UTILITAS MATHEMATICA 128 (2026) 219–236

Utilitas Mathematica
www.combinatorialpress.com/um

Combinatorial Press

Word-representable co-bipartite graphs: Representation number, speed, and entropy

Biswajit Das and Ramesh Hariharasubramanian*

ABSTRACT

A graph $G(V, E)$ is *word-representable* if there exists a word w over the alphabet V such that for distinct letters $x, y \in V$, x and y alternate in w if and only if they are adjacent in G. In general, determining whether a graph is word-representable is an NP-complete problem. A graph is *co-bipartite* if its complement is bipartite. Therefore, the vertex set of a co-bipartite graph can be partitioned into two disjoint subsets X and Y such that the subgraphs induced by X and Y are cliques. Necessary and sufficient conditions for a co-bipartite graph to be word-representable in terms of a vertex ordering are known. Based on this ordering, we study the representation number of word-representable co-bipartite graphs and analyse the speed and entropy of this graph class. We show that the representation number of any word-representable co-bipartite graph is at most 3, and permutation graphs are the only co-bipartite graphs with representation number 2. We prove that the speed is at most $2^{O(n \log n)}$ and the entropy is 0. In particular, we obtain an upper bound on the number of labelled graphs in this class, which is significantly smaller than the known bound for the class of all co-bipartite graphs. These results provide a better understanding of the structure and enumeration of word-representable co-bipartite graphs and show that vertex ordering is an effective tool for studying this class.

Keywords: word-representable graph, co-bipartite graph, representation number, speed

2020 Mathematics Subject Classification: 05C62, 05C30.

1. Introduction

The concept of word-representable graphs was introduced by Sergey Kitaev in connection with the study of the Perkins semigroup [30]. Since then, this notion has been widely

* Corresponding author.

Received 03 Jun 2026; Revised 10 Jul 2026; Accepted 16 Jul 2026; Published Online 22 Jul 2026.

DOI: 10.61091/um128-11

studied and has found connections to algebra, graph theory, computer science, and combinatorics on words. Word-representable graphs generalise several well-known graph classes, including circle graphs, comparability graphs, and all 3-colourable graphs. These graphs also arise in scheduling problems with repeating tasks, where certain pairs of tasks are required to alternate over time, and these alternation requirements naturally form an undirected graph. A general overview of the background and motivation for studying word-representable graphs is given in [27].

A graph is said to be k-word-representable if there exists a k-uniform word representing it. According to Corollary 2.11 and Proposition 2.12, determining whether a graph is word-representable, and whether it is k-word representable for any given $3 \leq k \leq \lceil n/2 \rceil$, are both NP-complete problems. From a k-word representation of a word-representable graph, one can derive an upper bound on the representation number of the graph. Therefore, it is important to study the word-representability of specific graph classes and to obtain polynomial-time bounds on their representation numbers.

Word-representability and representation numbers have been studied for several classes of graphs. In [28], outerplanar graphs and prism graphs were shown to be word representable, while odd wheel graphs were proved to be non-word representable. The word-representability of crown graphs and k-cube graphs was investigated in [19] and [9], respectively. The representation number of crown graphs was determined in [19], where it was shown that the crown graph $H_{k,k}$ has representation number $\lceil k/2 \rceil$. More recently, Hefty et al. [22] established several lower bounds on the representation numbers of various graph classes, including balanced bipartite graphs, hypercubes, and comparability graphs. The word-representability of split graphs was studied in [23, 26, 24, 11]. In [16], it was shown that the representation number of any word-representable split graph is at most three. In particular, [29] studied the word-representability of split graphs using vertex labelling and also provided a polynomial-time recognition algorithm for word-representable split graphs. In [14], the authors established necessary and sufficient conditions for a co-bipartite graph to be word-representable based on orientation. In [10], the authors explored K_m-K_n graphs (co-bipartite graphs) and identified forbidden induced subgraphs for K_m–K_3 and K_m–K_4. Later, a necessary and sufficient vertex-ordering characterisation of word-representable co-bipartite graphs was obtained in [15]. Vertex-ordering characterisations have been widely used to obtain forbidden-pattern characterisations of several important graph classes, including interval graphs, permutation graphs, and comparability graphs. For example, a graph G is an interval graph if and only if there exists a vertex ordering such that, for any three vertices $x \prec y \prec z$, adjacency of x and z implies adjacency of y and z [32]. Additional significant results on forbidden-pattern characterisations can be found in [13, 18] and [7, Section 7.4].

The class of co-bipartite graphs contains both word-representable and non-word representable graphs. In particular, the graphs G_1 and G_2 are minimal, with respect to the number of vertices, non-word-representable co-bipartite graphs, as shown in Figure 1.

$G_1 =$ $G_2 =$

Fig. 1. The minimal (by the number of vertices) non-word-representable co-bipartite graphs G_1[27] and G_2 [31]

The classes of split graphs and co-bipartite graphs are hereditary, since every induced subgraph of a graph in either class also belongs to the same class. Split graphs and co-bipartite graphs play an important role in the study of the speed of hereditary graph classes and their asymptotic structure, and these aspects have been extensively studied in the literature. These concepts are also closely related to the asymptotic enumeration of word-representable graphs, as discussed in [12]. In general, for every infinite hereditary class X of simple graphs, the following holds:

$$\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)},$$

where $k(X)$ is called the index of X. This index is defined using the graph classes $\mathcal{E}_{i,j}$, consisting of graphs whose vertices can be partitioned into at most i independent sets and j cliques. In particular, bipartite graphs, split graphs, and co-bipartite graphs correspond to the classes $\mathcal{E}_{2,0}$, $\mathcal{E}_{1,1}$, and $\mathcal{E}_{0,2}$, respectively. The index $k(X)$ is the maximum integer k such that X contains the class $\mathcal{E}_{i,j}$ for some integers i and j satisfying $i+j = k$. This result was independently established by Alekseev [1] and by Bollobás and Thomason [5, 6], and is now known as the Alekseev–Bollobás–Thomason Theorem (see also [2]). Recent work by Feng et al. [17] established a general framework for proving factorial speed via finite-language graph representations. This approach provides a powerful tool for establishing the factorial speed of hereditary graph classes and has been used to show that several graph classes have at most factorial speed.

