\infty}^\infty x^n \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$.
- 4) $\binom{2n}{n}/(n+1) = \frac{1}{2\pi} \int_0^4 x^n \sqrt{(4-x)/x} dx$.
- 5) $1/(n+1)^{k+1} = \int_0^1 x^n \frac{(-\log(x))^k}{\Gamma(k+1)} dx$.
- 6) $(\alpha)^{(n)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^n x^{\alpha-1} \exp(-x) dx$.
- 7) $\frac{(\alpha)^{(n)}}{(\alpha+\beta)^{(n)}} = \frac{1}{B(\alpha, \beta)} \int_0^1 x^n x^{\alpha-1} (1-x)^{\beta-1} dx$, where $B(\alpha, \beta)$ is the value of beta function at α and β , Now taking $\beta = k - \alpha$ we deduce that $\left\{ \frac{(\alpha)^{(n)}}{(k+n-1)!} \right\}$.
- 8) Fibonacci numbers are defined as the solution of the following difference equation $F_{n+2} = F_{n+1} + F_n$. with $F_0 = 0$, $F_1 = 1$. Hence

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Now

$$F_{n+2k+1} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{2k+1} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{2k+1} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Now notice that $\forall k \geq 0$ both $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{2k+1}$ and $-\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{2k+1}$ are positive. Now we apply assertion 1 of Proposition 2.14 We have also

$$F_{n+3} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^3 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

But $-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^3 = 1 - \frac{2}{\sqrt{5}} > 0$. Similarly we can consider F_{n+5} and generally F_{n+2k+1} .

They constitute pm sequence (no probabilistic in general) since $-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2k+1} > 0$ for $k \geq 0$. Applying third assertion of the above-mentioned Proposition 2.14, we deduce that $\{F_{2n+2}\}_{n \geq 0}$. The fact that $\{F_{n+1}/(n+1)\}_{n \geq 0}$ and $\{F_{2n+2}/(n+1)\}_{n \geq 0}$ are pm sequences follows third assertion of the above-mentioned Proposition 2.14, where $a_n = F_{n+1}$ or $a_n = F_{2n+2}$ and $b_n = 1/(n+1)$. Similarly, using the well known (see e.g. [9]) property of Fibonacci numbers $F_1 + \dots + F_n = F_{n+2} - 1$ and then (8) with $a_n = F_{n+1}$ and $b_n = 1$ we see that $\{(F_{n+2} - 1)/(n+1)\}_{n \geq 0}$ is a pm sequence. Now we apply assertion 3 of Proposition 2.14 and use the fact that $2n+1$. The last case i.e. proof that the sequence $\{1/F_{2k+n}\}_{n \geq 0}$ is a pm sequence for every natural k is treated and proved in the excellent paper by Berg [4].

- 9) We start with the observation that if X has the so-called Poisson distribution with parameter $\lambda \geq 0$, i.e.,

$$P(X = k) = \lambda^k \exp(-\lambda)/k!,$$

then

$$EX^n = \sum_{j=0}^n \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \lambda^j.$$

- 10) Now, we use the fact that the so-called Bell numbers B_n are defined as

$$B_n = \sum_{j=0}^n \left\{ \begin{matrix} n \\ j \end{matrix} \right\}.$$

We also use the well-known fact that

$$B_{n+1} = \sum_{j=0}^n \binom{n}{j} B_j,$$

and apply assertion 2) of the Proposition 2.14 with $h_{i,n}(\chi) = \binom{n}{i}$.

□

Lemma 2.19. *If $\{a_n\}_{n \geq 0}$ is a pm sequence, then the following sequence of polynomials indexed by $n \geq 0$*

$$\sum_{j=0}^{2n} a_j x^j / j!$$

assumes only nonnegative values for $x \in \mathbb{R}$.

