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Radio number of zero divisor graphs induced by product of three primes

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ABSTRACT

A zero divisor graph on a finite commutative ring $\mathfrak{R}$ is a graph with set of vertices consists of zero divisor elements $Z(\mathfrak{R})$ of the ring, and we have an edge between any two elements in $\mathfrak{R}$ if their product is the zero element. In this work, we will explore some zero divisor graph invariants constructed on the rings of the form $\mathfrak{R} = \mathbb{Z}_n$, when n is a product of square free primes. In particular, we will find the radio number for zero-divisor graphs constructed on $\mathbb{Z}_{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}$ where $\mathfrak{p}_1, \mathfrak{p}_2$, and $\mathfrak{p}_3$ are distinct primes with $2 \leq \mathfrak{p}_3 < \mathfrak{p}_2 < \mathfrak{p}_1$ and combining with the known results for $\mathbb{Z}_{\mathfrak{p}^3}$ and $\mathbb{Z}_{\mathfrak{p}_1^2\mathfrak{p}_2}$, It covers all possible cases when n is divisible by at most three primes.

Keywords: radio labeling, commutative rings, zero divisors

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1. Introduction

The motivation for the study of Radio labeling for graphs arises from Radio Networks. In a radio network, distinct electromagnetic wave frequencies, referred to as radio waves, are assigned to antennas. When we adjust the tuning of a radio receiver, we receive different radio signals. Each radio station is linked to a distinct channel, especially when the stations are relatively closed. In the case of two nearby radio stations, there can be potential interference. To avoid this interference, it is essential to ensure that the associated channels maintain a difference that should be greater than a specific value. The process of association of channels to different transmitters is known as the *Channel Assignment problem* or (CAP) problem.

William Hale [11], presented a model for a Channel Assignment (CAP) settings problem

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in 1980. In the literature, the (CAP) model problem can been converted into a problem of coloring of graphs or graph labeling, where vertices of the graph will represent transmitters and we label these transmitters with either alphabetic or numeric values. When all transmitters are in close proximity, the corresponding two vertices of the graph are said be adjacent. The allocated of transmitters of a channels serve as labels for the graph vertices. In this labeling, there is a fundamental requirement for a minimum gap between two transmitters, consequently, any two graph vertices having their assigned values. The terminal point provides an optimal coloring having a channel range that minimizes the labeling.

Graph labeling finds numerous applications in networks. This is because networks with different nodes possess varying transmission capacities for end-to-end encrypted communication through messages in wired or non-wired connected links. This is the reason why every node is labeled with a unique number, automatically generated by reducing the link label. Indirectly, the edge labeling of the graph indicates the pair of interconnected endpoints. For a device that relies on transmission, a positive (or no-negative in some cases) integer value is associated to various channel to prevent interference among them such channel interferences are particularly pronounced in close-distance communication, such as wireless networks, where the differences amongst channel assignments must be substantial.

A *radio labeling* [8] on a given graph Ω is a map $\psi : V(\Omega) \to \mathbb{Z}^+$ such that for any $u, v \in V(\Omega)$, the following radio condition holds.

$$|\psi(u) - \psi(v)| + d(u, v) \geq diam(\Omega) + 1. \tag{1}$$

The condition given in the above equation is known as the *radio condition*. In the radio condition, the *distance* of any two vertices u_1 and v_1 is denoted by $d(u_1, v_1)$ and the diameter (maximum distance between any two vertices) of a graph Ω is written by $diam(\Omega)$.

The distance between any subsets A and B of the vertex set $V(\Omega)$ is denoted by $d(A, B)$ and defined as the minimum distance $d(u, v)$ for any distinct vertices $u \in A$ and $v \in B$.

Let us denote the set of integers $\{k, k+1, \dots, K\}$ as $S(\Omega, \psi)$ with $k = \min_{u \in V(\Omega)} \psi(u)$ and $K = \max_{v \in V(\Omega)} \psi(v)$. The positive integer $span(\psi) = 1 + K - k$ is known as the span of ψ. The minimum such integer (span) of a graph Ω is written by $rn(\Omega)$, called the *radio number* of the graph Ω and the corresponding labeling for which $span(\psi) = rn(\Omega)$ is known as optimal radio labeling for Ω.

