UTILITAS MATHEMATICA 128 (2026) 127–139

Utilitas Mathematica
www.combinatorialpress.com/um

Combinatorial Press

Ramsey-type results on parameters related to domination

Jin Sun and Xinmin Hou*

ABSTRACT

The inequality chain $ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G)$ is known as the domination chain, where $ir(G), \gamma(G), i(G), \alpha(G), \Gamma(G)$ and $IR(G)$ are the lower irredundance number, the domination number, the independence domination number, the independence number, the upper domination number and the upper irredundance number of G, respectively. The Ramsey-type problem seeks to characterize the family $\mathcal{H}$ of graphs such that every $\mathcal{H}$-free graph G has a bounded parameter μ. The classical Ramsey's theorem states that every $\{K_n, E_n\}$-free graph has a bounded number of vertices. Furuya (Discrete Math.Theor 2018) characterized $\mathcal{H}$ such that every connected $\mathcal{H}$-free graph G has a bounded domination number. The characterization of the graph family $\mathcal{H}$ for which every connected $\mathcal{H}$-free graph G has a bounded independence number was due to Choi, Furuya, Kim, Park (Discrete math. 2020) and Chiba, Furuya (Electron. J. Combin., 2022). In this paper, we further characterize $\mathcal{H}$ such that every connected $\mathcal{H}$-free graph G has bounded $\mu(G)$ for μ belonging to the set $\{ir(G), i(G), \Gamma(G), \mathrm{IR}(G)\}$. This completes the characterization of $\mathcal{H}$ for which every connected $\mathcal{H}$-free graph G has bounded $\mu(G)$ for $\mu(G)$ along the domination chain. Additionally, we characterize $\mathcal{H}$ such that every connected $\mathcal{H}$-free graph G has bounded $\mu(G)$ for μ related to the domination number. Specifically, we consider the following parameters of G: open irredundance number $OIR(G)$, independence saturation number $IS(G)$ and irredundance saturation number $IRS(G)$.

Keywords: Ramsey-type problem, domination number, domination chain

2020 Mathematics Subject Classification: 05C55, 05C75.

* Corresponding author.

Received 29 Sep 2024; Revised 11 Jun 2026; Accepted 18 Jun 2026; Published Online 22 Jul 2026.

DOI: 10.61091/um128-07

1. Introduction

In this paper, we consider only finite and simple graphs. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $N_G(v) = \{u \in V : uv \in E\}$ denote the *(open) neighborhood* of v, and let $N_G[v] = N(v) \cup \{v\}$ denote the *closed neighborhood* of v (the subscript may be omitted if there is no confusion). For a subset $S \subseteq V$, let $N_G[S] = \bigcup_{v \in S} N_G[v]$. We denote by $G[S]$ the subgraph of G induced by edges in S. A set $S \subseteq V$ is called an *independent set* of G if there is no edge in $G[S]$. For a set $T \subseteq V$ disjoint from S, we denote by $G[S, T]$ the subgraph of G induced by edges between S and T. The more notation, see [7].

For graphs H_1 and H_2, we say $H_1 \prec H_2$ if H_1 is an induced subgraph of H_2. Let $\mathcal{H}$ be a family of graphs, we say G is $\mathcal{H}$-free, if there is no graph $H \in \mathcal{H}$ such that $H \prec G$. For graph families $\mathcal{H}_1$ and $\mathcal{H}_2$, we say $\mathcal{H}_1 \leq \mathcal{H}_2$ if for any $H_2 \in \mathcal{H}_2$ there exists $H_1 \in \mathcal{H}_1$ such that $H_1 \prec H_2$, i.e., each graph in $\mathcal{H}_2$ is not $\mathcal{H}_1$-free. The following straightforward result will be commonly used without explicit mention.

Observation 1.1. *The relation '$\leq$' is transitive. Therefore, if $\mathcal{H}_1 \leq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.*

As usual, let P_n, C_n, K_n and E_n denote a *path*, a *cycle*, a *complete graph*, and an *edgeless graph* of order n, respectively. Let $K_{s,t}$ be the *complete bipartite graph* with partitions of orders s and t.

The classical Ramsey's Theorem can be stated as follows:

(A) (Ramsey's Theorem [14], 1929) For a family $\mathcal{H}$ of graphs, there is a constant $c = c(\mathcal{H})$ such that $|V(G)| < c$ for every $\mathcal{H}$-free graph G if and only if $\mathcal{H} \leq \{K_m, E_n\}$ for some positive integers m and n;

(B) (The connected version, Proposition 9.4.1 in [7]) For a family $\mathcal{H}$ of graphs, there is a constant $c = c(\mathcal{H})$ such that $|V(G)| < c$ for every connected $\mathcal{H}$-free graph G if and only if $\mathcal{H} \leq \{K_n, K_{1,n}, P_n\}$ for some positive integer n.

The *Ramsey number* $R(m, n)$ is the least integer $c = c(\{K_m, E_n\})$ in (A), thus every graph G of order at least $R(m, n)$ contains either a complete graph K_m or an edgeless graph E_n.

In general, given a graph parameter μ, define

$$B\text{-}\mu = \{\mathcal{H} : \text{there is constant } c \text{ such that } \mu(G) < c \text{ for any connected } \mathcal{H}\text{-free graph } G\}.$$

A Ramsey-type problem of μ is to determine B-μ.

Problem 1.2. *Given a graph parameter μ, determine B-μ.*

In the following, we list more results on different parameters of Problem 1.2.