Proof. If $x = 0$, then the assertion is true. Hence assume that $x \neq 0$. We consider sequence of Hausdorff means with $\{a_n\}$, $\{b_n = x^{-n}n!\}$, $\alpha = 1/d$, $\beta = 1/(1-d)$, χ being the one point measure concentrated at point $d \in (0, 1)$, Consequently we have

$$h_{i,n}(\chi) = \binom{n}{i} d^i (1-d)^{n-i}.$$

Now following Proposition 2.14(2) we deduce that the following sequence is a pm sequence:

$$\sum_{j=0}^n \binom{n}{j} a_j x^{-n+j} (n-j)! = \frac{n!}{x^n} \sum_{j=0}^n a_j x^j / j!.$$

Now since all elements of a pm sequence with even indexes are positive and since $1/x^{2n} > 0$ for all $x \neq 0$ and $(2n)!$ is positive we get our assertion. $\square$

3. Moment sequences and linear difference equations

First, let us fix the terminology. The best one seems to be taken from the systems theory. So let us consider the following difference equation with constant parameters:

$$\sum_{j=0}^m d_j r_{n+j} = c_n, \quad (10)$$

with complex, in general parameters $\{c_n\}$, d_j , $j = 0, \dots, m$, and $d_m \neq 0$ and with also complex in general so-called initial conditions: $r_0 = p_0, \dots, r_{m-1} = p_{m-1}$.

If $\forall n \geq 0 : c_n = 0$, then the equation is called homogeneous otherwise it is called non-homogeneous.

Recall that we dealt with this type of difference equations, above, when discussing fR-sequences.

For the sake of the completeness of the paper, let us recall the basic properties of the linear difference equations with constant parameters.

Given the so-called system parameters: $\{d_j\}_{j=0}^m$, (sometimes one says about input-output system parameters) the input sequence $\{c_n\}_{n \geq 0}$ and initial conditions, the task is to find the so-called output sequence $\{r_n\}_{n \geq 0}$. The Eq. (10) with the set of initial values is called initial value problem. It is known that every the initial value problem has always a solution, i.e. the sequence $\{r_n\}$ is determined uniquely. In other words, the set of parameters, the input sequence and the set of initial conditions determine the output sequence uniquely.

Moreover, the parameter m is called the order of the difference equation and the following algebraic equation:

$$\sum_{j=0}^m d_j x^j = 0, \quad (11)$$

is called a characteristic equation of the difference Eq. (10). The roots of the characteristic equation are very important since they enable the construction of the solution of the homogeneous equation. The formula for this solution is very simple, if the roots are all different. The case of the different roots of the characteristic equation is also important for the problem of checking if the solution is a pm sequence. Hence we will present this solution for the case of different roots only. For the case of multiple roots we direct the reader to the monograph [8].

Notice that when roots of the characteristic Eq. (11) are all different and equal to $b_j, j = 1, \dots, m$ we have the alternative form of the characteristic equation namely

$$\prod_{j=1}^m (x - b_j) = 0.$$

On the other hand we have expansion

$$\prod_{j=1}^m (x - b_j) = \sum_{j=0}^m x^{m-j} (-1)^j S_j(\mathbf{b}),$$

where $S_j(\mathbf{b})$ denotes a j -th simple symmetric function of the numbers $b_1, \dots, b_m$ with $S_0(\mathbf{b}) = 1$. Hence, we have relationship between parameters $\{d_j\}_{j=0}^m$ and numbers $\{b_j\}_{j=1}^m$.

We start with the homogeneous linear difference equations with constant coefficients. Obviously the following sequence of numbers:

$$r_n = \sum_{k=1}^m \alpha_k (b_k)^n,$$

for different real b_k , and non-negative $\alpha_k, k = 1, \dots, m$ is a pm sequence. By the way, the support of the measure that generates this sequence has finite cardinality, equal to m and consists of points of the set $\{b_1, \dots, b_m\}$. On the other hand, it is known that this sequence, i.e., $\{r_n\}_{n \leq 0}$ is a solution of the following difference equation

$$\sum_{j=0}^m (-1)^j S_j(\mathbf{b}) r_{n+m-j} = 0, \quad (12)$$

with initial conditions $r_0 = p_0, \dots, r_{m-1} = p_{m-1}$ selected in such a way that coefficients $\{\alpha_k\}_{k=1, \dots, m}$ are non-negative. It is known that there exists a direct one-to-one relationship between numbers $\{p_0, \dots, p_{m-1}\}$ and the numbers $\{a_k\}_{k=1, \dots, m}$ with an obvious relationship:

$$\sum_{j=1}^m a_j b_j^k = p_k,$$

for $k = 0, \dots, m-1$.