The finding of an optimal radio labeling for Ω is an NP-hard problem even for small families of graphs [12].

Radio labeling amongst the network settings provides efficiency determine the timing of sensor communication or sensor connectivity. This criterion is crucial for secure transmission in cellular networks, security systems, and other similar platforms and known for a particular classes of graphs. The radio number for standard structures of graphs is already known, for example see [3, 14, 15], intensive work has been observed during the last couple of years [6, 9, 13, 16, 17, 18].

Algebraic structures associated with graph invariants are studied in various forms. We will associate a ring structure with graphs through zero divisors. This association was initially considered by Beck [7] in 1988, who explored problems related to coloring on such graphs.

An element $u \neq 0$ is a *zero divisor* (ZD) for any commutative ring $\Re$, if there is another element $v \neq 0$ in $\Re$ so that $u.v = 0$. The set of all (ZD) elements of the ring $\Re$ is generally denoted by $Z(\Re)$. Note that, in the rings of the form $\mathbb{Z}_n$, a non-zero element $v \in \mathbb{Z}_n$, is a (ZD) if and only if $\gcd(v, n) > 1$.

We construct a *(ZD) graph* $\Upsilon(\Re)$ associated to a finite commutative ring $\Re$ and having the set of vertices as $V(\Upsilon(\Re)) = Z(\Re) \setminus \{0\}$ and the corresponding edge set, $E(\Upsilon(\Re))$, is the set of pairs $\{(u, v)|\ u.v = 0\} \subseteq V(\Upsilon(\Re)) \times V(\Upsilon(\Re))$. In this text, we will consider only those (ZD) graphs of that are constructed on commutative finite rings of the type $\mathbb{Z}_n$ where n is a product of distinct primes. Amongst several studies on such graphs, a few can be seen in [1, 4, 5].

The radio number for graphs affiliated with commutative finite rings for some particular classes were studied in [2, 10, 12]. The work is still open for large families of (ZD) graphs and the problem to find a Radio number, even for a smaller networks is non-trivial. The difficulty arises with the fact that the (ZD) graph structure over $\mathbb{Z}_n$ varies independently with the variation of n. However, it is evident that the (ZD) graph $\Upsilon(\mathbb{Z}_n)$ structure depends only on the prime factorization of the integer n.

Let us consider the possible structures $\Upsilon(\mathbb{Z}_n)$ and their radio labeling if n is divisible by at most three primes. If n is just a prime number, then $\mathbb{Z}_n$ forms a field and the corresponding (ZD) graph $\Upsilon(\mathbb{Z}_n)$ is empty.

If the integer n is divisible by any two primes, then either $n = \mathfrak{p}^2$ or $n = \mathfrak{p}_1\mathfrak{p}_2$ and the corresponding (ZD) graph $\Upsilon(\mathbb{Z}_n)$ is either a complete graph $K_{\mathfrak{p}-1}$ or a complete bipartite graph $K_{\mathfrak{p}_1-1,\mathfrak{p}_2-1}$ respectively. Clearly, the radio number in these cases are $rn(K_{\mathfrak{p}-1}) = \mathfrak{p} - 1$ (trivial) and $rn(K_{\mathfrak{p}_1-1,\mathfrak{p}_2-1}) = \mathfrak{p}_1 + \mathfrak{p}_2 - 1$ (because the optimal radio labeling can be obtained by assigning the partition set having $\mathfrak{p}_1 - 1$ elements with labeling $1, 2 \ldots, \mathfrak{p}_1 - 1$ and the partition set having $\mathfrak{p}_2 - 1$ elements with labeling $\mathfrak{p}_1 + 1, \mathfrak{p}_1 + 2, \ldots \mathfrak{p}_1 + \mathfrak{p}_2 - 1$).

Consider a commutative ring of the form $\mathbb{Z}_n$, where n is the product of any three primes. All possible values of n in this case are, $n = \mathfrak{p}^3$, $\mathfrak{p}_1^2\mathfrak{p}_2$ and $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ for some primes $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$.

The radio number for the possibilities $\mathfrak{p}^3$ and $\mathfrak{p}_1^2\mathfrak{p}_2$ are known results.