(1) Chiba and Furuya determined B-μ when μ is the path cover/partition number in [4], and when μ is the induced star and path cover/partition number in [5].

(2) Choi, Furuya, Kim and Park [6] determined B-ν, where $\nu(G)$ is the matching number of G.

(3) Lozin [13] determined B-μ when μ is the neighborhood diversity or VC-dimension.

(4) Lozin and Razgon [12] determined B-μ, where $\mu(G)$ is the tree-width of graph G.

(5) Kierstead and Penrice [11]; and Scott, Seymour, and Spirkl [15] determined B-δ, where $\delta(G)$ is the minimal degree of graph G.

(6) Galvin, Rival and Sands [9]; and Atminas, Lozin and Razgon [3] determined B-μ, where $\mu(G)$ is the length of the longest path of graph G.

(7) Sun and Hou [16] determined B-μ, where $\mu(G)$ is the deficiency of graph G.

Let nG be the graph consisting of n disjoint copies of G. In order to state the forbidden subgraphs condition conveniently, we introduce additional kinds of graphs (as shown in Figure 1).

- $K_{1,n}^*$: The graph obtained by adding a pendant to every leaf of $K_{1,n}$.
- K_n^*: The graph obtained by adding a pendant to every vertex of K_n.
- CK_n: The graph obtained by adding a perfect matching between two disjoint copies of K_n.
- BS_n^p: The graph obtained by connecting the *centers* of two disjoint $K_{1,n}$ by a path $P = P_p$. We call the two ends of P_p as the centers of BS_n^p.
- F_n: The graph obtained by adding a universal vertex adjacent to each vertex of nK_2.

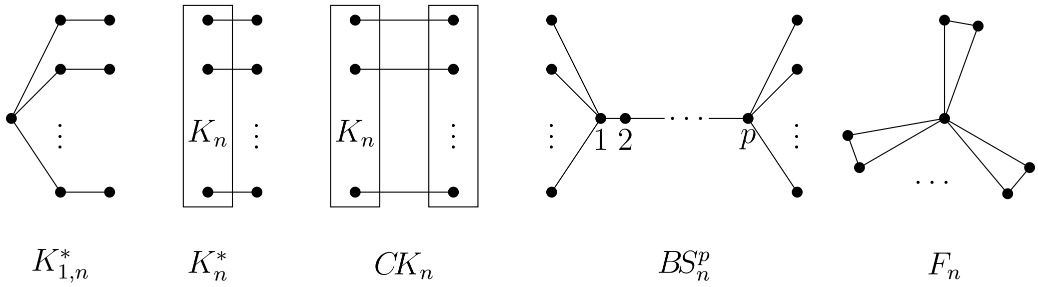

Fig. 1. The graphs $K_{1,n}^*$, K_n^*, CK_n, BS_n^p and F_n

In this paper, we mainly concern Problem 1.2 when μ are parameters related to domination.

1.1. The domination chain

Let $G = (V, E)$ be a graph. For a property P about subsets of V, we define a subset S as a *minimal* (resp. *maximal*) P-set if S satisfies P, and no proper subset (resp. superset) of S satisfies P. A set $S \subseteq V$ is called a *dominating set* of G if $V = N[S]$. Define the *domination number* of G as

$$\gamma(G) = \min\{|S|: S \text{ is a dominating set of } G\},$$

and the *upper domination number* of G as

$$\Gamma(G) = \max\{|S|: S \text{ is a minimal dominating set of } G\}.$$

Recall that the *independence number* of G

$$\alpha(G) = \max\{|S|\colon S \text{ is an independent set of } G\}.$$

Define the *independence domination number* of G as

$$i(G) = \min\{|S|\colon S \text{ is a maximal independent set of } G\}.$$

Let $S \subseteq V(G)$ and $v \in S$. The *private neighborhood* of v with respect to S is defined as

$$PN[v, S] = N_G[v] - N_G[S \setminus \{v\}].$$

A set $S \subseteq V(G)$ is called an *irredundant set* if every vertex s in S has at least one private neighbor, i.e., $PN[s, S] \neq \emptyset$ for any $s \in S$.

The *upper irredundance number*, $\mathrm{IR}(G)$, is defined as

$$\mathrm{IR}(G) = \max\{|S|\colon S \text{ is an irredundant set of } G\},$$

and the *lower irredundance number*, $ir(G)$, of G is defined as

$$ir(G) = \min\{|S|\colon S \text{ is a maximal irredundant set of } G\}.$$

Note that $\gamma(G)$ is also the minimum size of minimal dominating sets of G since a dominating set with minimum size must be a minimal dominating set. Similarly, $\alpha(G)$ (resp. $\mathrm{IR}(G)$) is the maximum size of maximal independent (resp. irredundant) sets of G. It can be checked directly that a maximal independent set is a minimal dominating set, and a minimal dominating set is a maximal irredundant set. Visually,

$$\{\text{maximal independent set}\} \subseteq \{\text{minimal dominating set}\} \subseteq \{\text{maximal irredundant set}\}.$$

By the definitions of $ir(G), \gamma(G), i(G), \alpha(G), \Gamma(G)$, and $IR(G)$, we have the following inequality chain.

Proposition 1.3 ([10]). *For any graph G, it holds that*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

The inequality chain above is called the *domination chain*. As pointed in [10], this inequality chain has become one of the strongest focal points for research in domination theory. Many aspects of this chain has been studied. For the history and more information of these parameters, the reader can survey the book [10] and references therein. For parameters μ defined by size of special subsets of $V(G)$, we will use μ-set to mean subset S realizing parameter μ; for example, a γ-set D of G is a minimum dominating set of G such that $\gamma(G) = |D|$.