The fact that in the case of pm sequences only different roots are concerned can be seen, when studying the following example. Let's take $m = 2$ and $b_1 = b_2 = a$ i.e. we consider the following equation

$$r_{n+2} - 2ar_{n+1} + a^2 r_n = 0,$$

with $r_0 = 1$ and $r_1 = r1$. It is not difficult to check that the following sequence

$$r_n = a^{n-1} (nr_1 - a(n-1)),$$

for $n \geq 0$ satisfies the equation. Now, as it is also easy to check, the Hankel transform of this sequence is $1, -(r1 - a)^2, 0, \dots$. Hence, unless $a = r1$, for no real a and $r1$, $\{r_n\}$ is a pm sequence. If $r1 = a$ we have the obviously trivial sequence $r_n = a^n$.

One has to underline that the case of two complex but conjugate roots of the characteristic equation also leads to the non-pm case. More precisely, if one considers, for example, the difference equation:

$$r_{n+2} + a^2 r_n = 0,$$

with $r_0 = 1$, $r_1 = r1$. Then $\{r_n\}$ is not a pm sequence since we have $r_2 = -a^2 < 0$.

The situation becomes more complicated and generally different when one considers nonhomogeneous linear equations with constant coefficients. That is when one considers the following difference equation:

$$\sum_{j=0}^m (-1)^j S_j(\mathbf{b}) r_{n+m-j} = c_n, \quad (13)$$

with $\{c_n\}$ being a pm sequence. First, let us consider the case when $\{c_n\}$ is a pm sequence generated by a discrete distribution with finite support.

As an example, let us take $m = 1$ and $c_n = d^n$.

That is, let us consider such an equation

$$r_{n+1} - ar_n = d^n,$$

with an initial condition $r_0 = p_0$. As it is commonly known the solution of such an equation is given by the formula:

$$r_n = \frac{a^n - d^n}{a - d} + p_0 a^n,$$

for $n \geq 0$. Finding Hankel transform of this sequence we get: $p_0, p_0(d - a) - 1, 0, \dots$. Hence $\{r_n\}$ is a pm sequence iff $p_0 \geq 0$ and $p_0(d - a) - 1 \geq 0$. In particular, if $d > a$ then $p_0 \geq \frac{1}{d-a}$.

Let us note, that if we consider a second-order nonhomogeneous equation of the following form:

$$r_{n+2} + a^2 r_n = b^n,$$

with $r_0 = r0$ and $r_1 = r1$. One can easily split solving this equation into two parts. First to consider the case of even n and then the odd n case. Anyway, solving this equation doesn't cause any difficulty. With the help of Mathematica, one obtains the following sequence of Hankel transform of the sequence $\{r_n\}$. Namely, we get:

$$r0, r0 - a^2 r0 - r1^2, -b^2 (r1^2 + a^2 r0^2) + 2r1b - 1 + 2a^2 r0 - a^2 (a^2 r0^2 + r1^2), 0, \dots$$

Now, it is enough to notice that the polynomial is $-b^2 (r1^2 + a^2 r0^2) + 2r1b - 1 + 2a^2 r0 - a^2 (a^2 r0^2 + r1^2)$ negative for all b . Consequently sequence $\{r_n\}$ cannot be a pm sequence for any b different from zero. We will see in a moment that this is not the case when one considers sequence $\{c_n\}$ being a moment sequence of absolutely continuous measure.