Theorem 1.1. [10] *For a prime number* $\mathfrak{p}$, *the radio number* $rn(\Upsilon(\mathbb{Z}_{\mathfrak{p}^3})) = \mathfrak{p}^2 + \mathfrak{p} - 2$.

Theorem 1.2. [2] *For any two distinct primes* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *such that* $\mathfrak{p}_1 > \mathfrak{p}_2 \geq 2$. *the radio number* $rn(\Upsilon(\mathbb{Z}_{\mathfrak{p}_1^2\mathfrak{p}_2})) = 2\mathfrak{p}_1^2 + 4\mathfrak{p}_2 - 7$.

In the Section 2, we will investigate a few fundamental invariants of the (ZD) graphs constructed on the rings of the form $\mathbb{Z}_n$, where n is the product of distinct prime numbers. In Section 3, we introduce the required preliminaries and structural properties of the zero-divisor graph associated with $\mathbb{Z}_{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}$. We will also develops the partition of the vertex set into six classes and establishes the relevant distance properties and graph invariants. In Section 4, we derive the lower bound for the radio number and provides an explicit radio-labeling construction yielding the corresponding upper bound. The exact value of the radio number is then established in the main theorem, followed by an illustrative example for $\mathbb{Z}_{30}$.

2. Zero divisor graphs induced by product of distinct primes

In this section, we will construct and study a class of (ZD) graphs constructed over the rings of the form $\mathbb{Z}_n$ for $n = \prod_{i=1}^{k} \mathfrak{p}_i$, product of distinct primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$. We will explore a few basic graph invariants that we will use in our later section.

Consider any k distinct primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$.

Let us denote the ring $\widehat{\mathcal{R}} = \mathbb{Z}_n$ for $n = \prod_{i=1}^{k} \mathfrak{p}_i$ and $K = \{1, 2, \ldots, k\}$ with the powers set $P(K)$ and $\overline{P(K)} := P(K) \setminus \{\phi, K\}$. We also consider a general index set from $I = \{i_1, i_2, \ldots, i_j\} \in \overline{P(K)}$.

For any zero-divisor element $u \in \widehat{\mathcal{R}} \setminus \{0\}$, at least one $\mathfrak{p}_i$ must divides u (not all). We partition the set of vertices of $\Upsilon(\widehat{\mathcal{R}})$ in the form of following subsets.

$$V_I = \Big\{ u \in \widehat{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_i \mid u \text{ for all } i \in I, \ \text{ but } \mathfrak{p}_j \nmid u \text{ if } j \notin I \Big\}. \tag{2}$$

Note that, for any two distinct vertices $u \in V_I$ and $u' \in V_{I'}$ (I and I' can be same), the distance $d(u, v) = d(V_I, V_{I'})$.

Lemma 2.1. *The number of elements in $V_I \subseteq V(\Upsilon(\widehat{\mathcal{R}}))$ defined in (2).*

$$|V_I| = \prod_{i \notin I} (\mathfrak{p}_i - 1). \tag{3}$$

Proof. Since all primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are distinct, therefore, by considering the map $x(\text{mod } n) \mapsto (x(\text{mod } \mathfrak{p}_1),\ x(\text{mod } \mathfrak{p}_2), \ldots, x(\text{mod } \mathfrak{p}_k))$ (Chinese Remainder Theorem),

$$\widehat{\mathcal{R}} \cong \mathbb{Z}_{\mathfrak{p}_1} \times \mathbb{Z}_{\mathfrak{p}_2} \times \ldots \times \mathbb{Z}_{\mathfrak{p}_k}.$$

Using the construction of V_I in (2), each vertex $u \in V_I$ has zero coordinates in positions indexed by I and nonzero coordinates elsewhere. Hence,

$$|V_I| = \prod_{i \notin I} (\mathfrak{p}_i - 1).$$

$\square$

There are $2^k - 2$ such subsets that forms a partition of the set of vertices, we have $V(\Upsilon(\widehat{\mathcal{R}})) = \coprod_{I \in \overline{P(K)}} V_I$. It leads us to the following result.