We first focus on Problem 1.2 for μ in the domination chain. Furuya [8] has determined B-γ.

Theorem 1.4 ([8]). *$\mathcal{H} \in B$-γ if and only if $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n\}$ for some positive integer n.*

Denote by γ_n the smallest constant depending only on n in Theorem 1.4, thus $\gamma(G) < \gamma_n$ for any $\{K^*_{1,n}, K^*_n, P_n\}$-free graph G.

Choi, Furuya, Kim, Park [6] and Chiba, Furuya [4] characterized B-α.

Theorem 1.5 ([6, 4])**.** *$\mathcal{H} \in B$-α if and only if $\mathcal{H} \leq \{K_{1,n}, K^*_n, P_n\}$ for some positive integer n.*

Denote by α_n the smallest constant depending only on n in Theorem 1.5, thus $\alpha(G) < \alpha_n$ for any $\{K_{1,n}, K^*_n, P_n\}$-free graph G.

We proceed to determine B-μ for μ belonging to the set $\{ir(G), i(G), \Gamma(G), \mathrm{IR}(G)\}$. This completes the characterization of B-μ for μ along the domination chain.

Theorem 1.6. (1) *$\mathcal{H} \in B$-ir if and only if $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n\}$ for some positive integer n.*

(2) *$\mathcal{H} \in B$-i if and only if $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n, K_{n,n}, BS^2_n\}$ for some positive integer n.*

(3) *$\mathcal{H} \in B$-IR if and only if $\mathcal{H} \in B$-Γ if and only if $\mathcal{H} \leq \{K_{1,n}, K^*_n, P_n, CK_n\}$ for some positive integer n.*

1.2. Parameters related to the domination chain

When we do not want a vertex can serve as the private neighbor of itself, we get an variant of IR-number. Given a graph G, a subset S of $V(G)$ is an *open irredundant set* if every vertex s in S has at least one private neighbor outside of S, i.e., $PN[s, S] - S \neq \emptyset$. The *open irredundance number* $OIR(G)$ is defined as

$$OIR(G) = \max\{|S| \colon S \text{ is an open irredundant set of } G\}.$$

Clearly, $OIR(G) \leq IR(G)$ for any graph G since an open irredundant set must be an irredundant set. We also determine B-OIR in the following.

Theorem 1.7 (OIR)**.** *$\mathcal{H} \in B$-OIR if and only if $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n, CK_n, F_n\}$, for some positive integer n.*

Arumugam, Favaron, Sudha [1] and Arumugam, Subramanian [2] introduced the independence saturation number and the irredundance saturation number respectively.

Let $G = (V, E)$ be a graph and $v \in V$. Let

$$IS(v) = \max\{|S| \colon S \text{ is an independent set of } G \text{ with } v \in S\},$$

and define the *independence saturation number* of G as

$$IS(G) = \min\{IS(v) : v \in V\}.$$

Let

$$IRS(v) = \max\{|S| \colon S \text{ is an irredundant set in } G \text{ with } v \in S\},$$

and define the *irredundance saturation number* of G as

$$IRS(G) = \min\{IRS(v) : v \in V\}.$$

Since all independent sets are irredundant, we have $IS(v) \le IRS(v)$. Thus $IS(G) \le IRS(G)$. Since $IS(G)$ is the size of some maximal independent set, $i(G) \le IS(G) \le \alpha(G)$ and similarly $ir(G) \le IRS(G) \le IR(G)$. We determine B-IS and B-IRS as follows.

Theorem 1.8. *IS $\mathcal{H} \in B$-IS if and only if*

$$\mathcal{H} \le \{K^*_{1,n}, K^*_n, P_n, K_{n,n}, BS^p_n : 2 \le p \le n-3\},$$

for some $n \ge 5$.

Theorem 1.9. *IRS $\mathcal{H} \in B$-IRS if and only if*

$$\mathcal{H} \le \{K^*_{1,n}, K^*_n, P_n, K_{n,n}, CK_n, BS^p_n : 2 \le p \le n-3\},$$

for some $n \ge 5$.

Remark 1.10. The methods employed in proving Theorems 1.7, 1.8, and 1.9 are novel, for example, the independence number is bounded hierarchically by induction, whereas the independence saturation number is bounded by contradiction, i.e., the forbidden induced subgraph would be extracted from a large $IS(G)$ set.

The rest of the paper is arranged as follows. In Section 2, We present some necessary lemmas and establish notation conventions for symbols that appear frequently throughout the argument. We give the proofs of Theorems 1.6, 1.7, 1.8, and 1.9 in Section 3.

2. Preliminaries

There exists a quality bound between the lower irredundance number ($ir(G)$) and the domination number ($\gamma(G)$).

Proposition 2.1 ([10])**.** *For any graph G, it holds that $ir(G) \le \gamma(G) \le 2ir(G) - 1$.*

The following lemma [3] serves as the bipartite graph version of Ramsey's theorem.

Lemma 2.2 ([3])**.** *For any positive integer n, there exists a smallest integer $BR(n)$ such that every bipartite graph $G = (V_1, V_2, E)$ with $|V_i| \ge BR(n)$ for $i = 1, 2$, there are subsets $U_i \subseteq V_i$ of order n satisfying that $G[U_1, U_2] \cong K_{n,n}$ or E_{2n}.*

Let $\mathcal{BS}^p_n$ be the family of graphs obtained by adding additional edges connecting the leaves adjacent to one center and the leaves adjacent to anther center of BS^p_n.

