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# The Leonardo triangle

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ABSTRACT

We introduce the Leonardo $k$-triangle and derive the explicit formula for generalized Leonardo numbers by using some properties of this triangle. These include elegant formulas for the generalized Leonardo numbers, although with our suggested notation as a tool of thought, we claim that Fibonacci numbers are a particular case of Leonardo numbers, rather than the other way around. Moreover, we introduce the dual Leonardo $k$-triangle to generalize the explicit formula for dual Leonardo $k$-numbers.

*Keywords:* Fibonacci number, Leonardo number, dual Leonardo number, Pascal's triangle

*2020 Mathematics Subject Classification:* 11B37, 11B39.

## 1. Introduction

As is well known, the sequences of Fibonacci $\{F_n\}_{n\geq 0}$ and Lucas numbers $\{L_n\}_{n\geq 0}$ are defined, respectively, by

$$F_0 = 0, F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1),$$

$$L_0 = 2, L_1 = 1 \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad (n \geq 1).$$

It is well known that the Fibonacci numbers can be derived by summing elements on the rising diagonal lines in Pascal's triangle

$$F_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}, \tag{1}$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x. In 1967, Feinberg [2] derived the expansion for the Lucas numbers by arranging the various coefficients of the polynomial

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$(x+1)^{n-1}(x+2)$ where $n \geq 1$, in triangular array which is called a Lucas triangle. He showed that the sum of elements on each rising diagonal line of the Lucas triangle is the Lucas number, which yields

$$L_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i}. \tag{2}$$

Kuhapatanakul and Chobsorn [4] presented the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$, for a fixed positive integer $k$, by

$$\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1 \quad \text{and} \quad \mathcal{L}_{k,n+1} = \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k \quad (n \geq 1).$$

For $k = 1$, $\mathcal{L}_{1,n} = Le_n$ is the Leonardo numbers. In [3], the dual Leonardo $k$-sequence $\{\mathcal{M}_{k,n}\}_{n\geq 0}$ is defined by

$$\mathcal{M}_{k,0} = 1-k, \mathcal{M}_{k,1} = k+3 \quad \text{and} \quad \mathcal{M}_{k,n+1} = \mathcal{M}_{k,n} + \mathcal{M}_{k,n-1} + 2k \quad (n \geq 1).$$

For $k = 1$, $\mathcal{L}_{1,n} = Le_n$ is the Leonardo numbers, and $\mathcal{M}_{1,n}$ is the dual Leonardo numbers, briefly $M_n$. The relationship between the generalized Leonardo numbers $\mathcal{L}_{k,n}$ and the dual Leonardo $k$-numbers $\mathcal{M}_{k,n}$

$$\mathcal{L}_{k,n-1} + \mathcal{L}_{k,n+1} = \mathcal{M}_{k,n} \quad \text{and} \quad \mathcal{M}_{k,n-1} + \mathcal{M}_{k,n+1} = 5\mathcal{L}_{k,n} + k,$$

see more the generalized Leonardo sequences [7, 8].

The connection between the Fibonacci, Lucas, generalized Leonardo numbers and dual Leonardo $k$-numbers are

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k \quad \text{and} \quad \mathcal{M}_{k,n} = (k+1)L_{n+1} - 2k.$$

In this work, we present the triangular arrays to derive the expansions for the generalized Leonardo numbers and the dual Leonardo $k$-numbers.

## 2. The Leonardo triangle

In this section, we introduce two triangular arrays and call *Leonardo $k$-triangle type* 1 and *Leonardo $k$-triangle type* 2. Both triangular arrays lead to the same expansion for the generalized Leonardo numbers.