Since co-bipartite graphs are complements of bipartite graphs, they can be recognised in polynomial time. Furthermore, several computational problems that are difficult for general graphs admit polynomial-time algorithms when restricted to co-bipartite graphs. For example, Brandstädt et al. [8] gave a polynomial-time algorithm to determine whether a graph contains a stable set whose removal results in a co-bipartite graph. Similarly, the *cd*-coloring problem can be solved in polynomial time for co-bipartite graphs [33]. However, not all computational problems become easier on this class. Bodlaender and Jansen [3] showed that the SIMPLE MAX CUT problem remains NP-complete for co-bipartite graphs. Moreover, the class of co-bipartite graphs has unbounded clique-width and contains at least $2^{n^2/4}$ distinct graphs on n vertices [4].

The study of word-representability of co-bipartite graphs is natural in the context of graph classes defined by vertex partitions. Bipartite graphs, where the vertex set can be partitioned into two independent sets, and split graphs, where the vertex set can be partitioned into one clique and one independent set, have already been studied in the context of word-representability. Co-bipartite graphs, where the vertex set can be partitioned into two cliques, form the complementary class of bipartite graphs. Therefore, it is natural to study the word-representability of co-bipartite graphs.

A necessary and sufficient condition for word-representable co-bipartite graphs in terms of a vertex ordering was established in [15]. Using this ordering, we present a constructive algorithm for obtaining a 3-uniform word-representation of every word-representable co-bipartite graph, thereby establishing that its representation number is at most 3. Furthermore, we prove that a word-representable co-bipartite graph has representation number 3 if and only if it is not a permutation graph. Since the class of word-representable graphs is hereditary, the class of word-representable co-bipartite graphs is also hereditary. We also determine the speed and entropy of the class of word-representable co-bipartite graphs, showing that its speed is at most $2^{O(n \log n)}$ and its entropy is 0. Thus, vertex ordering serves as an effective tool for studying the structural and combinatorial properties of word-representable co-bipartite graphs.

The main goal of this paper is to study the word-representability of co-bipartite graphs via vertex ordering. In Section 2, we introduce the necessary definitions and preliminaries related to word-representable graphs. In Section 3, based on the vertex ordering of word-representable co-bipartite graphs [15], we present an algorithm that constructs a 3-uniform word-representation for any word-representable co-bipartite graph. In Section 4, we study the speed and entropy of the class of word-representable co-bipartite graphs.

2. Preliminaries

In this section, we briefly describe the necessary definitions, notation, and basic results on word-representable graphs.

A simple graph $G(V, E)$ is an undirected graph with a vertex set V and an edge set E. For a vertex $v \in V$, N_v denotes the set of vertices in V that are adjacent (neighbours) to v. The *degree* of a vertex v is defined as the cardinality of N_v. A subset $S \subseteq V$ is called an *independent set* if no two distinct vertices $u, v \in S$ are adjacent. A subset $C \subseteq V$ is called a *clique* if every pair of distinct vertices $u, v \in C$ is adjacent. A clique C is said to be *maximal* if there is no vertex $v \in V \setminus C$ such that $C \cup \{v\}$ is also a clique in G.

For a graph $G(V, E)$, let V^+ denote the set of all nonempty words over the alphabet V. Suppose w is a word over an alphabet V. Two distinct letters x and y, with $x, y \in V$, are said to *alternate* in the word w if, after deleting all letters in w except the occurrences of x and y, the resulting word is of the form $xyxy\cdots$ or $yxyx\cdots$ (of either odd or even length). Both x and y must appear at least once in w for alternation to occur. If x and y do not alternate in w, then x and y are called *non-alternating* in w.

Definition 2.1 ([27] , Definition 3.0.5)**.** A simple graph $G(V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y, where $x \neq y$ and $x, y \in V$, alternate in w if and only if x and y are adjacent in G. If a word w *represents* G, then w contains each letter of V at least once.

Definition 2.2 ([27], Definition 3.2.1.). A *k-uniform word* is a word w in which every letter occurs exactly k times.

Definition 2.3 ([27], Definition 3.2.3.). A graph is *k-word-representable* (or *k-representable*) if there exists a k-uniform word representing it.

Theorem 2.4 ([28], Theorem 7). *A graph G is word-representable if and only if there exists an integer $k \geq 1$ such that G is k-representable.*

Definition 2.5 ([25], Definition 3). For a word-representable graph G, *the representation number* is the least k such that G is k-representable.

For a circle graph, the representation number is already known.

Theorem 2.6 ([25], Theorem 6).

$$\mathcal{R}_2 = \{G : G \textit{ is a non-complete circle graph}\}.$$

Proposition 2.7 ([27], Proposition 3.0.15). *Let $w = w_1xw_2xw_3$ be a word representing a graph G, where w_1, w_2 and w_3 are possibly empty words, and w_2 contains no x. Let X be the set of all letters that appear only once in w_2. Then, possible candidates for x to be adjacent in G are the letters in X.*

The *initial permutation* of w is the permutation obtained by removing all but the leftmost occurrence of each letter x in w, and it is denoted by $\pi(w)$. Similarly, the *final permutation* of w is the permutation obtained by removing all but the rightmost occurrence of each letter x in w, and it is denoted $\sigma(w)$. For a word w, $w_{\{x_1,\cdots,x_m\}}$ denotes the word after removing all letters except the letters $x_1, \ldots, x_m$ present in w.

Example 2.8. $w = 6345123215$, we have $\pi(w) = 634512$, $\sigma(w) = 643215$ and $w_{\{6,5\}} = 655$.

There exists a connection between graph orientations and word-representability. In the following, the definition of a semi-transitive orientation and its connection to word-representability are described.

Semi-transitive orientation is defined based on shortcuts in the papers [20] and [21]. A semi-cycle is the directed acyclic graph obtained by reversing the direction of one edge of a directed cycle [27]. An acyclic digraph is a *shortcut* if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, any shortcut

- is *acyclic* (that is, there are *no directed cycles*);
- has *at least* 4 vertices;
- has *exactly one* source (the vertex with no edges coming in), *exactly one* sink (the vertex with no edges coming out), and a *directed path* from the source to the sink that goes through *every* vertex in the graph;

- has an edge connecting the source to the sink that we refer to as the *shortcutting edge*;
- is *not* transitive (that is, there exist vertices u, v and z such that $u \to v$ and $v \to z$ are edges, but there is *no* edge $u \to z$).

Definition 2.9. [21] An orientation of a graph is *semi-transitive* if it is *acyclic* and *shortcut-free.*

From these definitions, it is clear that *any* transitive orientation is necessarily semi-transitive. The converse is *not* true. Thus, semi-transitive orientations generalise transitive orientations. A key result in the theory of word-representable graphs is the following theorem.