Lemma 2.3. $\{K_{n,n}, BS^p_n\} \le \mathcal{BS}^p_{BR(n)}$.

Proof. Denote by V_i $(i = 1, 2)$ the set of leaves adjacent to the corresponding two centers of $BS^p_{BR(n)}$, respectively. Then $|V_1| = |V_2| = BR(n)$. Recall that any graph $G \in \mathcal{BS}^p_{BR(n)}$ is obtained by adding some edges between V_1 and V_2 from $BS^p_{BR(n)}$. Applying Lemma 2.2 to the bipartite graph $G[V_1, V_2] = G(V_1 \cup V_2)$, since V_1 and V_2 are independents , we can find $U_i \subseteq V_i$ with $|U_i| = n$ for $i = 1, 2$ such that $G[U_1, U_2] \cong K_{n,n}$, thus $K_{n,n} \prec G$; or $G[U_1, U_2]$ is an edgeless graph, thus $BS^p_n \prec G$ in this case. Therefore, $\{K_{n,n}, BS^p_n\} \leq \mathcal{BS}^p_{BR(n)}$. □

The following result was given by Zverovich and Zverovich in [17].

Theorem 2.4 ([17]). *If a graph G is $\mathcal{BS}^2_{k-1}$-free, where $k \geq 3$, then*

$$i(G) \leq \gamma(G)(k-2) - (k-3).$$

The following lemma of Lozin [13] can be used to bound the matching number $\nu(G)$ of bipartite graphs.

Lemma 2.5 ([13]). *Let G be an $\{nK_2, K_{n,n}\}$-free bipartite graph. Then there exists a minimum $q(n)$, such that $\nu(G) < q(n)$.*

In order to extract induced subgraphs we need from an irredundant set in proofs of next section, we will use the following fact and notation frequently. Let S be an irredundant set of graph G. Suppose that $v \in S$ is not an isolated vertex of $G[S]$, i.e., there exists a vertex $w \in S$ satisfying $vw \in E(G)$. Take a vertex $v' \in PN[v, S] = N[v] - N[S \setminus \{v\}]$. Thus $v' \neq v$ as $v \in N(w) \subseteq N[S \setminus \{v\}]$. Moreover, $v' \notin S$ otherwise $v' \in N[v'] \subseteq N[S \setminus \{v\}]$, a contradiction. For $T \subseteq S$, let $T' = \{v' : v \in S\}$, where each $v' \in PN[v, S] - S$ is a private neighbor of v outside S. Note that $u' \neq v'$ as $N(u') \cap S = \{u\} \neq \{v\} = N(v') \cap S$. Thus $|T'| = |T|$. At this time, $G[T, T']$ is isomorphic to $|T|K_2$. For $X' \subseteq T'$, we will let $X = \{v \in T : v' \in X'\}$ implicitly.

3. Proofs of Theorems 1.6, 1.7, 1.8, and 1.9

Proof of Theorem 1.6. (1) It is a corollary of Theorem 1.4 and Proposition 2.1.

(2) We first prove the "only if" part. Suppose that there is a constant c such that every connected $\mathcal{H}$-free graph G satisfies $i(G) < c$. Let $n = 3c - 2$. Then $i(K^*_{1,n}) = i(K^*_n) = i(K_{n,n}) = n$, $i(P_n) = c$, and $i(BS^2_n) = n+1$. This implies that $K^*_{1,n}, K^*_n, P_n, K_{n,n}$, and BS^2_n are not $\mathcal{H}$-free. Therefore, $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n, K_{n,n}, BS^2_n\}$.

Next we prove the "if" part. Now suppose $\mathcal{H} \leq \{K^*_{1,n}, K^*_n, P_n, K_{n,n}, BS^2_n\}$. Given any connected $\mathcal{H}$-free graph G, we know that G is $\{K^*_{1,n}, K^*_n, P_n, K_{n,n}, BS^2_n\}$-free. Clearly G is $\{K^*_{1,n}, K^*_n, P_n\}$-free. By Theorem 1.4, $\gamma(G) < \gamma_n$. By Lemma 2.3, G is $\mathcal{BS}^2_{BR(n)}$-free. According to Theorem 2.4,

$$i(G) \leq \gamma(G)(BR(n)-1) - (BR(n)-2) < \gamma_n BR(n).$$

(3) Recall that we need to prove the following statements are equivalent (a) $\mathcal{H} \in B$-IR, (b) $\mathcal{H} \in B$-Γ, and (c) $\mathcal{H} \leq \{K_{1,n}, K^*_n, P_n, CK_n\}$ for some positive integer n.

$(a) \Rightarrow (b)$ is clear since $\Gamma(G) \le \mathrm{IR}(G)$. In order to prove $(b) \Rightarrow (c)$, we only need to show that $K_{1,n}, K_n^*, P_n$ and CK_n have unbounded upper domination number Γ as n increases. By Theorem 1.5, $K_{1,n}, K_n^*$ and P_n have unbounded independence number α, so does the upper domination number Γ. The maximum clique of CK_n forms a largest minimal dominating set of CK_n. Therefore, we have $\Gamma(CK_n) = n$.