To avoid unnecessary complications we will consider signed measures $d\mu$ defined on the real line that satisfy the so-called Cramer's condition, that is that there exists $\delta > 0$ such that

$$\int \exp(\delta |x|) d|\mu|(x) < \infty. \quad (14)$$

It is known, that if a measure satisfies this condition, then it can be identified by its moments. Let us call the set of such measures Cra .

Now let us consider a positive measure $dA \in Cra$ and a polynomial $Q(x)$ both such that

$$dB(x) = \frac{1}{Q(x)}dA(x). \quad (15)$$

Note that following Proposition 1 of [17] if only $dA \in Cra$ and

$$\int \frac{1}{Q(x)^2}dA(x) < \infty \text{ and } 1/Q(x) \geq 0, \quad (16)$$

on the $\text{supp } A$, then $dB \in Cra$ and it is a positive measure.

Theorem 3.1. *Let sequences, respectively $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be pm sequences generated by the measures dA and $dB \in Cra$, related to one another by (15) with polynomial Q defined by*

$$Q(x) = \sum_{j=0}^m c_j x^j,$$

with $c_m \neq 0$. Then, the sequences are related to one another by the following difference equation:

$$\sum_{j=0}^m c_j b_{n+j} = a_n, \quad (17)$$

with initial conditions

$$b_k = \int \frac{x^k}{Q(x)}dA(x), \quad (18)$$

$k = 0, \dots, m-1$. iff $Q(x) \geq 0$ on $\text{supp } A$.

Proof. We start with an obvious observation that for dA and dB to positive measures, we have to have $Q(x) \geq 0$ on the $\text{supp } A$. Further we have equality: $\forall n \geq 0$: $a_n = \int x^n dA(x)$. Now, we have $a_n = \int Q(x)x^n dB(x) = \sum_{j=0}^m c_j b_{n+j}$. To get unique solution of this equation, we need to set m initial conditions that can be found by calculating m numbers defined by (18). Conversely, assuming that both sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ are pm sequences of measures respectively $dA, dB \in Cra$ we deduce that for $n \geq 0$

$$\int x^n dA(x) = \int \left(\sum_{j=0}^m c_j x^j \right) x^n dB(x).$$

Now, since $dA, dB \in Cra$ we have uniqueness of the measure that generates given sequence of moments. Hence, we deduce that we have

$$dA(x) = Q(x)dB(x).$$

In particular that $Q(x) \geq 0$ on the $\text{supp } A$, since supports of the two measures are the same. $\square$

Remark 3.2. Notice that knowing, say the sequence $\{a_n\}_{n \geq 0}$ and the polynomial $Q(x)$ satisfying (16), we can expand $1/Q(x)$ in an infinite series

$$1/Q(x) = \sum_{j \geq 0} d_j x^j,$$

and then we have

$$b_k = \sum_{j \geq 0} d_j a_{j+k},$$

for $k = 0, \dots, m-1$.

Remark 3.3. In other words the difference Eq. (17) with an input of the form of a pm sequence $\{a_n\}_{n \geq 0}$ generated by a positive measure $dA \in Cra$ has the solution that is also a pm sequence generated by the measure dB also belonging to Cra iff its characteristic polynomial $Q(x) \geq 0$ on $\text{supp } A$. Notice also that from the fact that $dB \in Cra$ it follows that zeros of $Q(x)$ cannot occur at points where dA has atoms, similarly zeros of odd multiplicity cannot occur at points of increase of the measure dA .¹ In particular, it has roots of odd multiplicity outside the support of the measure dA .

4. Remarks and examples

Let us return to the examples that were analyzed above.