Theorem 2.10 ([21], Theorem 3)**.** *A graph is word-representable if and only if it admits a semi-transitive orientation. Moreover, each non-complete word-representable graph is* $2(n-\kappa)$*-word-representable where* κ *is the size of the maximum clique in* G*.*

The recognition problem of word-representable graphs and deciding the representation number of word-representable graphs are NP-complete problems.

Corollary 2.11 ([21], Corollary 2)**.** *The recognition problem for word-representable graphs is NP-complete.*

Proposition 2.12 ([21], Proposition 8.)**.** *Deciding whether a given graph is a k-word-representable graph, for any given* $3 \leq k \leq \lceil n/2 \rceil$*, is NP-complete.*

The following characterisation of word-representable split graphs is derived using vertex labelling. In what follows, for any two integers $a \leq b$, we denote the set of integers $\{a, a+1, \ldots, b\}$ by $[a, b]$.

Theorem 2.13. [29] *Let* $G(I \cup C, E)$ *be a split graph. Then,* G *is word-representable if and only if the vertices of* C *can be labelled from* 1 *to* $k = |C|$ *in such a way that for each* $a, b \in I$ *the following holds.*

(1) *Either* $N(a) = [1, m] \cup [n, k]$*, for* $m < n$*, or* $N(a) = [l, r]$*, for* $l \leq r$*.*

(2) *If* $N(a) = [1, m] \cup [n, k]$ *and* $N(b) = [l, r]$*, for* $m < n$ *and* $l \leq r$*, then* $l > m$ *or* $r < n$*.*

(3) *If* $N(a) = [1, m] \cup [n, k]$ *and* $N(b) = [1, m'] \cup [n', k]$*, for* $m < n$ *and* $m' < n'$*, then* $m' < n$ *and* $m < n'$*.*

A word u contains a word v as a *factor* if $u = xvy$ where x and y can be empty words. In this paper, $w = w_1 w_2 \cdots w_n$ denotes the word w contains $\{w_1, w_2, \ldots, w_n\}$ as factors where w_i is a word possibly empty. In a graph $G(V, E)$, we denote the edges between two vertices x and y as $x \sim y$. If there is no edge between them, we denote this as $x \nsim y$.

Since co-bipartite graphs are complements of bipartite graphs, let $B(X, Y)$ denote a bipartite graph with independent sets X and Y, where $|X|= m$, $|Y|= n$, and X is the larger independent set. We denote its complement, i.e., the co-bipartite graph, by $\overline{B}(K_m, K_n)$, where $K_m = \overline{B}[X]$ and $K_n = \overline{B}[Y]$ are cliques. Thus, two vertices are adjacent in $\overline{B}(K_m, K_n)$ if and only if they are non-adjacent in $B(X, Y)$. Since X is chosen as the larger independent set in $B(X, Y)$, every apex vertex (a vertex adjacent to all other vertices) of $\overline{B}(K_m, K_n)$ belong to the maximal clique K_m.

The word-representability of co-bipartite graphs is studied in [14], where necessary and sufficient conditions for semi-transitive orientations of co-bipartite graphs are established. The known results concerning the semi-transitive orientation of co-bipartite graphs are as follows.

Observation 2.14. [14] *Any semi-transitive orientation of $\overline{B}(K_m, K_n)$ subdivides the set of all vertices into three possibly empty groups of the types shown schematically in Figure 2.*

- *The vertical oriented paths are a schematic way to show (parts of) $\vec{P_1}$ for $V(K_m)$ and $\vec{P_2}$ for $V(K_n)$;*
- *For the vertices of K_m, the vertical oriented paths in the types A and B represent up to l consecutive vertices in $\vec{P_1}$, $l \leq n$. Similarly, for the vertices of K_n, the vertical oriented paths in the types A and B represent up to l consecutive vertices in $\vec{P_2}$, $l \leq m$.*
- *For the vertices of K_m, the vertical oriented path in type C contains all n vertices in $\vec{P_1}$. Similarly, for the vertices of K_n, the vertical oriented path in type C contains all m vertices in $\vec{P_2}$. These oriented paths P_1 and P_2 are subdivided into three groups of consecutive vertices, with the middle group possibly containing no vertices; the group of vertices containing the source (resp., sink) is the* source-group *(resp.,* sink-group*).*

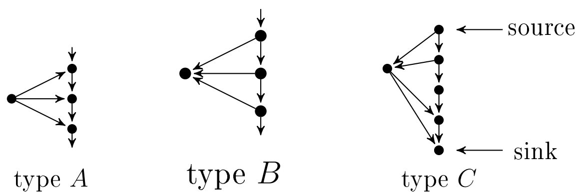

Fig. 2. Three types of vertices in K_m and K_n

Lemma 2.15. [14] *Suppose that the graph $\overline{B}(K_m, K_n)$ is word-representable and S is a semi-transitive orientation of the graph $\overline{B}(K_m, K_n)$. Suppose $\{x, y\} \in V(K_m)$ and x is a type A vertex and y is a type B vertex and the orientation of the edge between x and y is $y \to x$. Then, x and y do not have any common neighbour among the vertices of K_n.*

Lemma 2.16. [14] *Suppose the graph $\overline{B}(K_m, K_n)$ is word-representable and S is a semi-*

transitive orientation of the graph $\overline{B}(K_m, K_n)$. Suppose $\{x, y\} \in V(K_m)$ and $\{x_s, x_{s+1}\} \in V(K_n)$, and the orientation of the edges between $\{x, y\}$ is $x \to y$, and between $\{x_s, x_{s+1}\}$ is $x_s \to x_{s+1}$ in the semi-transitive orientation S. Then, the semi-transitive orientation S follows the following conditions:

(a) *If the orientation of the edges between $\{x_s, x\}$ is $x_s \to x$ and between $\{y, x_{s+1}\}$ is $y \to x_{s+1}$ in the semi-transitive orientation S, then $x \to x_{s+1}$ and $x_s \to y$ in the semi-transitive orientation S.*

(b) *If the orientation of the edges between $\{x, x_{s+1}\}$ is $x \to x_{s+1}$ and between $\{x_s, x\}$ is $y \to x_s$ in the semi-transitive orientation S, then $x \to x_s$ and $y \to x_{s+1}$ in the semi-transitive orientation S.*