### *2.1. The Leonardo $k$-triangle type* 1

Suppose that $g_0(x) = 1$ and the polynomial

$$g_n(x) = (x+1)^n + k\sum_{i=1}^{n} x^{i-1}(x+1)^{n-i}, \qquad (n \geq 1).$$

The polynomials $g_n(x)$ for $n = 1$ to $5$ are following.

$$g_1(x) = x + (k+1)$$

$$g_2(x) = x^2 + 2(k+1)x + (k+1)$$
$$g_3(x) = x^3 + 3(k+1)x^2 + 3(k+1)x + (k+1)$$
$$g_4(x) = x^4 + 4(k+1)x^3 + 6(k+1)x^2 + 4(k+1)x + (k+1)$$
$$g_5(x) = x^5 + 5(k+1)x^4 + 10(k+1)x^3 + 10(k+1)x^2 + 5(k+1)x + (k+1).$$

Now arrange the coefficients in the expansions of $g_n(x)$ to form a left-justified triangular array and call that *Leonardo $k$-triangle type* 1, see Table 1.

**Table 1.** Leonardo $k$-triangle type 1

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | $k+1$ | | | | | |
| 2 | 1 | $2(k+1)$ | $(k+1)$ | | | | |
| 3 | 1 | $3(k+1)$ | $3(k+1)$ | $(k+1)$ | | | |
| 4 | 1 | $4(k+1)$ | $6(k+1)$ | $4(k+1)$ | $(k+1)$ | | |
| 5 | 1 | $5(k+1)$ | $10(k+1)$ | $10(k+1)$ | $5(k+1)$ | $(k+1)$ | |
| 6 | 1 | $6(k+1)$ | $15(k+1)$ | $20(k+1)$ | $15(k+1)$ | $6(k+1)$ | $(k+1)$ |
| | | | | $\vdots$ | | | |

We give the following examples for the Leonardo $k$-triangle type 1 for $k = 1, 2$.

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 1 | 4 | 2 | | | | |
| 3 | 1 | 6 | 6 | 2 | | | |
| 4 | 1 | 8 | 12 | 8 | 2 | | |
| 5 | 1 | 10 | 20 | 20 | 10 | 2 | |
| 6 | 1 | 12 | 30 | 40 | 30 | 12 | 2 |
| | | | | $\vdots$ | | | |

Leonardo 1-triangle type 1

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 3 | | | | | |
| 2 | 1 | 6 | 3 | | | | |
| 3 | 1 | 9 | 9 | 3 | | | |
| 4 | 1 | 12 | 18 | 12 | 3 | | |
| 5 | 1 | 15 | 30 | 30 | 15 | 3 | |
| 6 | 1 | 18 | 45 | 60 | 45 | 18 | 3 |
| | | | | $\vdots$ | | | |

Leonardo 2-triangle type 1

The Leonardo $k$-triangle type 1 has many intriguing properties:

(1) The sum of numbers on row $n$ is $(2^n - 1)(k+1) + 1$.

(2) Any interior number in each row is the sum of the number above it and the number to the left of that one, except column 1.

(3) For $k = 0$, the Leonardo $k$-triangle type 1 is the Pascal's triangle.

Observe that the sum of elements on each rising diagonal line in the Leonardo $k$-triangle type 1 give the generalized Leonardo numbers, $\mathcal{L}_{k,n}$.

**Theorem 2.1.** *Suppose that $C_{n,i}$ is the element in the $n$-th row and $i$-th column of the*

*Leonardo $k$-triangle type* 1. *Then*

$$\mathcal{L}_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-i,i}. \tag{3}$$

**Proof.** Since $C_{n,i}$ is the coefficient of $x^{n-i}$ in $g_n(x)$, we get that

$$C_{n,i} = \begin{cases} 1, & i = 0 \\ (k+1)\binom{n}{i}, & 1 \le i \le n. \end{cases}$$