Lemma 2.2. *The order (number of vertices) of the (ZD) graph $V(\Upsilon(\widehat{\mathcal{R}}))$ is;*

$$|V(\Upsilon(\widehat{\mathcal{R}}))|=\sum_{I\in\overline{P(K)}}|V_I|=\sum_{I\in\overline{P(K)}}\left(\prod_{i\notin I}(\mathfrak{p}_i-1)\right).$$

Lemma 2.3. *Any two vertices $u,\ u'\in V(\Upsilon(\widehat{\mathcal{R}}))$ are adjacent if and only if there exists two subsets $I,\ I'\in\overline{P(K)}$ such that $u\in V_I,\ u'\in V_{I'}$ and $I\cup I'=K$.*

Proof. Consider two vertices $u,\ u'\in V(\Upsilon(\widehat{\mathcal{R}}))$, take $I=\{i\in K:\ \mathfrak{p}_i\mid u\}$ and $I'=\{j\in K:\ \mathfrak{p}_j\mid u'\}$, implies $u\in V_I$ and $u'\in V_{I'}$. Clearly, $I\cup I'\subseteq K$.

If u and u' are adjacent in $\Upsilon(\widehat{\mathcal{R}})$, then $u.u'=0$ in $\widehat{\mathcal{R}}\Rightarrow\prod_{t=1}^{k}\mathfrak{p}_t\mid(u.u')$. It implies that for any $t\in K=\{1,2,\ldots,k\}$, $\mathfrak{p}_t\mid(u.u')\Rightarrow\mathfrak{p}_t\mid u$ or $\mathfrak{p}_t\mid u'\Rightarrow t\in I$ or $t\in I'\Rightarrow t\in I\cup I'\Rightarrow I\cup I'=K$.

Conversely, if $I\cup I'=K$, then for every $t\in K\Rightarrow t\in I$ or $t\in I'\Rightarrow\mathfrak{p}_t\mid u$ or $\mathfrak{p}_t\mid u'\Rightarrow\mathfrak{p}_t\mid(u.u')\Rightarrow\prod_{t=1}^{k}\mathfrak{p}_t|(u.u')\Rightarrow u.u'=0$ in $\widehat{\mathcal{R}}\Rightarrow u$ and u' are adjacent in $\Upsilon(\widehat{\mathcal{R}})$. □

Lemma 2.4. *The degree of each vertex $u\in V_I\subseteq V(\Upsilon(\widehat{\mathcal{R}}))$ is;*

$$\deg(u)=\left(\prod_{i\in I}\mathfrak{p}_i\right)-1.$$

Proof. Following the similar argument given in Lemma 2.1. If a vertex $u\in V_I$ is connected with a vertex u', then $\mathfrak{p}_i|u'$ for all $i\notin I$ and hence maps to zero coordinates in positions indexed by $K\setminus I$. Clearly, we have $\prod_{i\in I}\mathfrak{p}_i$ number of such elements and hence we obtained the required result by excluding zero element. □

Lemma 2.5. *Consider the ring $\widehat{\mathcal{R}}=\mathbb{Z}_n$ for $n=\prod_{i=1}^{k}\mathfrak{p}_i,\ k\geq 3$. Diameter of the (ZD) graph $\Upsilon(\widehat{\mathcal{R}})$ is;*

$$diam(\Upsilon(\widehat{\mathcal{R}}))=3.$$

Proof. For any two vertices $u,v\in V(\Upsilon(\widehat{\mathcal{R}}))$ if $\mathfrak{p}_r|u$ and $\mathfrak{p}_s|v$ with $r\neq s$, then we have vertices $w=\frac{n}{\mathfrak{p}_r}$ and $w'=\frac{n}{\mathfrak{p}_s}$ such that we always have a path $(u-w-w'-v)$, it follows that every two vertices are connected by a path of length at most 3 and hence $diam(\Upsilon(\widehat{\mathcal{R}}))\leq 3$. In order to complete the proof, we only have to show the existence of a shortest path of length 3.