Lemma 2.17. [14] *Suppose the graph $\overline{B}(K_m, K_n)$ is word-representable and S is a semi-transitive orientation of the graph. Let $x \in V(\overline{B}(K_m, K_n))$, and x is a type C vertex. Suppose x_s and x_{s+1} are the last and first vertices of the source and sink group, respectively. Now, for $y \in V(\overline{B}(K_m, K_n))$, if x and y are in the same partition in the graph $V(\overline{B}(K_m, K_n))$, then the following conditions hold for y.*

(a) *If the edges between x and y are oriented as $x \to y$, y is not a type A vertex that is adjacent to both x_s and x_{s+1}. Also, if y is a type C vertex, then x_s cannot be contained in the sink group of y.*

(b) *If the edges between x and y are oriented as $y \to x$, y is not a type B vertex that is adjacent to both x_s and x_{s+1}. Also, if y is a type C vertex, then x_{s+1} cannot be contained in the source group of y.*

Theorem 2.18. [14] *A co-bipartite graph $\overline{B}(K_m, K_n)$ has a semi-transitive orientation if and only if*

- *K_m and K_n are oriented transitively,*
- *In the graph $\overline{B}(K_m, K_n)$, every vertex belongs to one of the three types described in Observation 2.14, and*
- *The conditions present in Lemmas 2.15, 2.16 and 2.17 are satisfied.*

3. Representation number

In the paper [15], a necessary and sufficient condition for a word-representable co-bipartite graph is derived based on the vertex ordering obtained from Observation 2.14. In the following theorems, we denote the neighbours of a type A or B vertex a_i (excluding the vertices of K_m) as $[x_i, y_i] = \{x_i, x_{i+1}, \ldots, y_{i-1}, y_i\}$, where $\{x_i, x_{i+1}, \ldots, y_{i-1}, y_i\} \subseteq V(K_n)$ and the neighbour of type C vertex c_i (excluding the vertices of K_m) as $[1, x_i] \cup [y_i, n] = \{1, 2, \ldots, x_i, y_i, \ldots, n\}$, where $\{1, 2, \ldots, x_i, y_i, \ldots, n\} \subseteq V(K_n)$.

Theorem 3.1. [15] *A co-bipartite graph $\overline{B}(K_m, K_n)$, where $V(K_n) = \{1, 2, \ldots, n\}$, is word-representable if and only if there exists a vertex ordering for the vertices of K_m that satisfies the following conditions:*

(a) *The ordering of type A, B and C vertices is $A < C < B$.*

(b) *Suppose c_i and c_j are type C vertices, $c_i < c_j$, $c_i, c_j \in V(K_m)$. Let $N_{c_i} \setminus V(K_m) = [1, x_i] \cup [y_i, n]$, $N_{c_j} \setminus V(K_m) = [1, x_j] \cup [y_j, n]$. Then $x_i \leq x_j$ and $y_i \leq y_j$.*

(c) *Suppose a_i and a_j are type A or type B vertices, $a_i < a_j$, $a_i, a_j \in V(K_m)$. Let $N_{a_i} \setminus V(K_m) = [x_i, y_i]$, $N_{a_j} \setminus V(K_m) = [x_j, y_j]$. Then $x_i \leq x_j$ and $y_i \leq y_j$.*

(d) *Suppose a_i is type A (or B) vertex and c_j is type C vertex, $a_i < c_j$(or $a_i > c_j$ for type B), $a_i, c_j \in V(K_m)$. Let $N_{a_i} \setminus V(K_m) = [x_i, y_i]$, $N_{c_j} \setminus V(K_m) = [1, x_j] \cup [y_j, n]$. Then $x_i \leq y_j$ and $x_j \leq y_i$.*

(e) *Suppose a_i is type A vertex and b_j is type B vertex. Let $N_{a_i} \setminus V(K_m) = [x_i, y_i]$, $N_{b_j} \setminus V(K_m) = [x_j, y_j]$. Then $x_j \leq x_i$ and $y_j \leq y_i$.*

(f) *Suppose $v \in V(K_m)$ and $N_v \setminus V(K_m) = \emptyset$. Let a_i be the last type A vertex and c_j be any type C vertex. Then $a_i < v < c_j$ and $y_i < y_j$, where $N_{a_i} \setminus V(K_m) = [x_i, y_i]$, $N_{c_j} \setminus V(K_m) = [1, x_j] \cup [y_j, n]$.*

From the labelling conditions, we observe that only type A and type C vertices are enough to obtain a semi-transitive orientation. In the following theorem, we prove this statement.

Theorem 3.2. [15] *Let G be a word-representable co-bipartite graph, and let S be a semi-transitive orientation of G containing vertices of types A, B, and C. Then G admits another semi-transitive orientation in which every vertex is of either type A or type C, and no vertex is of type B.*

We determine the maximum possible representation number of a co-bipartite graph using the ordering obtained from Theorems 3.1 and 3.2. We show that this number is at most 3 by providing an algorithm that constructs a 3-word representation for any word-representable co-bipartite graph. In this algorithm, we take the matrix $M_{m\times n}$ as input, described below.

$M_{m\times n}$ is a matrix representation of the co-bipartite graph $\overline{B}(K_m, K_n)$ where

- the rows of $M_{m\times n}$ correspond to the vertices of K_m,
- the columns of $M_{m\times n}$ correspond to the vertices of K_n, and
- $M[i][j] = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}$

In the matrix $M_{m\times n}$, we define two types of vertices, where $1 \leq j \leq n$.

- If a is a type A vertex then, it is represented as $[x_i, y_i]$ where

$$M_{a\times j} = \begin{cases} 1, & \text{if } x_i \leq j \leq y_i, \\ 0, & \text{otherwise.} \end{cases}$$

- If c is a type C vertex then, it is represented as $[1, x_i] \cup [y_i, n]$ where

$$M_{c\times j} = \begin{cases} 1, & \text{if } 1 \leq j \leq x_i, \\ 1, & \text{if } y_i \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

One key point to note is that any vertex v with $v \in V(K_m)$ and $N_v \setminus V(K_m) = \varnothing$ is treated as a type A vertex in the algorithm.

Theorem 3.3. *The representation number of word-representable co-bipartite graphs is at most* 3.

Proof. We provide an algorithm to generate a 3-uniform word that represents a word-representable co-bipartite graph. The construction starts with three identical permutations of the vertices of K_n. Vertices of K_m are inserted so that each vertex alternates exactly with its neighbourhood interval. Type C vertices are inserted first to preserve the nesting of neighbourhoods, followed by type A vertices inserted according to the ordering constraints of Theorem 3.1. This guarantees a valid 3-uniform word-representation.