Noting first that $\mathcal{L}_{k,0} = C_{0,0} = 1$ and $\mathcal{L}_{k,1} = C_{1,0} = 1$. Now assume (3) holds for $n \ge 1$. By the definition and the inductive hypothesis, we obtain

$$\begin{aligned}
\mathcal{L}_{k,n+1} &= \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k = \sum_{i=0}^{\lfloor n/2 \rfloor} C_{n-i,i} + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} C_{n-i-1,i} + k \\
&= C_{n,0} + C_{n-1,1} + \sum_{i=2}^{\lfloor n/2 \rfloor} C_{n-i,i} + C_{n-1,0} + \sum_{i=2}^{\lfloor (n+1)/2 \rfloor} C_{n-i,i-1} + k \\
&= 2 + C_{n-1,1} + (k+1)\sum_{i=2}^{\lfloor n/2 \rfloor} \binom{n-i}{i} + (k+1)\sum_{i=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-i}{i-1} + k \\
&= \begin{cases} 2 + k + C_{n-1,1} + (k+1)\displaystyle\sum_{i=2}^{\lfloor n/2 \rfloor} \binom{n-i+1}{i}; & n \text{ even} \\ 2 + k + C_{n-1,1} + (k+1)\displaystyle\sum_{i=2}^{\lfloor (n-1)/2 \rfloor} \binom{n-i+1}{i} + (k+1); & n \text{ odd} \end{cases} \\
&= 1 + C_{n,1} + (k+1)\sum_{i=2}^{\lfloor (n+1)/2 \rfloor} \binom{n-i+1}{i} = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} C_{n-i+1,i},
\end{aligned}$$

which shows that the identity (3) holds for $n+1$, thereby proving the theorem. $\square$

We can write the generalized Leonardo numbers $\mathcal{L}_{k,n}$ in terms of binomial coefficient sums.

**Theorem 2.2.** *For any nonnegative integer $n$, we have*

$$\mathcal{L}_{k,n} = 1 + (k+1)\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}. \tag{4}$$

**Proof.** By Theorem 2.1 and the definition of $C_{n,i}$, we get that

$$\mathcal{L}_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-i,i} = C_{n,0} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} C_{n-i,i} = F_{n+1} + k\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} + k \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \qquad \text{Using (1)}$$
$$= 1 + (k+1) \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

Therefore, we get the desired result. □

### *2.2. The Leonardo k-triangle type* 2

For fixed integer $k \geq 0$, suppose that the numbers

$$A_{n,i} = \begin{cases} F_{n+1}; & i = 0 \\ \binom{n}{i} k; & 1 \leq i \leq n \\ 0; & i > n \end{cases}.$$

**Definition 2.3.** Define the Leonardo $k$-triangle type 2 as follows:

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $A_{0,0}$ | | | | | | | |
| 1 | $A_{1,0}$ | $A_{1,1}$ | | | | | | |
| 2 | $A_{2,0}$ | $A_{2,1}$ | $A_{2,2}$ | | | | | |
| 3 | $A_{3,0}$ | $A_{3,1}$ | $A_{3,2}$ | $A_{3,3}$ | | | | |
| 4 | $A_{4,0}$ | $A_{4,1}$ | $A_{4,2}$ | $A_{4,3}$ | $A_{4,4}$ | | | |
| 5 | $A_{5,0}$ | $A_{5,1}$ | $A_{5,2}$ | $A_{5,3}$ | $A_{5,4}$ | $A_{5,5}$ | | |
| $\vdots$ | | | | $\vdots$ | | | | |
| $n$ | $A_{n,0}$ | $A_{n,1}$ | $A_{n,2}$ | $A_{n,3}$ | | $\cdots$ | | $A_{n,n}$ |

For clarity, we also give the following examples of the Leonardo $k$-triangle type 2 for $k = 2, 3$:

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 2 | 4 | 2 | | | | |
| 3 | 3 | 6 | 6 | 2 | | | |
| 4 | 5 | 8 | 12 | 8 | 2 | | |
| 5 | 8 | 10 | 20 | 20 | 10 | 2 | |
| 6 | 13 | 12 | 30 | 40 | 30 | 12 | 2 |
| 7 | 21 | 14 | 42 | 70 | $\cdots$ | | |
| 8 | 34 | 16 | 54 | $\cdots$ | | | |
| $\vdots$ | | | | $\vdots$ | | | |