Since $k\geq 3$, we consider two vertices $\mathfrak{p}_1$ and $\mathfrak{p}_k$. By Lemma 2.3, if $\mathfrak{p}_1$ is adjacent to a vertex u, then u must be a multiple of $\prod_{i=2}^{k}\mathfrak{p}_i$ and $\mathfrak{p}_1\nmid u$ (otherwise $u=0$). Similarly, if $\mathfrak{p}_k$ is adjacent to a vertex v, then v must be a multiple of $\prod_{i=1}^{k-1}\mathfrak{p}_i$ and $\mathfrak{p}_k\nmid v$ (otherwise $v=0$).

Hence a shortest path between $\mathfrak{p}_1$ and $\mathfrak{p}_k$ is $(\mathfrak{p}_1 - \prod_{i=2}^{k} \mathfrak{p}_i - \prod_{i=1}^{k-1} \mathfrak{p}_i - \mathfrak{p}_k)$ that is of length exactly 3, which completes the proof. $\square$

3. Zero divisor graphs induced by product of Three distinct primes

In this section we will explicitly discuss some (ZD) graph invariant constructed over a modulo ring induced by the product of three distinct primes.

Consider any three distinct primes $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ and without loss of generality, we assume that $2 \leq \mathfrak{p}_3 < \mathfrak{p}_2 < \mathfrak{p}_1$ in this and following section.

Let us denote the ring $\widetilde{\mathcal{R}} = \mathbb{Z}_n$ for $n = \prod_{i=1}^{3} \mathfrak{p}_i$. To compute the Radio number on the given (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$, we first study the structure and basic graph invariants for (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$. We will write a partition of the set of vertices $V(\Upsilon(\widetilde{\mathcal{R}}))$ in the form of subsets described in Eq. (2).

$$\begin{aligned}
V_1 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_1 \mid u, \ \mathfrak{p}_2 \mid u \text{ but } \mathfrak{p}_3 \nmid u\right\},\\
V_2 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_1 \mid u, \ \mathfrak{p}_3 \mid u \text{ but } \mathfrak{p}_2 \nmid u\right\},\\
V_3 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_2 \mid u, \ \mathfrak{p}_3 \mid u \text{ but } \mathfrak{p}_1 \nmid u\right\},\\
V_4 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_1 \nmid u, \ \mathfrak{p}_2 \nmid u \text{ but } \mathfrak{p}_3 \mid u\right\},\\
V_5 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_1 \nmid u, \ \mathfrak{p}_3 \nmid u \text{ but } \mathfrak{p}_2 \mid u\right\},\\
V_6 &= \left\{u \in \widetilde{\mathcal{R}} \setminus \{0\} : \ \mathfrak{p}_2 \nmid u, \ \mathfrak{p}_3 \nmid u \text{ but } \mathfrak{p}_1 \mid u\right\}.
\end{aligned}$$

For each set V_i of partition of $\Upsilon(\widetilde{\mathcal{R}})$, the divisibility condition, cardinality and degree of each vertex is provided in the following Table 1 that infers from the Eq. (3) and Lemma 2.4 for more clarity about the structure.

Table 1. Vertex classes of $\Upsilon(\widetilde{\mathcal{R}})$

Partition sets	Divisibility	Cardinality, $\lvert V_i \rvert$	Degree of each vertex
V_1	$\mathfrak{p}_1$ and $\mathfrak{p}_2$ only	$\mathfrak{p}_3 - 1$	$\mathfrak{p}_1\mathfrak{p}_2 - 1$
V_2	$\mathfrak{p}_1$ and $\mathfrak{p}_3$ only	$\mathfrak{p}_2 - 1$	$\mathfrak{p}_1\mathfrak{p}_3 - 1$
V_3	$\mathfrak{p}_2$ and $\mathfrak{p}_3$ only	$\mathfrak{p}_1 - 1$	$\mathfrak{p}_2\mathfrak{p}_3 - 1$
V_4	$\mathfrak{p}_3$ only	$(\mathfrak{p}_1 - 1)(\mathfrak{p}_2 - 1)$	$\mathfrak{p}_3 - 1$
V_5	$\mathfrak{p}_2$ only	$(\mathfrak{p}_1 - 1)(\mathfrak{p}_3 - 1)$	$\mathfrak{p}_2 - 1$
V_6	$\mathfrak{p}_1$ only	$(\mathfrak{p}_2 - 1)(\mathfrak{p}_3 - 1)$	$\mathfrak{p}_1 - 1$

Since the above sets gives a partition of the set of vertices, therefore, $V(\Upsilon(\widetilde{\mathcal{R}})) = \coprod_{i=1}^{6} V_i$.