Example 4.1. We start with the example with two complex conjugate roots of the characteristic equation. Now, let us consider the following equation:

$$r_{n+2} + a^2 r_n = b^n,$$

with $r_0 = \frac{1}{a^2+b^2}$, $r_1 = r_1$. It turns out that the Hankel transform of r_n is equal to $\frac{1}{a^2+b^2}$, $\frac{(b-r_1(a^2+b^2))(b+r_1(a^2+b^2))}{(a^2+b^2)^2}$, $\frac{-(b-r_1(a^2+b^2))^2}{a^2+b^2}$, $0, \dots$. Hence, unless $r_1 = \frac{b}{a^2+b^2}$ the solution of the above-mentioned equation cannot be a pm sequence. Thus, the fact that the sequence $\{r_n\}$ is a pm sequence heavily depends on the initial conditions.

Let us consider one more example.

Example 4.2. Namely, let us consider similar equation excited, this time by a sequence $\{\frac{1}{n+1}\}_{n \geq 0}$. That is, consider the following difference equation:

$$r_{n+2} + r_n = \frac{1}{n+1}.$$

Now let us recall, that we have:

$$\int_0^1 x^n dx = \frac{1}{n+1}.$$

¹ Point of increase of a positive measure μ is such a point x for which $\mu(B) > 0$ whenever $x \in B$ for an open set B .

So the measure dA has the density equal to 1 for $x \in [0, 1]$ and zero otherwise. According to the theorem above, the initial conditions should be:

$$\begin{aligned} r_0 &= \int_0^1 \frac{1}{x^2 + 1} dx = \frac{\pi}{4}, \\ r_1 &= \int_0^1 \frac{x}{x^2 + 1} dx = \frac{1}{2} \log 2. \end{aligned}$$

We see that given these initial conditions we see that

$$dB(x) = \frac{1}{1 + x^2} \mathbf{1}_{[0,1]}(x) dx,$$

and consequently

$$r_n = \int_0^1 \frac{x^n}{1 + x^2} dx.$$

Now we can get a few first elements of the Hankel transform of the sequence : 0.785398, 0.0484346, 0.000201726, 5.41176×10^{-8} , 9.22425×10^{-13} , 9.9286×10^{-19} , Now, if we only slightly change the value of, say, r_1 by considering, say $r_1 = .01 + .5 \times \log 2$, then, we get the following sequence of Hankel transforms: 0.785398, 0.0414032, 0.00082643, -0.0000171229 , -1.10472×10^{-7} , -5.22732×10^{-11} , -1.57765×10^{-15} .

Remark 4.3. This example suggests that this sensitivity either for the initial conditions or for the fact if $\frac{1}{Q(x)}dA(x)$ is a positive measure can be used in the numerical calculation to test if, for example, the roots of the odd multiplicity of the polynomial $Q(x)$ lie in the support of dA or finding the first values of the integral (18).

Let us return now to the question if a convolution of two pm sequences is a pm sequence. From the formula (8) it follows that the sequence of arithmetic averages of a convolution of two pm sequences is a pm sequence. However, if we consider the simple case of one sequence, say, $\{a_n\}_{n \geq 0}$ is a moments sequence of the distribution dA and the sequence say $\{b^n\}_{n \geq 0}$ then the solution of the difference equation

$$r_{n+1} - br_n = a_n,$$

with $r_0 = r0$ is given by the formula

$$r_n = b^n r0 + \sum_{j=0}^{n-1} b^{n-1-j} a_j.$$

Hence for $r0 = 0$ sequence $\{r_n\}$ is a sequence of convolutions of $\{b^n\}$ and $\{a_n\}$. Moreover, we know from the Theorem 3.1 that it is a moments sequence if only point b lies outside the support the measure dA , more precisely if $dA(x)/(x - b)$ is a positive measure as it follows from Theorem 3.1.