Leonardo 1-triangle type 2

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 3 | | | | | |
| 2 | 2 | 6 | 3 | | | | |
| 3 | 3 | 9 | 9 | 3 | | | |
| 4 | 5 | 12 | 18 | 12 | 3 | | |
| 5 | 8 | 15 | 30 | 30 | 15 | 3 | |
| 6 | 13 | 18 | 45 | 60 | 45 | 18 | 3 |
| 7 | 21 | 21 | 63 | $\cdots$ | | | |
| 8 | 34 | 24 | 84 | $\cdots$ | | | |
| $\vdots$ | | | | $\vdots$ | | | |

Leonardo 2-triangle type 2

Observe that the sum of elements on each rising diagonal line in the Leonardo $k$-triangle type 2 give the generalized Leonardo numbers, $\mathcal{L}_{k,n}$. We conjecture that the sum of elements on each rising diagonal line in the Leonardo $k$-triangle type 2 gives the generalized Leonardo numbers $\mathcal{L}_{k,n}$. We begin with some properties of the Leonardo $k$-triangle type 2.

**Lemma 2.4.** *For a nonnegative integer $n$ and $2 \leq i \leq n$, we have*

(i) $A_{n-1,0} + A_{n,0} = A_{n+1,0}$

(ii) $A_{n,1} + k = A_{n+1,1}$

(iii) $A_{n,i-1} + A_{n,i} = A_{n+1,i}$.

**Proof.** Since $A_{n_0} = F_{n+1}$ and $F_{n-1} + F_n = F_{n+1}$, we get the part (i). The parts (ii) and (iii) can be proven by using the Pascal identity, $\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$. □

**Theorem 2.5.** *Let $n$ be a nonnegative integer. Then*

$$\mathcal{L}_{k,n} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i}. \tag{5}$$

**Proof.** We will prove this result by induction on $n$, noting first that

$$\mathcal{L}_{k,0} = 1 = A_{0,0} \quad \text{and} \quad \mathcal{L}_{k,1} = 1 = A_{1,0}.$$

Suppose (5) holds for $n > 1$. We will show that this implies the identity holds for $n+1$. To this end,

$$\begin{aligned}
\mathcal{L}_{k,n+1} &= \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} + k \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i} + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} A_{n-i-1,i} + k \\
&= A_{n,0} + A_{n-1,1} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i} + A_{n-1,0} + \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} A_{n-i,i-1} + k \\
&= (A_{n,0} + A_{n-1,0}) + (A_{n-1,1} + k) + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i} + \sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} A_{n-i,i-1} \\
&= \begin{cases} A_{n+1,0} + A_{n,1} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} A_{n-i+1,i}; & n \text{ even} \\ A_{n+1,0} + A_{n,1} + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} A_{n-i+1,i} + A_{n+1,n+1}; & n \text{ odd} \end{cases} \\
&= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} A_{n-i+1,i},
\end{aligned}$$

so the proof is complete. □

We can write the generalized Leonardo numbers $\mathcal{L}_{k,n}$ in terms of binomial coefficient sums.

**Theorem 2.6.** *Let $n$ be a nonnegative integer. Then*

$$\mathcal{L}_{k,n} = 1 + (k+1)\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

**Proof.** By Theorem 2.5 and the definition of $A_{n,i}$, we get that

$$\begin{aligned}\mathcal{L}_{k,n} &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i} = A_{n,0} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} A_{n-i,i} = F_{n+1} + k\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} + k\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \qquad \text{Using (1)} \\ &= 1 + (k+1)\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.\end{aligned}$$

Therefore, we get the desired result. □

## 3. The dual Leonardo triangle

Define the folowing polynomial as

$$f_{n+1}(x) = 3(x+1)^n + x^2(x+1)^{n-1} + 2\sum_{j=2}^{n} x^{j+1}(x+1)^{n-j},$$

where $(x+1)^{-n} = 0$, for $n \geq 0$, we have that

$$\begin{aligned} f_1(x) &= 3 \\ f_2(x) &= x^2 + 3x + 3 \\ f_3(x) &= 3x^3 + 4x^2 + 6x + 3 \\ f_4(x) &= 5x^4 + 7x^3 + 10x^2 + 9x + 3 \\ f_5(x) &= 7x^5 + 12x^4 + 17x^3 + 19x^2 + 12x + 3 \\ f_6(x) &= 9x^6 + 19x^5 + 29x^4 + 36x^3 + 31x^2 + 15x + 3 \\ &\vdots \end{aligned}$$

One such triangle is generated by the coefficients of $f_n(x)$ and call this array that *dual Leonardo triangle*, see Table 2.