Now we prove that $(c) \Rightarrow (a)$. We claim that, for any connected $\{K_{1,n}, K_n^*, P_n, CK_n\}$-free graph G, it holds that $\mathrm{IR}(G) < R(R(n, \alpha_n), \alpha_n)$. Suppose not, thus there exists an IR-set S of G such that $|S| \ge R(R(n, \alpha_n), \alpha_n)$. According to Theorem 1.5, $\alpha(G) < \alpha_n$ as G is $\{K_{1,n}, K_n^*, P_n\}$-free. Since $\alpha(G[S]) \le \alpha(G) < \alpha_n$, there exists a clique $K \subseteq S$ of order $R(n, \alpha_n)$. Each vertex v in K is not an isolated vertex. Thus we can take $K' = \{v' : v \in K\}$, where each v' is a private neighbor of v outside S. Thus $|K'| = |K| = R(n, \alpha_n)$. Since $\alpha(G[K']) \le \alpha(G) < \alpha_n$, there exists a clique $X' \subseteq K'$ of order n. Let $X = \{v \in K : v' \in X'\}$. Hence $G[X \cup X']$ forms an induced CK_n. This leads to a contradiction. The proof is completed. $\square$

The proof of Theorem 1.7 as follows.

Proof of Theorem 1.7. The "only if" part can be checked directly from the fact that $OIR(K_{1,n}^*) = OIR(K_n^*) = OIR(CK_n) = OIR(F_n) = n$, and $OIR(P_n) = \lceil n/3 \rceil$.

Now we prove the "if" part. All we need to show is that if a connected graph G satisfies

$$OIR(G) \ge R(R(n, n), R(n, 2n\gamma_n)),$$

then G contains an induced $K_{1,n}^*, K_n^*, P_n, CK_n$ or F_n. Take an OIR-set S of G, and thus $|S| \ge R(R(n, n), R(n, 2n\gamma_n))$. Note that each element in S has at least one private neighbor outside of S. For any $v \in S$, fix one and denote it by v'. For any subset X of S, let $X' = \{x' : x \in X\}$.

If $G[S]$ contains a clique K of order $R(n, n)$, then $|K'| = |K| = R(n, n)$. Applying Ramsey's theorem to graph $G[K']$, there exists a subset $L' \subseteq K'$ of order n such that L' is a clique or an independent set. Let $L = \{v \in S : v' \in L'\}$. If L' is a clique, then $G[L \cup L']$ forms an induced CK_n; otherwise, L' is an independent set, in this case, $G[L \cup L']$ forms an induced K_n^*.

Now assume $G[S]$ contains an independent set E of order $R(n, 2n\gamma_n)$. Thus $|E'| = |E| = R(n, 2n\gamma_n)$. If there exists a clique $X' \subseteq E'$ of order n, then $G[X \cup X']$ forms K_n^*, where $X = \{v \in S : v' \in X'\}$; otherwise, $G[E']$ contains an independent set Y' of order $2n\gamma_n$. Let D be a $\gamma(G)$-set. If $|D| \ge \gamma_n$, then G contains an induced K_n^*, P_n or $K_{1,n}^*$ by Theorem 1.4. We are done. Thus we may assume $|D| < \gamma_n$. Since

$$Y' \subseteq V(G) = N[D] = \bigcup_{d \in D} N[d],$$

there exist a vertex $u \in D$ such that $|N[u] \cap Y'| \ge |Y'|/|D| > 2n$. Therefore, we can choose $Z' \subseteq N(u) \cap Y'$ with $|Z'| = 2n - 1$. Let $Z = \{v \in S : v' \in Z'\}$ denote the subset of S corresponding to Z'. Then $|Z| = |Z'| = 2n - 1$. It holds that $u \notin Z \cup Z'$ as $Z' \subseteq N(u)$ and $G[Z, Z']$ is isomorphic to nK_2.

If u dominates at least n vertices in Z, i.e., there exists vertices $z_1, \cdots, z_n \in Z \cap N(u)$, then $G[\{u, z_1, z_1', \cdots, z_n, z_n'\}]$ forms an induced F_n. Otherwise, there are at least n vertices of Z nonadjacent to u, i.e., there exists vertices $z_1, \cdots, z_n \in Z - N(u)$, in this case $G[\{u, z_1, z_1', \cdots, z_n, z_n'\}]$ forms an induced $K_{1,n}^*$. Therefore, for any connected $\mathcal{H}$-free graph G, it holds that

$$OIR(G) < R(R(n,n), R(n, 2n\gamma_n)).$$

□

Proof of Theorem 1.8. We first prove the "only if" part. $IS(G) \geq i(G)$ can be arbitrarily large for $G \in \{K_{1,n}^*, K_n^*, P_n, K_{n,n}\}$ by Theorem 1.6 (2). For graph BS_n^p, each vertex v together with the leaves adjacent to some center of BS_n^p can form an independent set of order $n+1$. Thus $IS(BS_n^p) = n+1$. Therefore, $\mathcal{H} \leq \{K_{1,n}^*, K_n^*, P_n, K_{n,n}, BS_n^p : 2 \leq p \leq n-3\}$ for some $n \geq 5$.

Next we prove the "if" part. Let

$$N = (\gamma_n - 1)(q(n) + 2BR(n) - 1) + 2,$$

where $q(n)$ is defined in Lemma 2.5 and $BR(n)$ is defined in Lemma 2.2. We will show that if a connected graph G satisfies $IS(G) \geq N$, then G contains an induced $K_{1,n}^*, K_n^*, P_n, K_{n,n}$ or BS_n^p for some integer p. (We do not require $p \leq n-3$ as $P_n \prec BS_n^p$ if $p > n-3$.) By Theorem 1.4, we may assume that G has a γ-set D with $|D| < \gamma_n$, otherwise, G contains an induce $K_{1,n}^*$, K_n^*, or P_n, we are done.