Algorithm (Word-representation of a word-representable co-bipartite graph).

Input: A matrix $M_{m\times n}$ that satisfies the conditions for a word-representable co-bipartite graph.
Let P be the permutation of the vertices of K_n according to their transitive orientation.
$w = P_1P_2P_3$, where $P_1 = P_2 = P_3 = P$.
Initialize empty dynamic arrays $A \leftarrow [\,]$ and $C \leftarrow [\,]$.
for each vertex $v \in V(K_m)$ in vertex ordering **do**
  if $N_v \setminus V(K_m) = [x, y]$ **then**
    append v to A
  else if $N_v \setminus V(K_m) = [1, x] \cup [y, n]$ **then**
    append v to C
  end if
end for
for $i = 1$ to length(C) **do**
  Let $c_i = C[i]$ and $N_{c_i} \setminus V(K_m) = [1, x_i] \cup [y_i, n]$
  if $i > 1$ and $N_{c_{i-1}} \setminus V(K_m) = [1, x_i] \cup [y_{i-1}, n]$ **then**
    replace x_ic_{i-1} with $x_ic_{i-1}c_i$ in P_1
  else
    replace x_i with x_ic_i in P_1.
  end if
  replace x_i with x_ic_i in P_2.
  replace y_i with c_iy_i in P_3.
end for
for $i = 1$ to length(A) **do**

Let $a_i = A[i]$ and $N_{a_i} \setminus V(K_m) = [x_i, y_i]$
Let $c_j = C[length(C)]$ and $N_{c_j} \setminus V(K_m) = [1, x_j] \cup [y_j, n]$
if $i > 1$ and $N_{a_{i-1}} \setminus V(K_m) = [x_{i-1}, y_i]$ **then**
  replace $y_i a_{i-1}$ with $y_i a_{i-1} a_i$ in P_1.
else
  replace y_i with $y_i a_i$ in P_1.
end if
if $x_i > x_j \,\&\, x_j c_j \in P_1$ **then**
  remove first occurrences of each C vertex.
  replace x_i with $a_i c_1 c_2 \cdots c_j x_i$ in P_1.
  replace y_i with $y_i a_i$ in P_2.
else
  Suppose $a_{i-1} = A[i-1]$ and $N_{a_{i-1}} \setminus V(K_m) = [x_{i-1}, y_{i-1}]$.
  if $y_{i-1} < x_i$ **then**
    remove second occurrences of each A vertex (up to a_{i-1}).
    replace x_i with $a_i a_1 \cdots a_{i-1} x_i$ in P_1.
    replace y_i with $y_i a_i$ in P_2.
  else
    replace x_i with $a_i x_i$ in P_1.
    replace y_i with $y_i a_i$ in P_2.
  end if
end if
end for
Return w

Our claim is that the word obtained from the algorithm is a word-representation of the word-representable co-bipartite graph.

Proof of Correctness

We prove that the algorithm returns a 3-uniform word representing the given word-representable co-bipartite graph G.

Step 1: Each vertex occurs exactly three times.

Initially, $w = P_1 P_2 P_3$, where $P_1 = P_2 = P_3 = P$. Therefore, every vertex of K_n occurs exactly once in each of P_1, P_2, and P_3, and hence exactly three times in w.

Now consider a vertex of K_m.

- Suppose c_i is a type C vertex. The algorithm inserts one occurrence of c_i into each of P_1, P_2, and P_3. Later, if the first occurrence of c_i is removed from P_1, it is immediately reinserted as part of the block $c_1 c_2 \cdots c_j$ in the same copy P_1. Hence, no occurrence of c_i is lost or duplicated, and therefore c_i occurs exactly three times in the final word.
- Suppose a_i is a type A vertex. The algorithm inserts one occurrence of a_i after y_i in P_1, one occurrence after y_i in P_2, and one occurrence before x_i in P_1. In the case $y_{i-1} < x_i$, the second occurrences of the previously inserted type A vertices are

removed and immediately reinserted before x_i. Consequently, no occurrence of any type A vertex is lost or duplicated. Hence, every type A vertex also occurs exactly three times in the final word.

Therefore, every vertex of G appears exactly three times in the final word. Since no vertex appears more or fewer than three times, the output word is 3-uniform.

Step 2: Every edge corresponds to alternation.

We now show that if u and v are adjacent in G, then u and v alternate in the final word.

- *Two vertices of K_n.*

  Since $w_{V(K_n)} = P_1P_2P_3 = P^3$, for any two vertices $u, v \in V(K_n)$, we have $w_{\{u,v\}} = uvuvuv$ or $w_{\{u,v\}} = vuvuvu$. Hence, every pair of vertices of K_n alternates.

- *A type A vertex and a vertex of K_n.*

  Let a_i be a type A vertex with $N_{a_i} \setminus V(K_m) = [x_i, y_i]$. The algorithm places one occurrence of a_i before x_i in P_1 and the remaining two occurrences after y_i in P_1 and P_2. Therefore, for every vertex $u \in [x_i, y_i]$, we obtain $w_{\{a_i,u\}} = a_iua_iua_iu$. Hence, a_i alternates with every vertex of its neighbourhood.

- *A type C vertex and a vertex of K_n.*

  Let c_i be a type C vertex with $N_{c_i} \setminus V(K_m) = [1, x_i] \cup [y_i, n]$. The algorithm inserts the occurrences of c_i after the first and second occurrences of x_i and before the third occurrence of y_i. Therefore, for every vertex $u \in [1, x_i] \cup [y_i, n]$, we obtain $w_{\{c_i,u\}} = c_iuc_iuc_iu$ or $w_{\{c_i,u\}} = uc_iuc_iuc_i$. Hence, c_i alternates with every vertex of its neighbourhood.

- *Two vertices of K_m.*

  - Suppose $u = a_i$ and $v = a_j$ are type A vertices. By Theorem 3.1, $x_i \leq x_j$ and $y_i \leq y_j$.

    If $y_i \geq x_j$, then the occurrences of a_i and a_j are naturally interleaved, and $w_{\{a_i,a_j\}} = a_ia_ja_ia_ja_ia_j$.

    If $y_i < x_j$, the algorithm relocates the second occurrence of a_i immediately after the first occurrence of a_j. Consequently, $w_{\{a_i,a_j\}} = a_ia_ja_ia_ja_ia_j$.

    Hence, a_i and a_j alternate.