Let $B_{n,i}$ be the element in the $n$th row and $i$th column of the dual Leonardo triangle or the coefficient of $x^{n-i}$ in $f_n(x)$. We see that the sum of numbers on row $n$ of the dual Leonardo triangle is $9 \cdot 2^{n-2} - 2$ for $n \geq 2$, see sequence $A176449$ in [5]. We see that

**Table 2.** The dual Leonardo triangle

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | | | | | | | | |
| 2 | 1 | 3 | 3 | | | | | | | |
| 3 | 3 | 4 | 6 | 3 | | | | | | |
| 4 | 5 | 7 | 10 | 9 | 3 | | | | | |
| 5 | 7 | 12 | 17 | 19 | 12 | 3 | | | | |
| 6 | 9 | 19 | 29 | 36 | 31 | 15 | 3 | | | |
| 7 | 11 | 28 | 48 | 65 | 67 | 46 | 18 | 3 | | |
| 8 | 13 | 39 | 56 | 113 | 132 | 113 | 64 | 21 | 3 | |
| $\vdots$ | | | | | $\vdots$ | | | | | |

(i) $B_{1,0} = 0, B_{n,n} = 3$ and $B_{n,0} = 2n - 3$ for $n \geq 2$,

(ii) $B_{n,i} = B_{n-1,i} + B_{n-1,i-1}$ for $0 < i < n$.

Observe that the sum of elements on each rising diagonal line in the dual Leonardo triangle give the dual Leonardo number, that is, for $n \geq 1$,

$$M_{n-1} = \sum_{i=0}^{\lfloor n/2 \rfloor} B_{n-i,i}.$$

Indeed, we will show the generalized version of the dual Leonardo triangle. We begin by introducing polynomials $f_{k,i}(x)$ as

$$\begin{aligned}
f_{k,1}(x) &= (1-k)x + 2k + 1 \\
f_{k,2}(x) &= (x+1)f_{k,1}(x) + x^2 \\
f_{k,i}(x) &= (x+1)f_{k,i-1}(x) + 2kx^i \qquad \text{for } i \geq 3.
\end{aligned}$$

It is easy to check that, for $n \geq 3$, we obtain

$$f_{k,n}(x) = (x+1)^{n-2} f_{k,2} + 2k \sum_{j=0}^{n-3} x^{n-j}(x+1)^j.$$

The first few terms of the polynomials $f_{k,n}(x)$ for $n = 1, 2, 3, 4, 5$ are shown in the following

$$\begin{aligned}
f_{k,1}(x) &= (1-k)x + 2k + 1 \\
f_{k,2}(x) &= (2-k)x^2 + (k+2)x + 2k + 1 \\
f_{k,3}(x) &= (2+k)x^3 + 4x^2 + (3k+3)x + 2k + 1 \\
f_{k,4}(x) &= (2+3k)x^4 + (k+6)x^3 + (3k+7)x^2 + (5k+4)x + 2k + 1 \\
f_{k,5}(x) &= (2+5k)x^5 + (4k+8)x^4 + (4k+13)x^3 + (8k+11)x^2 + (7k+5)x + 2k + 1.
\end{aligned}$$

Now arrange the coefficients in the expansions of $f_{k,n}(x)$ to form a left-justified triangular array and call this array that *dual Leonardo k-triangle*, see Table 3.