Hence the total number of vertices in the (ZD) graph $V(\Upsilon(\widetilde{\mathcal{R}}))$ are;

$$|V(\Upsilon(\widetilde{\mathcal{R}}))| = \sum_{i=1}^{6} |V_i| = \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - \mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3. \tag{4}$$

By Lemma 2.5, the $diam(\Upsilon(\widetilde{\mathcal{R}})) = 3$.

Note that, every vertex of V_1 is adjacent to each vertex of V_2, V_3 and V_4, and every vertex of V_2 is adjacent to each vertex of V_1, V_3 and V_5. Also, every vertex of V_3 is adjacent to each vertex of V_1, V_2 and V_6.

A distance matrix between the vertices of partition classes, that is $d(V_i, V_j)$, is provided to create a foundation for the radio-labeling proof in the following section.

Table 2. Distance matrix between the vertex classes

$d(V_i, V_j)$	V_1	V_2	V_3	V_4	V_5	V_6
V_1	2	1	1	1	2	2
V_2	1	2	1	2	1	2
V_3	1	1	2	2	2	1
V_4	1	2	2	2	3	3
V_5	2	1	2	3	2	3
V_6	2	2	1	3	3	2

4. Radio Labeling on Zero divisor graphs

In this section, we will find the radio number for the case where n is the product of three distinct primes, thereby substantiating our claim.

In a radio labeling of a graph, a value is called a forbidden value for a vertex u if it violates the distance constraint in the inequality (1) for any labeled vertex. The set of all forbidden values in a radio labeling is denoted by F and it is determined by the distance restrictions of the graph. Clearly, $rn(G) \geq |V(G)| + |F|$ for any graph G.

This will help us to determine a lower bound for the radio number of (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$.

Let us denote the vertices $v_{i,n} \in V_i$, where $n = 1, 2, \ldots, |V_i|$. From now on any vertex $v_{i,n}$ means n^{th} vertex in the partition set V_i.

Theorem 4.1. *A lower bound (LB) of the radio number of the (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$ is:*

$$rn(\Upsilon(\widetilde{\mathcal{R}})) \geq |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2.$$

Proof. For any radio labeling ψ on the vertices of the (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$, the $d(u, v) + |\psi(u) - \psi(v)| \geq 1 + diam(\Upsilon(\widetilde{\mathcal{R}})) = 4$ for all $u, v \in V(\Upsilon(\widetilde{\mathcal{R}}))$.

Since $d(v_{4,n}, v_{6,k}) = d(v_{4,n}, v_{5,m}) = d(v_{5,m}, v_{6,k}) = diam(\Upsilon(\widetilde{\mathcal{R}})) = 3$, therefore, for a radio labeling ψ, $|\psi(v_{i,n}) - \psi(v_{j,m})| \geq 1$ for $i \neq j$ and $i, j \in \{4, 5, 6\}$. Due to our assumption of primes order $2 \leq \mathfrak{p}_3 < \mathfrak{p}_2 < \mathfrak{p}_1$ and Table 1, it is clear that $|V_4| > |V_5| > |V_6|$. Implies that, $||V_4| - |V_5| - |V_6|| = |(\mathfrak{p}_1 - 1)(\mathfrak{p}_2 - 1) - (\mathfrak{p}_1 + \mathfrak{p}_2 - 2)(\mathfrak{p}_3 - 1)|$ number of elements cannot be assigned consecutive values because $d(v_{i,n}, v_{i,m}) = 2$ and $|\psi(v_{i,n}) - \psi(v_{i,m})| \geq 2$ for any $v_{i,n}, v_{i,n} \in V_i$. Clearly, we have $|(\mathfrak{p}_1 - 1)(\mathfrak{p}_2 - 1) - (\mathfrak{p}_1 + \mathfrak{p}_2 - 2)(\mathfrak{p}_3 - 1)| - 1$ number of forbidden values amongst sets V_4, V_5 and V_6 plus 1 forbidden value exist while switching the labels to remaining sets V_1, V_2 or V_3.