  - Suppose $u = c_i$ and $v = c_j$ are type C vertices. By Theorem 3.1, $x_i \leq x_j$ and $y_i \leq y_j$. Since the insertion rule preserves this nesting, $w_{\{c_i,c_j\}} = c_ic_jc_ic_jc_ic_j$ or its reverse. Therefore, c_i and c_j alternate.

  - Suppose $u = a_i$ and $v = c_j$. By Theorem 3.1, $x_i \leq y_j$ and $x_j \leq y_i$.

    If $x_i \leq x_j$, then $w_{\{a_i,c_j\}} = a_ic_ja_ic_ja_ic_j$.

    If $x_i > x_j$, the first occurrences of the type C vertices are relocated before inserting a_i, and again $w_{\{a_i,c_j\}} = a_ic_ja_ic_ja_ic_j$.

    Hence, a_i and c_j alternate.

Therefore, every edge of G corresponds to alternation.

Step 3: Every non-edge corresponds to non-alternation.

We now show that if u and v are non-adjacent in G, then u and v do not alternate in the final word.

- *Non-neighbours of a type A vertex.*

  Let a_i be a type A vertex with $N_{a_i} \setminus V(K_m) = [x_i, y_i]$.

  For every vertex $u \in \{1, \dots, x_i - 1\}$, we have $w_{\{a_i,u\}} = ua_ia_iua_iu$, which contains the factor a_ia_i. Hence, a_i and u do not alternate.

  For every vertex $v \in \{y_i + 1, \dots, n\}$, if the second occurrence of a_i is not relocated, then $w_{\{a_i,v\}} = a_ia_iva_ivv$, which contains the factor a_ia_i or vv.

  If the second occurrence of a_i is relocated in Line 38, then for every $v_1 \in \{y_i + 1, \dots, x_j - 1\}$ and $v_2 \in \{x_j, \dots, n\}$, the words $w_{\{a_i,v_1\}}$ and $w_{\{a_i,v_2\}}$ contain consecutive occurrences of either a_i or the corresponding vertex. In particular, $w_{\{a_i,v_1\}} = a_iv_1a_ia_iv_1v_1$ and $w_{\{a_i,v_2\}} = a_ia_iv_2a_iv_2v_2$, showing that a_i does not alternate with v_1 or v_2.

  Hence, a_i does not alternate with any vertex in $\{1, \dots, x_i - 1, y_i + 1, \dots, n\}$.

- *Non-neighbours of a type C vertex.*

  Let c_i be a type C vertex with $N_{c_i} \setminus V(K_m) = [1, x_i] \cup [y_i, n]$.

  For every vertex $u \in \{x_i + 1, \dots, y_i - 1\}$, if the first occurrence of c_i is not relocated, then $w_{\{c_i,u\}} = c_iuc_iuuc_i$, which contains the factor uu.

  If the first occurrence of c_i is relocated in Line 32, then the second and third occurrences of c_i remain unchanged, and therefore $w_{\{c_i,u\}}$ still contains consecutive occurrences of u. Hence, c_i and u do not alternate.

  Therefore, c_i does not alternate with any vertex in $\{x_i + 1, \dots, y_i - 1\}$.

The final word returned by the algorithm is a 3-uniform word representing G. Consequently, representation number of G is at most 3. □

It has been shown that permutation graphs are precisely the co-bipartite circle graphs.

Theorem 3.4. [34] *If a bipartite graph G is the complement of a circle graph, then both G and its complement are permutation graphs.*

Since complete graphs have representation number 1, the following corollary characterises non-complete word-representable co-bipartite graphs with representation number 2.

Corollary 3.5. *Let G be a non-complete word-representable co-bipartite graph. Then G has representation number 2 if and only if G is a permutation graph. Consequently, if G is not a permutation graph, then its representation number is 3.*

Proof. Assume that G has representation number 2. By Theorem 2.6, G is a circle graph. Since G is co-bipartite, Theorem 3.4 implies that G is a permutation graph.

Conversely, suppose that G is a permutation graph. Every permutation graph is a circle graph. Hence, by Theorem 2.6, G admits a 2-uniform word-representation, and therefore its representation number is 2.

Thus, a non-complete word-representable co-bipartite graph has representation number 2 if and only if it is a permutation graph.

Now, let G be a non-complete word-representable co-bipartite graph that is not a permutation graph. By the above equivalence, G does not have representation number 2. Since G is non-complete, its representation number is greater than 1. Furthermore, Theorem 3.3 states that every word-representable co-bipartite graph has representation number at most 3. Hence, the representation number of G is 3. □

After determining the representation number, we now study the number of labelled word-representable co-bipartite graphs on n vertices. In the next subsection, we determine the speed and entropy of this class.

4. Speed and entropy

In this section, we study the asymptotic enumeration of word-representable co-bipartite graphs using the notions of speed and entropy.

Definition 4.1. Let $\mathcal{X}$ be a class of labelled graphs and let $\mathcal{X}_n$ denote the number of graphs in $\mathcal{X}$ on n vertices. The function $\mathcal{X}_n$, viewed as a function of n, is called the *speed* of the class $\mathcal{X}$.

Definition 4.2. Let $\mathcal{X}$ be a class of labelled graphs with speed $\mathcal{X}_n$. The *entropy* of the class $\mathcal{X}$ is defined as

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{X}_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)},$$

where $k(X)$ is called the *index* of the class X. The index is defined in terms of the graph classes $\mathcal{E}_{i,j}$, which consist of graphs whose vertex set can be partitioned into at most i independent sets and j cliques. The index $k(X)$ is the maximum integer k such that X contains the class $\mathcal{E}_{i,j}$ for some integers i and j satisfying $i + j = k$.

The speed and entropy measure the growth rate and structural complexity of a graph class. For example, the class of all graphs has speed $2^{\binom{n}{2}} = 2^{\Theta(n^2)}$ and entropy 1. The class of co-bipartite graphs also has speed $2^{\Theta(n^2)}$ and entropy $\frac{1}{2}$, since co-bipartite graphs belong to the class $\mathcal{E}_{0,2}$ in the Alekseev–Bollobás–Thomason theorem. Similarly, it was shown in [12] that the entropy of word-representable graphs is $\frac{2}{3}$. By the Alekseev–Bollobás–Thomason theorem [1, 5], it follows that the speed of word-representable graphs is $2^{\Theta(n^2)}$.

We now show that the class of word-representable co-bipartite graphs grows much slower than both the class of all co-bipartite graphs and the class of all word-representable graphs.