**Table 3.** The dual Leonardo $k$-triangle

| $n\backslash i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $1-k$ | $2k+1$ | | | | | | |
| 2 | $2-k$ | $k+2$ | $2k+1$ | | | | | |
| 3 | $2+k$ | $4$ | $3k+3$ | $2k+1$ | | | | |
| 4 | $2+3k$ | $k+6$ | $3k+7$ | $5k+4$ | $2k+1$ | | | |
| 5 | $2+5k$ | $4k+8$ | $4k+13$ | $8k+11$ | $7k+5$ | $2k+1$ | | |
| 6 | $2+7k$ | $9k+10$ | $8k+21$ | $12k+24$ | $15k+16$ | $9k+6$ | $2k+1$ | |
| $\vdots$ | | | | | $\vdots$ | | | |

Let $B^k_{n,i}$ denote the entry in row $n$ and column $i$ of the dual Leonardo $k$-triangle. It is easy to see that some properties of the dual Leonardo $k$-triangle.

$$B^k_{n,0} = (2n-5)k+2 \quad \text{and} \quad B^k_{n,i} = B^k_{n-1,i} + B^k_{n-1,i-1}, \quad (1 \le i < n).$$

Note that, for $k = 1$, $B^1_{n,i} = B_{n,i}$ and the dual Leonado 1-triangle is just the dual Leonado triangle. Observe that sums of elements on each rising diagonal line in the dual Leonardo $k$-triangle give the dual Leonardo $k$-number. We begin to provide a following theorem.

**Theorem 3.1.** *For all integers $n \ge 1$,*

$$\mathcal{M}_{k,n-1} = \sum_{i=0}^{\lfloor n/2 \rfloor} B^k_{n-i,i}. \tag{6}$$

**Proof.** We proceed by induction on $n$, noting first that

$$\mathcal{M}_{k,0} = B^k_{1,0} = 1-k \quad \text{and} \quad \mathcal{M}_{k,1} = B^k_{2,0} + B^k_{1,1} = k+3.$$

Now assume (6) holds for $n > 1$. By the definition and the inductive hypothesis, we obtain

$$\begin{aligned}
\mathcal{M}_{k,n} &= \mathcal{M}_{k,n-1} + \mathcal{M}_{k,n-2} + 2k = \sum_{i=0}^{\lfloor n/2 \rfloor} B^k_{n-i,i} + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} B^k_{n-i-1,i} + 2k \\
&= 2k + B^k_{n,0} + \sum_{i=1}^{\lfloor n/2 \rfloor} B^k_{n-i,i} + \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} B^k_{n-i,i-1} \\
&= \begin{cases} B^k_{n+1,0} + \sum\limits_{i=1}^{\lfloor n/2 \rfloor} \left(B^k_{n-i,i} + B^k_{n-i,i-1}\right); & n \text{ even} \\ B^k_{n+1,0} + \sum\limits_{i=1}^{\lfloor (n-1)/2 \rfloor} \left(B^k_{n-i,i} + B^k_{n-i,i-1}\right) + B^k_{\frac{n-1}{2},\frac{n-1}{2}}; & n \text{ odd} \end{cases} \\
&= \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} B^k_{n-i+1,i},
\end{aligned}$$

which shows that the identity (6) holds for $n+1$, thereby proving the theorem. □

**Theorem 3.2.** *For all integers $n \geq 1$, we have*

$$\mathcal{M}_{k,n-1} = (k+1)\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i} - 2k. \tag{7}$$

**Proof.** Since $B_{n,i}^k$ is the coefficient of $x^{n-i}$ in $f_{k,n}(x)$, where $n \geq 1$ and $i \geq 0$, we get

$$B_{n,i}^k = 2k\binom{n-2}{i+1} + (2-k)\binom{n-2}{i} + (k+2)\binom{n-2}{i-1} + (2k+1)\binom{n-2}{i-2}.$$

Using the Pascal's identity, we can write

$$B_{n,i}^k = 2k\binom{n-2}{i+1} + (1-k)\binom{n-1}{i} + (2k+1)\binom{n-1}{i-1} + \binom{n-2}{i}.$$