Let u be a vertex of G with maximum *local independence number* $\alpha_l(u) := \alpha(G[N(u)])$. Choose $U \subseteq N(u)$ be an independent set of order $\alpha_l(u)$. Since $IS(u) \geq IS(G) \geq N$, there exists an $IS(u)$-set V_1 such that $u \in V_1$, $|V_1| \geq N$, and V_1 is an independent set. Let $V_2 = V_1 - \{u\}$. Thus $|V_2| \geq N-1$ and $N(u) \cap V_2 = \emptyset$ as V_1 is an independent set. Since

$$V_2 \subseteq V(G) = N[D] = \bigcup_{d \in D} N[d],$$

by the pigeonhole principle, there is a vertex $v \in D$ such that

$$\left| N[v] \cap V_2 \right| \geq \left\lceil \frac{|V_2|}{|D|} \right\rceil \geq \left\lceil \frac{N-1}{\gamma_n - 1} \right\rceil \geq q(n) + 2BR(n) > 1.$$

$v \notin V_2$ as V_2 is an independent set. Thus $N[v] \cap V_2 = N(v) \cap V_2$. Note that $v \neq u$ as $N(u) \cap V_2 = \emptyset$. Let $V = N(v) \cap V_2$. Thus $\alpha_l(v) := \alpha(G[N(v)]) \geq |V| \geq q(n) + 2BR(n)$. Recall that vertex u has maximum local independence number, thus

$$|U| = \alpha_l(u) \geq \alpha_l(v) \geq q(n) + 2BR(n).$$

As established above, we have found vertices u and v, independent sets $U \subseteq N(u)$ and $V \subseteq N(v)$. $u \notin V$ but maybe $v \in U$. $U \cap V = \emptyset$ as $V \cap N(u) = \emptyset$.

Case 1. $N_G(v) \cap (U \cup \{u\}) \neq \emptyset$.

Let $U_1 = N(v) \cap U$ and $U_2 = U - U_1$.

Claim 1. $|U_2 - \{v\}| \geq BR(n)$.

Proof. Note that bipartite graph $G[U_1, V]$, defined as the graph induced by edges between U_1 and V, is actually the induced subgraph $G[U_1 \cup V]$, since both U_1 and V are independent sets. Thus $\alpha_l(v) = \alpha(G[N(v)]) \geq \alpha(G[U_1, V])$ as $U_1 \cup V \subseteq N(v)$. Apply König's Theorem to the bipartite graph $G[U_1, V]$, we have

$$\alpha(G[U_1, V]) + \nu(G[U_1, V]) = |U_1| + |V|.$$

If $\nu(G[U_1, V]) \geq q(n)$, then, by Lemma 2.5, $nK_2 \prec G[U_1, V]$. Thus the nK_2 in $G[U_1, V]$ together with u forms an induced $K^*_{1,n}$ in G as $U_1 \subseteq N(u)$ and $V \cap N(u) = \emptyset$, we are done. Therefore, we may assume $\nu(G[U_1, V]) < q(n)$. Since $\alpha_l(u)$ is maximum, we have

$$\begin{aligned} 0 \geq & \alpha_l(v) - \alpha_l(u) \\ \geq & \alpha(G[U_1, V]) - |U| \\ = & (|U_1| + |V| - \nu(G[U_1, V]) - (|U_1| + |U_2|) > |V| - q(n) + |U_2|. \end{aligned}$$

Therefore, $|U_2| > |V| - q(n) \geq 2BR(n)$. Thus $|U_2 - \{v\}| \geq BR(n)$. □

If $uv \in E(G)$, then some graph in $\mathcal{BS}^2_{BR(n)}$ appears in $G[\{u, v\} \cup U_2 \cup V]$. By Lemma 2.3, G contains an induced $K_{n,n}$ or BS_n. Now assume $uv \notin E(G)$. Thus there exists a vertex $p \in U$ satisfying $pv \in E(G)$ as $N(v) \cap (U \cup \{u\}) \neq \emptyset$. If $|N(p) \cap V| \geq BR(n)$, then some graph in $\mathcal{BS}^2_{BR(n)}$ appears in

$$G[\{u, p\} \cup (U \setminus \{p\}) \cup (N(p) \cap V)].$$

If $|N(p) \cap V| < BR(n)$, Then

$$|V \setminus N(p)| = |V| - |N(p) \cap V| > q(n) + 2BR(n) - BR(n) \geq BR(n).$$

Thus some graph in $\mathcal{BS}^3_{BR(n)}$ appears in $G[\{u, p, v\} \cup U_2 \cup (V \setminus N(p))]$. In a word, we are done from Lemma 2.3.

Case 2. $N_G(v) \cap (U \cup \{u\}) = \emptyset$.

Case 2.1 There exist $p \in U$ and $q \in V$ satisfying $pq \in E(G)$.