Remark 4.4. Now, let us notice that the scheme in that we have two positive measures absolutely continuous with respect to one another is a general situation. One considers it

in the series of papers [14, 15, 16, 19]. These papers provide many consequences of such assumptions, including infinite expansions of the Radon-Nikodym derivative:

$$\frac{dB}{dA}(x) = \sum_{i \geq 0} c_i \alpha_i(x), \quad (19)$$

and $\{\alpha_i(x)\}_{i \geq 0}$ is the sequence of polynomials orthogonal with respect to the measure dA . The expansion (19) converges in mean-square mod dA provided $\int \left(\frac{dB}{dA}(x)\right)^2 dA(x) < \infty$. Knowing sequences $\{\alpha_i\}$ and $\{\beta_i\}$ one is able to find a numerical sequence $\{c_i\}$ and thus get the expansion (19). Moreover, following the above mentioned positions of literature, there exists a finite linear relationship between two sets of polynomials $\{\alpha_i\}$ and $\{\beta_i\}$ orthogonal respectively to dA and dB , provided $\frac{dB}{dA}(x) = \frac{1}{Q(x)}$. This observation was first made by Pascal Maroni in a more general but more confining context, not necessarily concerning measures. Maroni's approach, followed by his associates in the case of measures concerns mostly polynomials of order at most 4. For details see [12] and [2] or other, later papers of Maroni et al.. More precisely, there exists a table $\{w_{i,n}\}_{N \geq n \geq 0, 0 \leq i \leq n}$ of real numbers such that:

$$\alpha_n(x) = \sum_{j=0}^N w_{j,n} \beta_{n-j}(x),$$

where N is the order of the polynomial $Q(x)$. This might lead to a new difference equation, this time with non-constant coefficients.

Remark 4.5. Note, also, that we could have calculated the moment sequence of dB by expanding $\frac{1}{Q(x)}$ in an infinite power series and integrating term by term getting:

$$\int x^n dB(x) = \sum_{j \geq 0} d_j \int x^{j+n} dA(x), \quad (20)$$

where, of course, $\sum_{j \geq 0} d_j x^j$ is the postulated expansion of $\frac{1}{Q(x)}$. If the series on the right-hand side of the above-mentioned formula is convergent, we have the other relationship combining moments of the measures dA and dB . Such an approach of calculating the sequence of moments of some measure in two different ways can lead to discovering new, interesting relationships, not only in the moment sequences in question but also between sequences describing these moment sequences. This was done for example in the paper [18].

As an example, let us consider two distributions : the semicircle with the density $\frac{1}{2\pi} \sqrt{4-x^2}$ for $|x| \leq 2$ and zero otherwise and the so-called arcsine distribution with the density $\frac{1}{\pi \sqrt{4-x^2}}$ for $|x| < 2$ and 0 otherwise. It is elementary to notice that since these distributions are symmetric their odd moments are equal to zero. Further, it is well-known that even elements of the pm sequence generated by these distributions are respectively the so-called Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ and the so-called central binomial coefficients. Further, we have

$$\frac{1}{2\pi} \sqrt{4-x^2} \times \frac{2}{(4-x^2)} = \frac{1}{\pi \sqrt{4-x^2}}.$$

Hence, taking into account that

$$\frac{2}{(4-x^2)} = \frac{1}{2} \sum_{j \geq 0} x^2/4^j,$$

for $|x| < 2$, the (20) takes the following form:

$$\binom{2n}{n} = \frac{1}{2} \sum_{j \geq 0} C_{n+j}/4^j.$$

Note that the convergence here is very slow since $C_n \cong 4^n/n^{3/2}$.

On the other hand, since we have:

$$\frac{1}{2\pi} \sqrt{4-x^2} = \left(2 - \frac{1}{2}x^2\right) \frac{1}{\pi \sqrt{4-x^2}},$$

we get an obvious relationship:

$$2\binom{2n}{n} - \frac{1}{2}\binom{2n+2}{n+1} = C_n.$$

We also see that the sequence $\{\binom{2n}{n}\}_{n \geq 0}$ is the solution of the following difference equation:

$$s_{n+1} - 4s_n = -2C_n,$$

with an initial condition : $s_0 = 1$. Taking into account general solution of the above-mentioned equation, we end up with the following identity:

$$\binom{2n}{n} = 4^n - \frac{1}{2} \sum_{j=0}^{n-1} 4^{n-j} C_j.$$

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