Consider

$$\sum_{i=0}^{\lfloor n/2 \rfloor} 2k\binom{n-i-2}{i+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} 2k\binom{n-i-1}{i} - 2k,$$

we get that

$$\sum_{i=0}^{\lfloor n/2 \rfloor}\left(2k\binom{n-i-2}{i+1} - k\binom{n-i-1}{i} + k\binom{n-i-1}{i-1}\right) = \sum_{i=0}^{\lfloor n/2 \rfloor} k\binom{n-i}{i} - 2k,$$

and

$$\sum_{i=0}^{\lfloor n/2 \rfloor}\left(\binom{n-i-1}{i} + \binom{n-i-2}{i}\right) = \sum_{i=0}^{\lfloor n/2 \rfloor}\binom{n-i}{i}.$$

Thus, we obtain

$$\begin{aligned}
\mathcal{M}_{k,n-1} &= \sum_{i=0}^{\lfloor n/2 \rfloor} B_{n-i,i}^k \\
&= (k+1)\sum_{i=0}^{\lfloor n/2 \rfloor}\left(\binom{n-i}{i} + \binom{n-i-1}{i-1}\right) - 2k \\
&= (k+1)\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i} - 2k,
\end{aligned}$$

as desired. □

**Corollary 3.3.** *For all integers $n \geq 1$, we have*

$$M_{n-1} = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i} - 2. \tag{8}$$

## 4. Concluding Comments

The preceding ideas suggest that Leonardo multi-nacci $k$-sequences (LMkS) $\{L_{n,k}^{(r)}\}$ can be generated by the linear recurrence relation of order $r$ :

$$L_{n,k}^{(r)} = L_{n-1,k}^{(r)} + L_{n-2,k}^{(r)} + \cdots + L_{n-r,k}^{(r)} + k.$$

For examples,

(i) for $k = 1$, $L_{n,1}^{(2)} = L_{n-1,1}^{(2)} + L_{n-2,1}^{(2)} + 1$ is the classical Leonardo numbers

(ii) for $k = 0$, $L_{n,0}^{(2)} = L_{n-1,0}^{(2)} + L_{n-2,0}^{(2)} + 0$ is the classical Fibonacci numbers, assuming initial conditions of unity.

(iii) for $k = 0$, $L_{n,0}^{(3)} = L_{n-1,0}^{(3)} + L_{n-2,0}^{(3)} + L_{n-3,0}^{(3)} + 0$ yield the Feinberg's tribonacci numbers, or the tri-nacci Leonardo numbers.

See some examples, with italicised initial values :

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Sloane |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_{n,0}^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | *A*000023; Units numbers |
| $L_{n,1}^{(1)}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | *A*000027; Natural numbers |
| $L_{n,0}^{(2)}$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | *A*000045; Fibonacci numbers |
| $L_{n,1}^{(2)}$ | 1 | 1 | 3 | 5 | 9 | 15 | 25 | 41 | 67 | 109 | *A*128587; Leonardo numbers |
| $L_{n,0}^{(3)}$ | 1 | 1 | 1 | 3 | 5 | 9 | 17 | 31 | 57 | 105 | *A*000213; Tribonacci numbers |
| $L_{n,1}^{(3)}$ | 1 | 1 | 1 | 4 | 7 | 13 | 25 | 46 | 85 | 157 | *A*248098; Tri-nacci Leonardo numbers |
| $L_{n,0}^{(4)}$ | 1 | 1 | 1 | 1 | 4 | 7 | 13 | 25 | 49 | 94 | *A*000288; Tetranacci numbers |
| $L_{n,1}^{(4)}$ | 1 | 1 | 1 | 1 | 5 | 9 | 17 | 33 | 65 | 125 | Quadri-nacci Leonardo numbers |

This, in turn, suggests further searches for patterns of intersections among these sequences [6, 9] and varieties of combinations of these sequences [1].

## Conflicts of Interest

The authors declare no conflicts of interest.

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## Data Availability

This study is theoretical and does not involve the generation or analysis of datasets.

## Author Contributions

All the authors contributed equally. All authors have read and agreed to the published version of the manuscript.

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