If $|N(p) \cap V| < BR(n)$, then some graph in $\mathcal{BS}^2_{BR_n}$ appears in

$$G[\{u, p\} \cup (U \setminus \{p\}) \cup (N(p) \cap V)].$$

If $|N(q) \cap U| < BR(n)$, then some graph in $\mathcal{BS}^2_{BR_n}$ appears in

$$G[\{v, q\} \cup (N(q) \cap U) \cup (V \setminus \{q\})].$$

Now we may assume $|N(p) \cap V| \geq BR(n)$ and $|N(q) \cap U| < BR(n)$. Thus

$$|V - N(p)| = |V| - |N(p) \cap V| > q(n) + 2BR(n) - BR(n) \geq BR(n),$$

and similarly $|U - N(q)| \geq BR(n)$. At this time some graph in $\mathcal{BS}^4_{BR(n)}$ appears in

$$G[\{u, p, q, v\} \cup (U - N(q)) \cup (V - N(p))].$$

Therefore, we are done from Lemma 2.3 in each possibility.

Case 2.2 $E_G[\{u\} \cup U, \{v\} \cup V] = \emptyset$.

Let $P = x_1 \dots x_p$ be a shortest path connecting $\{u\} \cup U$ and $\{v\} \cup V$ such that

$$N(x_1) \cap (\{u\} \cup U) \neq \emptyset \text{ and } N(x_p) \cap (\{v\} \cup V) \neq \emptyset.$$

Since $E_G[\{u\} \cup U, \{v\} \cup V] = \emptyset$, P is well-defined, i.e., $p \geq 1$.

Assume $|U \setminus N(x_1)| < n$ and $|V \setminus N(x_p)| < n$ first. If $x_1 = x_p$, then

$$\begin{aligned}
\alpha_l(x_1) &\geq |N(x_1) \cap (U \cup V)| && \text{as } U \cup V \text{ is an independent set} \\
&= |N(x_1) \cap U| + |N(x_p) \cap V| && \text{as } x_1 = x_p \\
&= (|U| - |U \setminus N(x_1)|) + (|V| - |V \setminus N(x_p)|) \\
&> (|U| - n) + (|V| - n) > |U| = \alpha_l(u),
\end{aligned}$$

a contradiction to that u has maximum local independence number. Thus $x_1 \neq x_p$. At this time we can get an induced BS_n^p from $\{x_1, \dots, x_p\} \cup (N(x_1) \cap U) \cup (N(x_p) \cap V)$.

Second, we assume $|U \setminus N_G(x_1)| \geq n$ by symmetry. If $x_1 u \in E(G)$, define

$$T_{p+1} = \{u, x_1, \cdots, x_p\} \cup (U \setminus N(x_1)).$$

Otherwise, if $x_1 u \notin E(G)$, choose $x_0 \in U \cap N(x_1)$ as $N(x_1) \cap (\{u\} \cup U) \neq \emptyset$. Define

$$T_{p+2} = \{u, x_0, x_1, \cdots, x_p\} \cup (U \setminus N(x_1)).$$

Uniformly, write T_{p+1} or T_{p+2} as T_i for $i \in \{p+1, p+2\}$. If $|V \setminus N_G(x_p)| < n$, then $T_i \cup (N(x_p) \cap V)$ contains an induced BS_n^i. Now we assume $|V \setminus N_G(x_p)| \geq n$. If $x_p v \in E(G)$, then $T_i \cup \{v\} \cup (V \setminus N(x_p))$ contains an induced BS_n^{i+1}. Otherwise, $v \notin N(x_p)$. Choose $x_{p+1} \in N(x_p) \cap V$. At this time, $T_i \cup \{x_{p+1}, v\} \cup (V \setminus N(x_p))$ contains an induced BS_n^{i+2}. Anyway, G contains an induced BS_n^j for some $j \in \{p+1, p+2, p+3, p+4\}$. The proof is complete. □

Denote by IS_n the smallest constant depending only on n in Theorem 1.8, thus $IS(G) < IS_n$ for any $\{K_{1,n}^*, K_n^*, P_n, K_{n,n}, BS_n^p : 2 \leq p \leq n-3\}$-free graph G.

Proof of Theorem 1.9. We first prove the "only if" part. Recall that $IRS(G) \geq IS(G)$ for any graph G. Thus, by Theorem 1.8, $IRS(G)$ is unbounded for any $G \in \{K_{1,n}^*, K_n^*, P_n, K_{n,n}, BS_n^p : 2 \leq p \leq n-3\}$ as n increases. Note that each vertex in CK_n belongs to an irredundant set of order n. Thus $IRS(CK_n) = n$. Therefore, $\mathcal{H} \leq \{K_{1,n}^*, K_n^*, P_n, K_{n,n}, CK_n, BS_n^p : 2 \leq p \leq n-3\}$ for some $n \geq 5$.

Next we prove the "if" part. All we need to show is that if a connected graph G satisfies $IRS(G) \geq IS_n + R(R(n,n), R(n, IS_n + n))$, then G contains an induced $K_{1,n}^*$, K_n^*, P_n, $K_{n,n}$, CK_n, or BS_n^p for some positive integer p. By Theorem 1.8, we may assume that there is a vertex $v \in V(G)$ such that $IS(v) = IS(G) < IS_n$.