Theorem 4.3. *Let $\mathcal{W}$ be the class of word-representable co-bipartite graphs. Then the number $\mathcal{W}_n$ of labelled graphs in $\mathcal{W}$ on n vertices satisfies*

$$\mathcal{W}_n \leq 2^{O(n \log n)}.$$

In particular, the entropy of the class $\mathcal{W}$ is 0.

Proof. Let $G \in \mathcal{W}$ be a word-representable co-bipartite graph on n vertices. Then the vertex set of G can be partitioned into two cliques K_a and K_b such that $a + b = n$ and $0 \leq b \leq a \leq n$. Since all edges between vertices within each clique are present by definition, these edges are fixed and do not contribute to the counting. Therefore, it suffices to count the possible edge sets between the two cliques.

Fix an ordering of the vertices of K_b as $V(K_b) = \{1, 2, \dots, b\}$. By the vertex ordering characterisation, for each vertex $v \in K_a$, the neighbourhood of v in K_b is either of the form

$$N(v) \setminus K_a = [x, y] \quad \text{or} \quad N(v) \setminus K_a = [1, x] \cup [y, b],$$

for some integers $1 \leq x \leq y \leq b$.

We first count the number of possible interval subsets. An interval $[x, y]$ is uniquely determined by choosing its left endpoint x and right endpoint y with $1 \leq x \leq y \leq b$. For a fixed x, there are exactly $(b - x + 1)$ choices for y. Therefore, the total number of interval subsets is

$$\sum_{x=1}^{b} (b - x + 1) = \frac{b(b+1)}{2} = O(b^2).$$

Similarly, a circular interval subset of the form $[1, x] \cup [y, b]$ is uniquely determined by the pair (x, y) satisfying $1 \leq x \leq y \leq b$. Since there are at most b^2 such pairs, the number of circular interval subsets is also $O(b^2)$.

Thus, each vertex in K_a has at most $cb^2 \leq cn^2$ possible neighbourhoods, for some constant c.

Since the neighbourhood of each vertex in K_a is determined by at most $O(b^2)$ possible intervals satisfying the vertex ordering conditions, the total number of possible neighbourhood assignments is at most $(cb^2)^a$. Therefore, for a fixed partition into cliques of sizes a and b, the number of possible graphs is at most $(cb^2)^a$. For each choice of a, there are $\binom{n}{a}$ ways to select the vertices of the clique K_a from the n vertices. Thus,

$$\mathcal{W}_n \leq \sum_{a=0}^{n} \binom{n}{a} (cb^2)^a \leq \sum_{a=0}^{n} \binom{n}{a} (cn^2)^a = (1 + cn^2)^n.$$

Taking logarithms,

$$\log_2 \mathcal{W}_n \leq n \log_2(1 + cn^2) = O(n \log n).$$

Hence,

$$\mathcal{W}_n \leq 2^{O(n \log n)}.$$

We now determine the entropy of the class $\mathcal{W}$. By the Alekseev–Bollobás–Thomason theorem, it suffices to determine the index $k(\mathcal{W})$.

Since $\mathcal{W}$ contains all complete graphs, it contains the class $\mathcal{E}_{0,1}$. Hence, $k(\mathcal{W}) \geq 1$.

We next show that $k(\mathcal{W}) < 2$. Observe that $\mathcal{W}$ does not contain the class $\mathcal{E}_{0,2}$, since $\mathcal{E}_{0,2}$ is precisely the class of all co-bipartite graphs, whereas not every co-bipartite graph is word-representable.

Further, $\mathcal{W}$ does not contain the classes $\mathcal{E}_{2,0}$ and $\mathcal{E}_{1,1}$, which consist of all bipartite graphs and all split graphs, respectively.

Therefore, $\mathcal{W}$ contains none of the classes $\mathcal{E}_{i,j}$ with $i + j = 2$. Hence, $k(\mathcal{W}) = 1$.

Substituting $k(\mathcal{W}) = 1$ into the Alekseev–Bollobás–Thomason formula,

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{W}_n}{\binom{n}{2}} = 1 - \frac{1}{k(\mathcal{W})},$$

we obtain

$$\lim_{n\to\infty} \frac{\log_2 \mathcal{W}_n}{\binom{n}{2}} = 0.$$

Hence, the entropy of the class $\mathcal{W}$ is 0. □

5. Conclusion

In this paper, using necessary and sufficient conditions for a co-bipartite graph to be word-representable in terms of vertex orderings, we show that every word-representable co-bipartite graph has a representation number at most 3. We also determine the speed and entropy of the class of word-representable co-bipartite graphs. As future work, this vertex-ordering characterisation can be used to analyse the graph parameters of this class.

References

[1] V. E. Alekseev. Range of values of entropy of hereditary classes of graphs. Russian. *Diskretnaya Matematika*, 4(2):148–157, 1992.

[2] N. Alon, J. Balogh, B. Bollobás, and R. Morris. The structure of almost all graphs in a hereditary property. *Journal of Combinatorial Theory, Series B*, 101(2):85–110, 2011. https://doi.org/10.1016/j.jctb.2010.10.001.

[3] H. L. Bodlaender and K. Jansen. On the complexity of the maximum cut problem. *Nordic Journal of Computing*, 7(1):14–31, 2000.

[4] R. Boliac and V. Lozin. On the clique-width of graphs in hereditary classes. In *Algorithms and Computation*, volume 2518 of *Lecture Notes in Computer Science*, pages 44–54, Berlin, Heidelberg. Springer, 2002. 10.1007/3-540-36136-7_5.

[5] B. Bollobás and A. Thomason. Projections of bodies and hereditary properties of hypergraphs. *Bulletin of the London Mathematical Society*, 27(5):417–424, 1995. https://doi.org/10.1112/blms/27.5.417.

[6] B. Bollobás and A. Thomason. Hereditary and monotone properties of graphs. In R. L. Graham and J. Nešetřil, editors, *The Mathematics of Paul Erdős II*. Volume 14, Algorithms and Combinatorics, pages 70–78. Springer, Berlin, Heidelberg, 1997. 10.1007/978-3-642-60406-5_7.

[7] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey*, volume 3 of *SIAM Monographs on Discrete Mathematics and Applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999. 10.1137/1.9780898719796.

[8] A. Brandstädt, V. B. Le, and T. Szymczak. The complexity of some problems related to graph 3-colorability. *Discrete Applied Mathematics*, 89(1–3):59–73, 1998. https://doi.org/10.1016/S0166-218X(98)00116-4.