Similarly, for any radio labeling ψ, $|\psi(v_{i,n}) - \psi(v_{j,m})| \geq 3$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$ because $d(v_{1,n}, v_{2,m}) = d(v_{2,m}, v_{3,k}) = d(v_{1,n}, v_{3,k}) = 1$. Therefore, we have 2 forbidden values while switching labeling, say, from V_1 to V_2 and from V_2 to V_3. Moreover, $d(v_{i,n}, v_{i,m}) = 2$ and hence $|\psi(v_{i,n}) - \psi(v_{i,m})| \geq 2$ for any $v_{i,n}, v_{i,n} \in V_i$, implies, we cannot assign consecutive labeling within the elements of each set of V_1, V_2 and V_3 infers $|V_3| + |V_2| + |V_1| - 1 = (\mathfrak{p}_1 - 1) + (\mathfrak{p}_2 - 1) + (\mathfrak{p}_3 - 1) - 1 = \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 - 4$ forbidden values.

Thus the total forbidden values are;

$$\begin{aligned}
|F| &= |(\mathfrak{p}_1 - 1)(\mathfrak{p}_2 - 1) - (\mathfrak{p}_1 + \mathfrak{p}_2 - 2)(\mathfrak{p}_3 - 1)| - 1 + 1 + 2 + (\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 - 4) \\
&= |(\mathfrak{p}_1 - 1)(\mathfrak{p}_2 - 1) - (\mathfrak{p}_1 + \mathfrak{p}_2 - 2)(\mathfrak{p}_3 - 1)| + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 - 2 \\
&= |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 - 2.
\end{aligned}$$

Since

$$\begin{aligned}
rn(\Upsilon(\widetilde{\mathcal{R}})) &\geq |V| + |F| \\
&= (\mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - \mathfrak{p}_1 - \mathfrak{p}_2 - \mathfrak{p}_3) \\
&\quad + (|\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 - 2).
\end{aligned}$$

$$\Rightarrow rn(\Upsilon(\widetilde{\mathcal{R}})) \geq |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2.$$

□

Theorem 4.2. *An upper bound (UB) for the Radio number of the (ZD) graph* $\Upsilon(\widetilde{\mathcal{R}})$ *is*

$$rn(\Upsilon(\widetilde{\mathcal{R}})) \leq |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2.$$

Proof. To get an upper bound of radio number of (ZD) graph $\Upsilon(\widetilde{\mathcal{R}})$, we define a radio labeling:

$$\psi : V(\Upsilon(\widetilde{\mathcal{R}})) \to \{1, 2, 3, \ldots, |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2\},$$

such that $d(u, v) + |\psi(u) - \psi(v)| \geq 1 + diam(\Upsilon(\widetilde{\mathcal{R}})) = 4$ for all $u, v \in V(\Upsilon(\widetilde{\mathcal{R}}))$.

We know that, $|V_4| > |V_5| > |V_6|$. Let us first assign labeling to the elements of the sets V_4, V_5 and V_6, we have two cases.

Case 1. If $|V_4| \geq |V_5| + |V_6|$.

Define $\psi(v_{4,n}) = 2n - 1$ and $\psi(v_{6,n}) = 2n$ for all $n = 1, 2, \ldots, |V_6|$, $\psi(v_{4,m+|V_6|}) = 2(|V_6| + m) - 1$, and $\psi(v_{5,m}) = 2(|V_6| + m)$, for all $m = 1, 2, \ldots, |V_5|$.

For the remaining $|V_4| - (|V_5| + |V_6|)$ number of elements in the set V_4, define

$$\psi(v_{6,|V_6|+|V_5|+k}) = 2(|V_6| + |V_5| + k) - 1, \qquad \forall \quad k = 1, 2, \ldots, |V_4| - (|V_6| + |V_5|).$$

Case 2: If $|V_4| < |V_5| + |V_6|$.