Let S be an $IRS(v)$-set. Thus $v \in S$. Let I be the set of isolated vertices in $G[S]$. Then $I \cup \{v\}$ is an independent set. Thus $|I \cup \{v\}| \leq IS(v) < IS_n$. Let $X = S \setminus (I \cup \{v\})$. Then every vertex in X has at least one private neighbor outside of S. For any $x \in X$, fix one of its private neighbor as x'. Let $X' = \{x' : x \in X\}$. Then

$$|X'| = |X| = |S| - |I \cup \{v\}| \geq IRS(G) - IS_n \geq R(R(n,n), R(n, IS_n + n)).$$

Assume first there exists a clique $K' \subseteq X'$ of order $R(n,n)$. Let $K = \{x \in X : x' \in K'\}$. Thus $|K| = |K'| = R(n,n)$. Applying Ramsey's Theorem to $G[K]$, there exists a subset $L \subseteq K$ of order n such that L is a clique or an independent set. If L is a clique, then $G[L \cup L']$ forms CK_n; otherwise L is an independent set, at this case, $G[L \cup L']$ forms K_n^*. Now assume that $G[X']$ contains an independent set Y' of order $R(n, IS_n + n)$. Let $Y = \{x \in X : x' \in Y'\}$. Thus $|Y| = |Y'| = R(n, IS_n + n)$. Applying Ramsey's Theorem to $G[Y]$, there exists either a clique C of order n or an independent set Z of order $IS_n + n$. The former case implies that $G[C \cup C']$ forms an induced K_n^*. For the latter, $|N(v) \cap Z| \geq n$ since $IS(v) < IS_n$. Let $z_1, \cdots, z_n \in N(v) \cap Z$, in this case $G[\{v, z^{--}1, z_1' \cdots, z_n, z_n'\}]$ forms an induced $K_{1,n}^*$. The proof is completed. □

Acknowledgments

This work was supported by the National Key Research and Development Program of China (2023YFA1010200), the National Natural Science Foundation of China (No. 12471336, and No. 12071453), and the Innovation Program for Quantum Science and Technology (2021ZD0302902).

References

[1] S. Arumugam, O. Favaron, and S. Sudha. Irredundance saturation number of a graph. *Australasian Journal of Combinatorics*, 46:37–49, 2010.

[2] S. Arumugam and M. Subramanian. Independence saturation and extended domination chain in graphs. *AKCE International Journal of Graphs and Combinatorics*, 4(2):171–181, 2007. https://doi.org/10.1080/09728600.2007.12088831.

[3] A. Atminas, V. V. Lozin, and I. Razgon. Linear time algorithm for computing a small biclique in graphs without long induced paths. In *Algorithm Theory – SWAT 2012*. Volume 7357, Lecture Notes in Computer Science, pages 142–152. Springer, Berlin and Heidelberg, 2012. 10.1007/978-3-642-31155-0_13.

[4] S. Chiba and M. Furuya. Ramsey-type results for path covers and path partitions. *The Electronic Journal of Combinatorics*, 29(4):P4.8, 2022. https://doi.org/10.37236/10639.

[5] S. Chiba and M. Furuya. Ramsey-type problems on induced covers and induced partitions toward the gyárfás–sumner conjecture. *Journal of Graph Theory*, 107(2):419–441, 2024. https://doi.org/10.1002/jgt.23124.

[6] I. Choi, M. Furuya, R. Kim, and B. Park. A ramsey-type theorem for the matching number regarding connected graphs. *Discrete Mathematics*, 343(2):111648, 2020. https://doi.org/10.1016/j.disc.2019.111648.

[7] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, 5th edition, 2017. 10.1007/978-3-662-53622-3.

[8] M. Furuya. Forbidden subgraphs for constant domination number. *Discrete Mathematics & Theoretical Computer Science*, 20(1):19, 2018. https://doi.org/10.23638/DMTCS-20-1-19.

[9] F. Galvin, I. Rival, and B. Sands. A ramsey-type theorem for traceable graphs. *Journal of Combinatorial Theory, Series B*, 33(1):7–16, 1982. https://doi.org/10.1016/0095-8956(82)90053-3.

[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[11] H. A. Kierstead and S. G. Penrice. Radius two trees specify χ-bounded classes. *Journal of Graph Theory*, 18(2):119–129, 1994. https://doi.org/10.1002/jgt.3190180203.

[12] V. Lozin and I. Razgon. Tree-width dichotomy. *European Journal of Combinatorics*, 103:103517, 2022. https://doi.org/10.1016/j.ejc.2022.103517.

[13] V. V. Lozin. Graph parameters and ramsey theory. In *Combinatorial Algorithms*. Volume 10765, Lecture Notes in Computer Science, pages 185–194. Springer, Cham, 2018. 10.1007/978-3-319-78825-8_15.

[14] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930. https://doi.org/10.1112/plms/s2-30.1.264.

[15] A. Scott, P. Seymour, and S. Spirkl. Polynomial bounds for chromatic number. i. excluding a biclique and an induced tree. *Journal of Graph Theory*, 102(3):458–471, 2023. https://doi.org/10.1002/jgt.22880.

[16] J. Sun and X. Hou. A ramsey-type theorem on deficiency. *Journal of Graph Theory*, 110(3):313–321, 2025. https://doi.org/10.1002/jgt.23271.

[17] I. E. Zverovich and V. E. Zverovich. An induced subgraph characterization of domination perfect graphs. *Journal of Graph Theory*, 20(3):375–395, 1995. https://doi.org/10.1002/jgt.3190200313.

Jin Sun
School of Mathematical Sciences, Anhui University, Hefei, Anhui 230621, China
E-mail jinsun@mail.ustc.edu.cn

Xinmin Hou
School of Mathematical Sciences, University of Science and Technology of China
Hefei, Anhui 230026, China
Hefei National Laboratory, University of Science and Technology of China
Hefei 230088, Anhui, China
E-mail xmhou@ustc.edu.cn