[9] B. Broere and H. Zantema. The k-cube is k-representable. *Journal of Automata, Languages and Combinatorics*, 24(1):3–12, 2019.

[10] H. Z. Q. Chen, H. Hameed, and S. Kitaev. On the word-representability of K_m-K_n graphs. *Discussiones Mathematicae Graph Theory*, 46(2):321–339, 2026. https://doi.org/10.7151/dmgt.2605.

[11] H. Z. Q. Chen, S. Kitaev, and A. Saito. Representing split graphs by words. *Discussiones Mathematicae Graph Theory*, 42(4):1263–1280, 2022. https://doi.org/10.7151/dmgt.2344.

[12] A. Collins, S. Kitaev, and V. V. Lozin. New results on word-representable graphs. *Discrete Applied Mathematics*, 216:136–141, 2017. https://doi.org/10.1016/j.dam.2014.10.024.

[13] P. Damaschke. Forbidden ordered subgraphs. In R. Bodendiek and R. Henn, editors, *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, pages 219–229. Physica-Verlag, Heidelberg, 1990. 10.1007/978-3-642-46908-4_25.

[14] B. Das and R. Hariharasubramanian. Word-representability of co-bipartite graph, 2025. 10.48550/arXiv.2501.10112. arXiv: 2501.10112 [math.CO].

[15] B. Das and R. Hariharasubramanian. Word-representable co-bipartite graphs: vertex ordering, representation number, speed, and entropy, 2025. 10.48550/arXiv.2509.03064. arXiv: 2509.03064 [math.CO].

[16] T. Dwary, K. Mozhui, and K. V. Krishna. Representation number of word-representable split graphs, 2025. 10.48550/arXiv.2502.00872. arXiv: 2502.00872 [math.CO].

[17] Z. Feng, H. Fernau, P. Fleischmann, P. Kindermann, and S. C. Sacher. Determining factorial speed fast, 2026. 10.48550/arXiv.2602.24064. arXiv: 2602.24064 [cs.DM].

[18] L. Feuilloley and M. Habib. Graph classes and forbidden patterns on three vertices. *SIAM Journal on Discrete Mathematics*, 35(1):55–90, 2021. https://doi.org/10.1137/19M1280399.

[19] M. Glen, S. Kitaev, and A. Pyatkin. On the representation number of a crown graph. *Discrete Applied Mathematics*, 244:89–93, 2018. https://doi.org/10.1016/j.dam.2018.03.013.

[20] M. M. Halldórsson, S. Kitaev, and A. Pyatkin. Alternation graphs. In P. Kolman and J. Kratochvíl, editors, *Graph-Theoretic Concepts in Computer Science*, volume 6986 of *Lecture Notes in Computer Science*, pages 191–202, Berlin, Heidelberg. Springer, 2011. 10.1007/978-3-642-25870-1_18.

[21] M. M. Halldórsson, S. Kitaev, and A. Pyatkin. Semi-transitive orientations and word-representable graphs. *Discrete Applied Mathematics*, 201:164–171, 2016. https://doi.org/10.1016/j.dam.2015.07.033.

[22] Z. Hefty, P. Horn, C. Muir, and A. Owens. Word-representation numbers of graphs: bottlenecks and bounds. *Journal of Combinatorial Theory, Series A*, 223:106215, 2026. https://doi.org/10.1016/j.jcta.2026.106215.

[23] K. Iamthong. Word-representability of split graphs generated by morphisms. *Discrete Applied Mathematics*, 314:284–303, 2022. https://doi.org/10.1016/j.dam.2022.02.023.

[24] K. Iamthong and S. Kitaev. Semi-transitivity of directed split graphs generated by morphisms. *Journal of Combinatorics*, 14(1):111–138, 2023. https://doi.org/10.4310/JOC.2023.v14.n1.a5.

[25] S. Kitaev. A comprehensive introduction to the theory of word-representable graphs. In É. Charlier, J. Leroy, and M. Rigo, editors, *Developments in Language Theory*, volume 10396 of *Lecture Notes in Computer Science*, pages 36–67, Cham. Springer, 2017. 10.1007/978-3-319-62809-7_2.

[26] S. Kitaev, Y. Long, J. Ma, and H. Wu. Word-representability of split graphs. *Journal of Combinatorics*, 12(4):725–746, 2021. https://doi.org/10.4310/JOC.2021.v12.n4.a6.

[27] S. Kitaev and V. Lozin. *Words and Graphs*. Monographs in Theoretical Computer Science. An EATCS Series. Springer, Cham, 2015. 10.1007/978-3-319-25859-1.

[28] S. Kitaev and A. Pyatkin. On representable graphs. *Journal of Automata, Languages and Combinatorics*, 13(1):45–54, 2008.

[29] S. Kitaev and A. Pyatkin. On semi-transitive orientability of split graphs. *Information Processing Letters*, 184:106435, 2024. https://doi.org/10.1016/j.ipl.2023.106435.

[30] S. Kitaev and S. Seif. Word problem of the Perkins semigroup via directed acyclic graphs. *Order*, 25(3):177–194, 2008. https://doi.org/10.1007/s11083-008-9083-7.

[31] S. Kitaev and H. Sun. Human-verifiable proofs in the theory of word-representable graphs. *RAIRO–Theoretical Informatics and Applications*, 58:10, 2024. https://doi.org/10.1051/ita/2024004. Article 10, 10 pages.

[32] S. Olariu. An optimal greedy heuristic to color interval graphs. *Information Processing Letters*, 37(1):21–25, 1991. https://doi.org/10.1016/0020-0190(91)90245-D.

[33] M. A. Shalu, S. Vijayakumar, and T. P. Sandhya. On the complexity of CD-coloring of graphs. *Discrete Applied Mathematics*, 280:171–185, 2020. https://doi.org/10.1016/j.dam.2018.03.004.

[34] E. Srinivasan and R. Hariharasubramanian. Bipartite complement of circle graphs using forbidden patterns. SSRN Electronic Journal, 2025. https://doi.org/10.2139/ssrn.5128202. SSRN 5128202.

Biswajit Das
Department of Mathematics, Indian Institute of Technology Guwahati
Guwahati, Assam 781039, India
E-mail d.biswajit@iitg.ac.in

Ramesh Hariharasubramanian
Department of Mathematics, Indian Institute of Technology Guwahati
Guwahati, Assam 781039, India
E-mail ramesh_h@iitg.ac.in