Define $\psi(v_{4,n}) = 2n - 1$ and $\psi(v_{6,n}) = 2n$ for all $n = 1, 2, \ldots, |V_6|$,
$\psi(v_{5,m}) = 2(|V_6| + m) - 1$ and $\psi(v_{6,m+|V_6|}) = 2(|V_6| + m)$ for all $m = 1, 2, \ldots, |V_4| - |V_6|$.

For the remaining $(|V_6| + |V_5|) - |V_4|$ number of elements in the set V_5, define $\psi(v_{5,k}) = 2|V_4| + 2k - 1$ for all $k = 1, 2, \ldots, (|V_6| + |V_5|) - |V_4|$.

Since $d(v_{4,m}, v_{6,k}) = d(v_{4,m}, v_{5,n}) = d(v_{5,n}, v_{6,k}) = 3$ and $d(v_{i,n}, v_{i,m}) = 2$, clearly $d(u, v) + |\psi(u) - \psi(v)| \geq 4$ for all $u, v \in V_4 \cup V_5 \cup V_6$. The radio condition is satisfied in both cases amongst the sets V_4. V_5 and V_6.

Now irrespective of above cases, we assign labeling to the elements of the sets V_1, V_2 and V_3. As we have already assigned labels till $t := ||V_4| - |V_5| - |V_6|| + |V_4| + |V_5| + |V_6| - 1$ that cannot be assigned to remaining sets.

Define $\psi(v_{3,a}) = t + 2a$ for all $a = 1, 2, \ldots, |V_3|$, $\psi(v_{2,b}) = t + 2(|V_3| + b) + 1$ for all $b = 1, 2, \ldots, |V_2|$, $\psi(v_{1,c}) = t + 2(|V_3| + |V_2| + c) + 2$ for all $c = 1, 2, \ldots, |V_1|$. Clearly, the radio condition (1) is satisfied with the maximum assigned label:

$$\begin{aligned} t + 2(|V_3| + |V_2| + |V_1|) + 2 &= ||V_4| - |V_5| - |V_6|| + |V_4| + |V_5| + |V_6| - 1 + 2(|V_3| + |V_2| + |V_1|) + 2 \\ &= |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2. \end{aligned}$$

Hence $rn(\Upsilon(\widetilde{\mathcal{R}})) \leq |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2$. □

From Theorem 4.1 and Theorem 4.2, we obtained the following result.

Theorem 4.3. *The* $rn(\Upsilon(Z_{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3})) = |\mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 + 2\mathfrak{p}_3 - 1| + \mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_3 - 2$ *for* $2 \leq \mathfrak{p}_3 < \mathfrak{p}_2 < \mathfrak{p}_1$.

Example 4.4. For $n = 30$, we take $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = (5, 3, 2)$. The six vertex partition sets described in Table 1 and their corresponding radio labeling on the vertices of $\Upsilon(\mathbb{Z}_{30})$ is provided in the following table.

Table 3. Vertex classes of $\Upsilon(\mathbb{Z}_{30})$ and their radio labeling

Sets V_i	Elements in V_i	Radio Labeling $\psi(u)$
V_1	$\{15\}$	31
V_2	$\{10, 20\}$	26, 28
V_3	$\{6, 12, 18, 24\}$	17, 19, 21, 23
V_4	$\{2, 4, 8, 14, 16, 22, 26, 28\}$	1, 3, 5, 7, 9, 11, 13, 15
V_5	$\{3, 9, 21, 27\}$	6, 8, 10, 12
V_6	$\{5, 25\}$	2, 4

The proposed labeling in Table 3 and using the distance Table 2, one can verify the radio condition 1 is satisfied and the corresponding radio number is $|15 - 10 - 6 + 2(2) - 1| + 15 + 10 + 6 - 2 = 2 + 2 + 31 - 2 = 31$ that satisfies the result in Theorem 4.3.

5. Conclusion and future work

In this work, we have studied zero divisor graphs on commutative rings $\mathbb{Z}_n$ where n was a product of square-free primes. We computed the radio number and extended results to cases where n is a product of at most three distinct primes.

The work provided in this article can be extended in a few directions including the results over the rings $\mathbb{Z}_n$, when n is a product of more than three primes, other graph invariants can be studied on these rings and topological indices such as energy or Wiener index can be studied on these graphs.

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